On cyclic decompositions of some complete directed graphs into antidirected cycles

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Abstract

An antidirected cycle of length n is the digraph obtained by orienting the edges of a cycle of length n such that no directed path of length 2 is formed. For every positive integer x, we show that there exists a cyclic decomposition of the complete symmetric digraph K^*_{nx+1} into antidirected cycles of length n.

1 Introduction

Let D and H be digraphs. A *D-decomposition* of H is a partition of the arc set of H into isomorphic copies of D, called *D-blocks*. Of particular interest is the case where H is K^*_n, the complete symmetric digraph on n vertices (containing a directed 2-cycle between every pair of distinct vertices). Throughout this note, we will consider the vertex set of K^*_n to be $\mathbb{Z}_n$, the group of integers modulo n. An *antidirected cycle* of length n, denoted AC_n, is obtained by orienting the edges of a cycle of length n such that no directed path of length 2 is induced. Antidirected cycles necessarily have an even number of arcs.

A D-decomposition of K^*_n is called *cyclic* if applying the isomorphism $i \mapsto i + 1$ preserves the D-blocks of the decomposition. Necessary and sufficient conditions for the existence of complete symmetric digraphs into directed cycles were found by Alspach, Gavlas, Šajna, and Verrall in [1]. Decompositions of certain complete symmetric digraphs into antidirected cycles are investigated in [2]. In [3], Bogdanowicz investigated decompositions of certain digraph circulants into antidirected Hamilton cycles. In [7],

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Jordon and Morris give necessary and sufficient conditions for the existence of cyclic directed Hamiltonian cycle systems of the complete symmetric digraph. In this note, we show that there exists a cyclic AC_n-decomposition of K^*_{nx+1} for every positive integer x.

Cyclic D-decompositions of K^*_n are often obtained by labeling the vertices of D. In order to construct cyclic AC_k-decompositions of K^*_n we adapt some techniques traditionally used in the labeling of undirected graphs.

The *length* of an arc (i,j) in K^*_n is defined to be $j-i$ if $i<j$, and $n+j-i$ otherwise. If the vertices of K^*_n are viewed as if placed in order around an regular n-gon, the length of an arc in K^*_n can be interpreted as the clockwise distance around the polygon from the initial vertex to the terminal vertex of the arc. Now consider a digraph D with n vertices. Given two integers x and y with $x<y$, denote the integer interval $\{x, x+1, \dots, y\}$ by $[x,y]$. Let $f\colon V(D)\to[0,n]$ be an injective function (called a *labeling* of D). Note that f induces a function $\bar{f}\colon E(D)\to[1,n]$ defined by $\bar{f}(u,v)=f(v)-f(u)$ if $f(v)>f(u)$, and $\bar{f}(u,v)=n+1+f(v)-v(u)$, otherwise. When $\{\bar{f}(a,b):(a,b)\in E(D)\}=[1,n]$, f is called a *directed ρ-labeling* of D. Notice that a directed ρ-labeling of D is an embedding of D into K^*_{n+1} such that each edge length is used exactly once.

Suppose we have a bipartite digraph D with vertex bipartition (A,B) and directed ρ-labeling f. If $f(a)<f(b)$ for all $(a,b)\in E(D)$ where $a\in A$ and $b\in B$, then we call f a *directed ρ^+-labeling*. The connection between directed ρ^+-labelings and cyclic decompositions is shown in the following theorems.

Theorem 1 (Kaplan et al. [8])**.** *If D is a digraph with n arcs that admits a directed ρ-labeling, then a cyclic D-decomposition of K^*_{n+1} exists.*

Theorem 2 (Bunge et al. [5])**.** *If D is a bipartite digraph with n arcs that admits a directed ρ^+-labeling, then a cyclic D-decomposition of K^*_{nx+1} exists for all positive integers x.*

2 Labelings of graphs

All of the structures defined for digraphs in the previous section have analogous definitions for graphs. We also consider the vertex set of the complete graph on n vertices, denoted K_n, to be $\mathbb{Z}_n$. The length of an edge $\{i,j\}$ in the complete graph K_n is $\min\{|i-j|, n-|i-j|\}$, which can be interpreted as the shortest distance between i and j around the polygon of n vertices.

There are many graph labelings that give rise to various types of graph decompositions (see [6]). In particular, we will use α-labelings to construct cyclic AC_n-decompositions. An α-labeling of a bipartite graph G with n edges and vertex bipartition (A,B) is an injective function $f\colon V(G)\to[0,n]$

such that the induced edge labeling defined by $\bar{f}(a,b) = |f(a) - f(b)|$ has the property $\{\bar{f}(e) : e \in E(G)\} = [1, n]$, and there exists some positive integer λ such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

The following two theorems conerning graph decompositions will be useful for constructing cyclic decompositions of antidirected cycles.

Theorem 3 (Rosa [10])**.** *Let n be a positive integer with $n \equiv 0 \pmod 4$. Then the cycle of length n admits an α-labeling.*

Theorem 4 (Kotzig [9])**.** *Let n be a positive integer with $n \equiv 2 \pmod 4$. Then the vertex-disjoint union of two cycles of length n admits an α-labeling.*

Note that the vertex set of AC_n can be partitioned into two sets A and B such that the initial vertex of each arc is in A. Thus, an α-labeling of the underlying undirected graph of AC_n gives rise to a directed ρ^+-labeling of AC_n. This immediately leads to the following two corollaries.

