On decomposing regular graphs into star forests

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Abstract

Let G be a forest with n edges. El-Zanati conjectures that G necessarily decomposes every $2n$-regular graph and every n-regular bipartite graph. We confirm these conjectures in the case when G consists of two stars.

1 Introduction

For integers a and b with $a \leq b$, let $[a, b] = \{a, a+1, \ldots, b\}$. For a positive integer n, let $\mathbb{Z}_n$ denote the group of integers modulo n. For a graph G with vertex set $V(G)$ and edge set $E(G)$, the *order* of G is $|V(G)|$ and the *size* of G is $|E(G)|$. The graph $K_{1,k}$ is known as a *k-star* and is often denoted by S_k. A *double-star* is a tree with exactly two vertices of degree greater than 1. The two vertices of degree greater than 1 are called the *centers* of the double-star and the edge joining them is called the *central-edge*. If T is a double-star where the two centers have degrees $k_1 + 1$ and $k_2 + 1$, then T is denoted by S_{k_1,k_2}. Note that S_{k_1,k_2} has $k_1 + k_2 + 1$ edges and is isomorphic to S_{k_2,k_1}. For a graph G and a positive integer t, let tG denote the vertex disjoint union of t copies of G.

Let H and G be graphs with G a subgraph of H. A *G-decomposition* of H is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of H each of which is isomorphic to G and such that each edge of H appears in exactly one G_i. If there exists a G-decomposition of H, then we say G *decomposes* H.

A large amount of research has been done on the topic of graph decompositions over the last five decades (see [2] and [3] for surveys). Much investigation has been motivated by a conjecture of Ringel [15] on decomposing complete graphs into trees.

Conjecture 1. *Every tree T with n edges decomposes the complete graph K_{2n+1}.*

A folklore conjecture similar to Ringel's relates to decomposing complete bipartite graphs into trees.

Conjecture 2. *Every tree T with n edges decomposes the complete bipartite graph $K_{n,n}$.*

Both of the above conjectures are special cases of conjectures due to Graham and Häggkvist (see [9]).

Conjecture 3. *Every tree T with n edges decomposes every $2n$-regular graph H.*

Conjecture 4. *Every tree T with n edges decomposes every n-regular bipartite graph H.*

*Research supported by National Science Foundation Grant No. A1063038

Despite persistent attacks over the last 40 years, Ringel's conjecture and variations thereof, such as the Graceful Tree Conjecture (see [8]), still stand today. Much less work has been done on the Graham and Häggkvist conjectures.

Results confirming Conjecture 3, in certain cases, can be found in [9] by Häggkvist, in [4], and in Snevily's Ph.D. thesis [17]. Some recent extensions of Snevily's results can be found in a paper by Jao, Kostochka, and West [14]. In [13], Jacobson, Truszczyński, and Tuza confirm Conjecture 4 for double-stars and for the path with 4 edges. Fink [7] confirms Conjecture 4 when H is the n-cube. Also, it is easy to see that S_n decomposes every $2n$-regular graph as well as every n-regular bipartite graph.

El-Zanati proposes that the conjectures by Graham and Häggkvist hold for forests with n edges.

Conjecture 5. *Every forest G with n edges decomposes every $2n$-regular graph H.*

Conjecture 6. *Every forest G with n edges decomposes every n-regular bipartite graph H.*

In this note, we provide some evidence in support of Conjectures 5 and 6. In particular, we show that the conjectures hold when G is the vertex-disjoint union of two stars.

2 Known Results

We begin by defining three graph labelings introduced by Rosa [16] as means for attacking problems like Ringel's Conjecture. Let G be a graph with n edges and let $f\colon V(G) \to [0, 2n]$ and $g\colon V(G) \to [0, n]$ be one-to-one functions. Then f is a *σ-labeling* of G if $\{|f(v) - f(u)|\colon \{u, v\} \in E(G)\} = [1, n]$ and g is a *β-labeling* if $\{|g(v) - g(u)|\colon \{u, v\} \in E(G)\} = [1, n]$. If in addition G is bipartite with vertex bipartition $\{A, B\}$, then a β-labeling g of G is an *α-labeling* if $\max\{g(u)\colon u \in A\} < \min\{g(v)\colon v \in B\}$. Thus an α-labeling of G is also a β-labeling which is also a σ-labeling of G. We have the following results (see [16] and [5]).

Theorem 7. *Let G be a graph with n edges. If G admits a σ-labeling, then there exists a G-decomposition of K_{2n+1} and of $K_{2n+2} - I$, where I is a 1-factor. If in addition, G is bipartite and G admits an α-labeling, then there also exists a G-decomposition of $K_{n,n}$.*

It is known that paths, stars, and all caterpillars in general admit α-labelings (see [16]). It is also known that trees with up to 35 edges admit β-labelings (see [8]). We also have the following result from [10].

