On decomposing regular graphs into star forests

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Abstract

Let $G$ be a forest with $n$ edges. El-Zanati conjectures that $G$ necessarily decomposes every $2n$-regular graph and every $n$-regular bipartite graph. We confirm these conjectures in the case when $G$ consists of two stars.

1 Introduction

For integers $a$ and $b$ with $a \leq b$, let $[a, b] = \{a, a + 1, \ldots, b\}$. For a positive integer $n$, let $\mathbb{Z}_n$ denote the group of integers modulo $n$. For a graph $G$ with vertex set $V(G)$ and edge set $E(G)$, the order of $G$ is $|V(G)|$ and the size of $G$ is $|E(G)|$. The graph $K_{1,k}$ is known as a $k$-star and is often denoted by $S_k$. A double-star is a tree with exactly two vertices of degree greater than 1. The two vertices of degree greater than 1 are called the centers of the double-star and the edge joining them is called the central-edge. If $T$ is a double-star where the two centers have degrees $k_1 + 1$ and $k_2 + 1$, then $T$ is denoted by $S_{k_1,k_2}$. Note that $S_{k_1,k_2}$ has $k_1 + k_2 + 1$ edges and is isomorphic to $S_{k_2,k_1}$. For a graph $G$ and a positive integer $t$, let $tG$ denote the vertex disjoint union of $t$ copies of $G$.

Let $H$ and $G$ be graphs with $G$ a subgraph of $H$. A $G$-decomposition of $H$ is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of $H$ each of which is isomorphic to $G$ and such that each edge of $H$ appears in exactly one $G_i$. If there exists a $G$-decomposition of $H$, then we say $G$ decomposes $H$.

A large amount of research has been done on the topic of graph decompositions over the last five decades (see [2] and [3] for surveys). Much investigation has been motivated by a conjecture of Ringel [15] on decomposing complete graphs into trees.

Conjecture 1. Every tree $T$ with $n$ edges decomposes the complete graph $K_{2n+1}$.

A folklore conjecture similar to Ringel’s relates to decomposing complete bipartite graphs into trees.

Conjecture 2. Every tree $T$ with $n$ edges decomposes the complete bipartite graph $K_{n,n}$.

Both of the above conjectures are special cases of conjectures due to Graham and Häggkvist (see [9]).

Conjecture 3. Every tree $T$ with $n$ edges decomposes every $2n$-regular graph $H$.

Conjecture 4. Every tree $T$ with $n$ edges decomposes every $n$-regular bipartite graph $H$.

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Despite persistent attacks over the last 40 years, Ringel’s conjecture and variations thereof, such as the Graceful Tree Conjecture (see [8]), still stand today. Much less work has been done on the Graham and Häggkvist conjectures.

Results confirming Conjecture 3, in certain cases, can be found in [9] by Häggkvist, in [4], and in Snevily’s Ph.D. thesis [17]. Some recent extensions of Snevily’s results can be found in a paper by Jao, Kostochka, and West [14]. In [13], Jacobson, Truszczyński, and Tuza confirm Conjecture 4 for double-stars and for the path with 4 edges. Fink [7] confirms Conjecture 4 when $H$ is the $n$-cube. Also, it is easy to see that $S_n$ decomposes every $2n$-regular graph as well as every $n$-regular bipartite graph.

El-Zanati proposes that the conjectures by Graham and Häggkvist hold for forests with $n$ edges.

**Conjecture 5.** Every forest $G$ with $n$ edges decomposes every $2n$-regular graph $H$.

**Conjecture 6.** Every forest $G$ with $n$ edges decomposes every $n$-regular bipartite graph $H$.

In this note, we provide some evidence in support of Conjectures 5 and 6. In particular, we show that the conjectures hold when $G$ is the vertex-disjoint union of two stars.