Corollary 5. *Let n be a positive integer with $n \equiv 2 \pmod 4$. Then a cyclic AC_n-decomposition of K^*_{nx+1} exists for all even positive integers x.*

Corollary 6. *Let n be a positive integer with $n \equiv 0 \pmod 4$. Then a cyclic AC_n-decomposition of K^*_{nx+1} exists for all positive integers x.*

In light of the above corollaries, it remains to be shown that cyclic AC_n-decompositions of K^*_{nx+1} exists for any even n and all odd positive integers x. That will constitute our main result.

3 Main Results

We denote the path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \leq i \leq k-1$, by $(x_0, x_1, \ldots, x_k)$. If $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$ are directed paths with $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$.

Lemma 7. *Let $n \equiv 2 \pmod 4$. Then there exists a cyclic AC_n-decomposition of K^*_{n+1}.*

Proof. Let $n = 4k+2$ where k is a positive integer. Consider the following labeling of AC_n obtained by identifying the endpoints of the following labeled path of length n.

$$\begin{aligned}
&(4k+1, 0, 4k, 1, 4k-1, 2, \ldots, 3k+3, k-2, 3k+2, k-1) \\
&+ (k-1, k, k+1) \\
&+ (k+1, 3k, k+2, 3k-1, \ldots, 2k+3, 2k-1, 2k+2, 2k) \\
&+ (2k, 4k+2, 2k+1, 4k+1)
\end{aligned}$$

It is easy to verify that the above labeling is a directed ρ-labeling of AC_n. By Theorem 1, there exists a cyclic AC_n-decomposition of K^*_{n+1}. ■

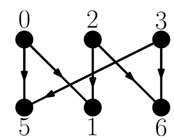

Figure 1: A directed ρ-labeling of AC_6 from Lemma 7.

We now give our main result.

Theorem 8. *There exists a cyclic AC_n-decomposition of K^*_{nx+1} for all even integers $n \geq 4$ and all positive integers x.*

Proof. If $n \equiv 0 \pmod 4$, the result follows by Corollary 6. Otherwise, let $n = 4k+2$ where k is a positive integer and let x be a positive integer. If x is even, then the existence of a cyclic AC_n-decomposition of K^*_{nx+1} follows from Corollary 5. If $x = 1$, then the result follows from Lemma 7. So we can assume that $x \geq 3$ is odd.

Let D be the vertex-disjoint union of two antidirected cycles of length n, and let D have vertex partition (A, B) such that the initial vertex of each arc lies in A. Let f be the α-labeling of the underlying undirected graph of D, guaranteed to exist by Theorem 4, such that the vertex labels in B are larger than the vertex labels in A. We are considering f to be a labeling of D. Note that the arc labels induced by $\bar{f}$ form the set $[1, 2n]$.

For each $1 \leq i \leq \frac{x-1}{2}$ let B_i be a copy of B. For each $b \in B$, the vertex corresponding to b in B_i will be denoted by b_i. Now, let D_i be a copy of D with vertex partition (A, B_i). Define the digraph $D' = \bigcup_{i=1}^{(x-1)/2} D_i$.

Define the labeling f' on the vertices of D' as follows:

$$f'(v) = \begin{cases} f(a) & \text{if } v = a \in A, \\ f(b) + n - 1 + (i-1)2n & \text{if } v = b_i \in B_i. \end{cases}$$

The original labeling of f induces arc labels $[1, 2n]$. So for each $1 \leq i \leq \frac{x-1}{2}$, the edge labels induced by f' on D_i are $[1+n-1+(i-1)2n, 2n+n-1+(i-1)2n]$. Based on the definition of the labeling, we have that $\{f'(e) : e \in E(D')\} = [n, 3n-1] \cup [3n, 5n-1] \cup \cdots \cup [n(x-2), nx-1] = [n, nx-1]$

Now let H be a single copy of AC_n and let g be a directed ρ-labeling of AC_n obtained from the proof of Lemma 7. It is important to note that before being reduced modulo $nx+1$, g contains the edge labels $[1, n-1]$ and -1.

Finally, define the labeling h on $H \cup D'$ as follows.

$$h(v) = \begin{cases} f'(v) & \text{if } v \in V(D'), \\ g(v) & \text{if } v \in V(H). \end{cases}$$

Then $\{\bar{h}(e) : e \in E(H \cup D')\} = [1, nx]$, and h is a directed ρ-labeling of $H \cup D'$. Thus, there exists a cyclic AC_n-decomposition of K^*_{nx+1}. ■

Example 1. *As an example, we will show how to obtain a cyclic AC_6-decomposition of K^*_{31}.*

The directed ρ-labeling of AC_6 is shown in Figure 1. Figure 2 shows an α-labeling of the vertex-disjoint union of two cycles of length 6, which will be used to obtain the stretched labeling on the remaining four copies of AC_6. Figure 3 shows the labeling described in the proof of Theorem 8 which is obtained by stretching the α-labeling. The 5 labeled copies of AC_6 contained in Figures 1 and 3 are used as the base blocks for a cyclic AC_6-decomposition of K^*_{31}.

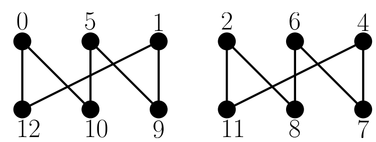

Figure 2: An α-labeling of the vertex-disjoint union of two cycles of length 6.

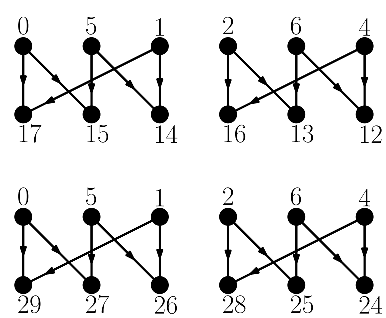

Figure 3: These four copies of AC_6 combined with the copy in Figure 1 lead to a cyclic AC_6-decomposition of K^*_{31}.

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