Theorem 8. *The disjoint union of a graph with a β-labeling, together with a collection of graphs with α-labelings, has a σ-labeling.*

An example of a σ-labeling of a star forest with 7 components and 15 edges is given in Figure 1.

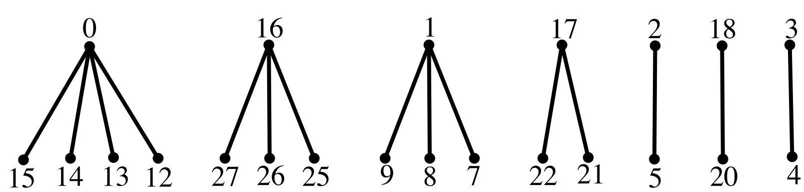

Figure 1: A σ-labeling of a star forest.

In the context of providing evidence in support of Conjecture 5, we have the following corollary to Theorem 8.

Corollary 9. *Let G be a forest with n edges. If one component of G is a tree on up to 36 vertices and all other components are caterpillars, then G decomposes K_{2n+1} and $K_{2n+2} - I$, where I is a 1-factor.*

As for Conjecture 6, a consequence of a result by Horak, Širáň, and Wallis [11] ensures that every forest with n edges decomposes the n-cube.

Also, Conjectures 5 and 6 hold when $G = nK_2$ as a consequence of a result by Alon [1].

Lemma 10. *For every graph G and every $t \geq 1$, tK_2 decomposes G if and only if t divides $|E(G)|$ and $\chi'(G) \leq |E(G)|/t$.*

Corollary 11. *Let $G = nK_2$ and suppose H is either n-regular and bipartite or $2n$-regular. Then G decomposes H.*

3 Main Results

We give some additional definitions before proceeding with our main results. An *orientation* of a graph H is an assignment of directions to the edges of H. An *Eulerian orientation* of H is a orientation where the indegree at each vertex is equal to the outdegree. It is simple to see that a graph with all even degrees has an Eulerian orientation.

Theorem 12. *Let G be a double-star with n edges and let H^+ be a $2n$-regular multigraph with edge-multiplicity at most 2. Suppose the subgraph M of H^+ consisting of the edges of multiplicity 2 is either empty or 2-regular. Let F be a 2-factor of H^+ that contains every component of M. Then there exists a G-decomposition Δ of H^+ with the property that the edges of F are the center-edges of the double-stars in Δ.*

Proof. Let G be the double-star S_{k_1,k_2} with center vertices a and b, where the degree of a is $k_1 + 1$ and the degree of b is $k_2 + 1$. Let H^+, D, and F be as in the hypothesis.

Orient the edges of G so that each leaf has indegree 1. Orient edge $\{a, b\}$ from a to b. Since F is a 2-factor in H^+, it has an Eulerian orientation. Since $H^+ - F$ is $(2n - 2)$-regular, it has an Eulerian orientation. Consider any cycle C in F, and let D_C denote the digraph consisting of all arcs with tail in $V(C)$. Thus every vertex in D_C will have outdegree either $k_1 + k_2 + 1$ (if the vertex is in C) or 0. The proof will be complete if we can show that each such subgraph D_C has a G-decomposition.

Let cycle C have length p and consist of alternating vertices and arcs labeled v_0, e_1, v_1, e_2, $\ldots$, v_{p-1}, e_p, $v_p = v_0$.

For the first copy G_1 of G in the decomposition, we use e_1 as the central arc, and identify v_0 with a and v_1 with b. Choose k_2 arcs other than e_2 with tail at v_1; label as X the set of endvertices of these k_2 arcs. The remaining k_1 arcs with tail at v_0 in G_1 in this construction will be determined at the end.

We construct the remaining copies $G_2, G_3, \ldots, G_p$ sequentially. After G_{i-1} is determined we construct G_i as follows:
The central arc of G_i is e_i, with v_{i-1} identified with a from G , and v_i identified with b. The remaining arcs with tail at v_{i-1} are all such arcs of $D_C - C$ that were not chosen to be in G_{i-1}. From the remaining $k_1 + k_2$ arcs with tail at v_i, we choose k_2 so that:

i) no arc is chosen that is adjacent with an arc chosen at this step to have tail v_{i-1} (to avoid an immediate triangle), and

ii) we include in the pool all arcs with head a vertex in X.