### 2 Known Results

We begin by defining three graph labelings introduced by Rosa [16] as means for attacking problems like Ringel’s Conjecture. Let $G$ be a graph with $n$ edges and let $f : V(G) \to [0, 2n]$ and $g : V(G) \to [0, n]$ be one-to-one functions. Then $f$ is a $\sigma$-labeling of $G$ if $\{|f(v) - f(u)| : \{u, v\} \in E(G)\} = [1, n]$ and $g$ is a $\beta$-labeling if $\{|g(v) - g(u)| : \{u, v\} \in E(G)\} = [1, n]$. If in addition $G$ is bipartite with vertex bipartition $\{A, B\}$, then a $\beta$-labeling $g$ of $G$ is an $\alpha$-labeling if $\max\{g(u) : u \in A\} < \min\{g(v) : v \in B\}$. Thus an $\alpha$-labeling of $G$ is also a $\beta$-labeling which is also a $\sigma$-labeling of $G$. We have the following results (see [16] and [5]).

**Theorem 7.** Let $G$ be a graph with $n$ edges. If $G$ admits a $\sigma$-labeling, then there exists a $G$-decomposition of $K_{2n+1}$ and of $K_{2n+2} - I$, where $I$ is a 1-factor. If in addition, $G$ is bipartite and $G$ admits an $\alpha$-labeling, then there also exists a $G$-decomposition of $K_{n,n}$.

It is known that paths, stars, and all caterpillars in general admit $\alpha$-labelings (see [16]). It is also known that trees with up to 35 edges admit $\beta$-labelings (see [8]). We also have the following result from [10].

**Theorem 8.** The disjoint union of a graph with a $\beta$-labeling, together with a collection of graphs with $\alpha$-labelings, has a $\sigma$-labeling.

An example of a $\sigma$-labeling of a star forest with 7 components and 15 edges is given in Figure 1.

![Figure 1: A $\sigma$-labeling of a star forest.](image-url)
Corollary 9. Let $G$ be a forest with $n$ edges. If one component of $G$ is a tree on up to 36 vertices and all other components are caterpillars, then $G$ decomposes $K_{2n+1}$ and $K_{2n+2} - I$, where $I$ is a 1-factor.

As for Conjecture 6, a consequence of a result by Horak, Širáň, and Wallis [11] ensures that every forest with $n$ edges decomposes the $n$-cube.

Also, Conjectures 5 and 6 hold when $G = nK_2$ as a consequence of a result by Alon [1].

Lemma 10. For every graph $G$ and every $t \geq 1$, $tK_2$ decomposes $G$ if and only if $t$ divides $|E(G)|$ and $\chi'(G) \leq |E(G)|/t$.

Corollary 11. Let $G = nK_2$ and suppose $H$ is either $n$-regular and bipartite or $2n$-regular. Then $G$ decomposes $H$.

3 Main Results

We give some additional definitions before proceeding with our main results. An orientation of a graph $H$ is an assignment of directions to the edges of $H$. An Eulerian orientation of $H$ is an orientation where the indegree at each vertex is equal to the outdegree. It is simple to see that a graph with all even degrees has an Eulerian orientation.

Theorem 12. Let $G$ be a double-star with $n$ edges and let $H^+$ be a $2n$-regular multigraph with edge-multiplicity at most 2. Suppose the subgraph $M$ of $H^+$ consisting of the edges of multiplicity 2 is either empty or 2-regular. Let $F$ be a 2-factor of $H^+$ that contains every component of $M$. Then there exists a $G$-decomposition $\Delta$ of $H^+$ with the property that the edges of $F$ are the center-edges of the double-stars in $\Delta$.

Proof. Let $G$ be the double-star $S_{k_1,k_2}$ with center vertices $a$ and $b$, where the degree of $a$ is $k_1 + 1$ and the degree of $b$ is $k_2 + 1$. Let $H^+, D,$ and $F$ be as in the hypothesis.

Orient the edges of $G$ so that each leaf has indegree 1. Orient edge $\{a, b\}$ from $a$ to $b$. Since $F$ is a 2-factor in $H^+$, it has an Eulerian orientation. Since $H^+ - F$ is $(2n - 2)$-regular, it has an Eulerian orientation. Consider any cycle $C$ in $F$, and let $D_C$ denote the digraph consisting of all arcs with tail in $V(C)$. Thus every vertex in $D_C$ will have outdegree either $k_1 + k_2 + 1$ (if the vertex is in $C$) or 0. The proof will be complete if we can show that each such subgraph $D_C$ has a $G$-decomposition.

Let cycle $C$ have length $p$ and consist of alternating vertices and arcs labeled $v_0, e_1, v_1, e_2, \ldots, v_p, e_p, v_0$. For the first copy $G_1$ of $G$ in the decomposition, we use $e_1$ as the central arc, and identify $v_0$ with $a$ and $v_1$ with $b$. Choose $k_2$ arcs other than $e_2$ with tail at $v_1$; label as $X$ the set of endpoints of these $k_2$ arcs. The remaining $k_1$ arcs with tail at $v_0$ in $G_1$ in this construction will be determined at the end.

We construct the remaining copies $G_2, G_3, \ldots, G_p$ sequentially. After $G_{i-1}$ is determined we construct $G_i$ as follows:

The central arc of $G_i$ is $e_i$, with $v_{i-1}$ identified with $a$ from $G$, and $v_i$ identified with $b$. The remaining arcs with tail at $v_{i-1}$ are all such arcs of $D_C - C$ that were not chosen to be in $G_{i-1}$. From the remaining $k_1 + k_2$ arcs with tail at $v_i$, we choose $k_2$ so that:

i) no arc is chosen that is adjacent with an arc chosen at this step to have tail $v_{i-1}$ (to avoid an immediate triangle), and
ii) we include in the pool all arcs with head a vertex in \( X \).

The selection process above can always be implemented because in \( G_{i-1} \) we chose all possible arcs with tail at \( v_{i-1} \) and head at a vertex in \( X \), so no such arc appears in \( G_i \).

It remains only to complete the construction of \( G_1 \). After \( G_p \) has been constructed, \( k_1 \) arcs with tail at \( v_0 \) have yet to be assigned; we include these arcs in \( G_1 \). Because of the pattern noted above, none of these arcs has as a head a vertex in \( X \). Thus \( G_1 \) also has no triangles and is therefore isomorphic to \( G \).

**Theorem 13.** Let \( k_1, k_2 \) be positive integers and let \( G = \tilde{S}_{k_1} \cup \tilde{S}_{k_2} \). Let \( n = k_1 + k_2 \) and suppose that \( H \) is a \( 2n \)-regular graph. Then \( G \) decomposes \( H \).

**Proof.** Let \( H \) have order \( p \). If \( H \) is the complete graph \( K_p \), the result is covered by Corollary 9. Hereafter, we assume that \( H \) is not complete.

If \( H \) has odd order, then \( H^c \), the complement of \( H \), is even regular and thus contains a 2-factor \( F \). Let \( H^+ \) denote the graph with vertex set \( V(H) \) and edge set \( E(H) \cup E(F) \). By Theorem 12, there is an \( S_{k_1,k_2} \)-decomposition \( \Delta \) of \( H^+ \) with the property that the edges of \( F \) are the center-edges of the double-stars in \( \Delta \). By removing the center edges from the double-stars in \( \Delta \), we obtain a \( G \)-decomposition of \( H \).

If \( H \) has even order, then \( H^c \) is odd regular. Let \( 2H^c \) be the multigraph obtained from \( H^c \) by doubling all its edges. Let \( F \) be a 2-factor in \( 2H^c \) and let \( H^+ \) be as in the previous case. Note that \( H^+ \) and \( F \) satisfy the conditions of Theorem 12. We proceed as in the previous case.

Horsley [12] recently proved that Conjecture 6 holds when \( G \) is a star forest. We provide proofs of two results subsumed by Horsley’s result because they parallel our results for \( 2n \)-regular graphs.

**Theorem 14.** Every star forest \( G \) with \( n \) edges decomposes \( K_{n,n} \).

**Proof.** Let \( k_1, k_2, \ldots, k_t \) be positive integers with sum \( n \) and let \( G \) be a star forest with \( t \) components where component \( i \) has size \( k_i \) for \( i \in [1,t] \). Let \((A,B)\) be a bipartition of \( V(K_{n,n}) \), where \( A = \{a_1, a_2, \ldots, a_n\} \) and \( B = \mathbb{Z}_n \). Let \( k'_0 = 0 \) and for each \( j \in [1,t] \), let \( k'_j = \sum_{i=1}^{j} k_i \).

Let \( G_1, G_2, \ldots, G_n \) be copies of \( G \) in \( K_{n,n} \) constructed as follows. For \( i \in [1,t] \), component \( i \) of \( G_1 \) is centered at vertex \( c_{i,1} = i - 1 \) in \( B \) and has leaves \( a_{k'_i+1}, a_{k'_i+2}, \ldots, a_{k_i} \) in \( A \). Thus the last component of \( G_1 \) has center \( c_{t,1} = t - 1 \) and leaves \( a_{k'_t+1}, a_{k'_t+2}, \ldots, a_{n} \). For \( i \in [1,t] \) and \( j \in [2,n] \), we will let \( c_{i,j} \) denote the center of component \( i \) in \( G_j \). For \( j \in [2,n] \), let \( G_j \) be the copy of \( G \) where each component has the same set of leaves as in \( G_1 \), but is centered at \( c_{i,j} + 1 \) (mod \( n \)) in \( B \). It is easy to verify that the \( n \) copies of \( G \) are edge-disjoint and thus \( \Delta = \{G_i; i \in [1,n]\} \) is a \( G \)-decomposition of \( K_{n,n} \).

**Theorem 15.** Let \( k_1, k_2 \) be positive integers and let \( n = k_1 + k_2 + 1 \). Suppose that \( H \) is an \( n \)-regular bipartite graph and let \( I \) be a 1-factor in \( H \). Then \( \tilde{S}_{k_1,k_2} \) decomposes \( H \) with the edges of \( I \) as the center edges of the double-stars in the decomposition.

**Proof.** Let \((A,B)\) be a bipartition of \( V(H) \), where \( A = \{x_1, x_2, \ldots, x_t\} \) and \( B = \{y_1, y_2, \ldots, y_t\} \). Without loss of generality, let \( E(I) = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_t, y_t\}\} \). Let \( H' = H - I \). Let \( I_1, I_2, \ldots, I_k \) be \( k \) edge-disjoint 1-factors in \( H' \) and let \( F = \bigcup_{i=1}^{k} I_i \). For each \( i \in [1,t] \), let \( S(x_i; k_i) \) be the \( k_i \)-star with center \( x_i \) induced by the edges in \( F \) incident with \( x_i \). For each \( j \in [1,t] \), let \( S(y_j; k_2) \) be the \( k_2 \)-star with center \( y_j \) induced by the edges in \( H' - F \) incident with \( y_j \). For each \( \ell \in [1,t] \), let \( G_{\ell} = S(x_{\ell}; k_1) \cup \{x_{\ell}, y_{\ell}\} \cup S(y_{\ell}; k_2) \). Each \( G_{\ell} \) is isomorphic to \( S_{k_1,k_2} \) and \( \Delta = \{G_1, G_2, \ldots, G_t\} \) is an \( S_{k_1,k_2} \)-decomposition of \( H \) with the edges of \( I \) as the center edges of the double-stars in the decomposition.
Theorem 16. Let $k_1, k_2$ be positive integers and let $G = S_{k_1} \cup S_{k_2}$. Let $n = k_1 + k_2$ and suppose that $H$ is an $n$-regular bipartite graph. Then $G$ decomposes $H$.

Proof. Let $(A, B)$ be a bipartition of $V(H)$, where $A = \{x_1, x_2, \ldots, x_t\}$ and $B = \{y_1, y_2, \ldots, y_t\}$. If $H$ is the complete bipartite graph $K_{n,n}$, then the result is covered by Theorem 14. Otherwise, let $I$ be a 1-factor in $K_{A,B} - H$ and let $H^* = H \cup I$. By Theorem 15, there exists an $S_{k_1,k_2}$-decomposition $\Delta$ of $H^*$ with the edges of $I$ as the center edges of the double-stars in the decomposition. By removing the center edges from the double-stars in $\Delta$, we obtain a $G$-decomposition of $H$. \hfill $\blacksquare$

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