The selection process above can always be implemented because in G_{i-1} we chose all possible arcs with tail at v_{i-1} and head at a vertex in X, so no such arc appears in G_i.

It remains only to complete the construction of G_1. After G_p has been constructed, k_1 arcs with tail at v_0 have yet to be assigned; we include these arcs in G_1. Because of the pattern noted above, none of these arcs has as a head a vertex in X. Thus G_1 also has no triangles and is therefore isomorphic to G. ∎

Theorem 13. *Let k_1, k_2 be positive integers and let $G = S_{k_1} \cup S_{k_2}$. Let $n = k_1 + k_2$ and suppose that H is a $2n$-regular graph. Then G decomposes H.*

Proof. Let H have order p. If H is the complete graph K_p, the result is covered by Corollary 9. Hereafter, we assume that H is not complete.

If H has odd order, then H^c, the complement of H, is even regular and thus contains a 2-factor F. Let H^+ denote the graph with vertex set $V(H)$ and edge set $E(H) \cup E(F)$. By Theorem 12, there is an S_{k_1,k_2}-decomposition Δ of H^+ with the property that the edges of F are the center-edges of the double-stars in Δ. By removing the center edges from the double-stars in Δ, we obtain a G-decomposition of H.

If H has even order, then H^c is odd regular. Let ${}^2H^c$ be the multigraph obtained from H^c by doubling all its edges. Let F be a 2-factor in ${}^2H^c$ and let H^+ be as in the previous case. Note that H^+ and F satisfy the conditions of Theorem 12. We proceed as in the previous case. ∎

Horsley [12] recently proved that Conjecture 6 holds when G is a star forest. We provide proofs of two results subsumed by Horsley's result because they parallel our results for $2n$-regular graphs.

Theorem 14. *Every star forest G with n edges decomposes $K_{n,n}$.*

Proof. Let $k_1, k_2, \ldots, k_t$ be positive integers with sum n and let G be a star forest with t components where component i has size k_i for $i \in [1,t]$. Let (A,B) be a bipartition of $V(K_{n,n})$, where $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \mathbb{Z}_n$. Let $k'_0 = 0$ and for each $j \in [1,t]$, let $k'_j = \sum_{i=1}^{j} k_i$.

Let $G_1, G_2, \ldots, G_n$ be copies of G in $K_{n,n}$ constructed as follows. For $i \in [1,t]$, component i of G_1 is centered at vertex $c_{i,1} = i-1$ in B and has leaves $a_{k'_{i-1}+1}, a_{k'_{i-1}+2}, \ldots, a_{k'_i}$ in A. Thus the last component of G_1 has center $c_{t,1} = t-1$ and leaves $a_{k'_{t-1}+1}, a_{k'_{t-1}+2}, \ldots, a_n$. For $i \in [1,t]$ and $j \in [2,n]$, we will let $c_{i,j}$ denote the center of component i in G_j. For $j \in [2,n]$, let G_j be the copy of G where each component has the same set of leaves as in G_1, but is centered at $c_{i,j}+1 \pmod n$ in B. It is easy to verify that the n copies of G are edge-disjoint and thus $\Delta = \{G_i \colon i \in [1,n]\}$ is a G-decomposition of $K_{n,n}$. ∎

Theorem 15. *Let k_1, k_2 be positive integers and let $n = k_1 + k_2 + 1$. Suppose that H is an n-regular bipartite graph and let I be a 1-factor in H. Then S_{k_1,k_2} decomposes H with the edges of I as the center edges of the double-stars in the decomposition.*

Proof. Let (A,B) be a bipartition of $V(H)$, where $A = \{x_1, x_2, \ldots, x_t\}$ and $B = \{y_1, y_2, \ldots, y_t\}$. Without loss of generality, let $E(I) = \{\{x_1,y_1\}, \{x_2,y_2\}, \ldots, \{x_t,y_t\}\}$. Let $H' = H - I$. Let $I_1, I_2, \ldots, I_{k_1}$ be k_1 edge-disjoint 1-factors in H' and let $F = \bigcup_{i=1}^{k_1} I_i$. For each $i \in [1,t]$, let $S(x_i;k_1)$ be the k_1-star with center x_i induced by the edges in F incident with x_i. For each $j \in [1,t]$, let $S(y_j;k_2)$ be the k_2-star with center y_j induced by the edges in $H' - F$ incident with y_j. For each $\ell \in [1,t]$, let $G_\ell = S(x_\ell;k_1) \cup \{x_\ell, y_\ell\} \cup S(y_\ell;k_2)$. Each G_ℓ is isomorphic to S_{k_1,k_2} and $\Delta = \{G_1, G_2, \ldots, G_t\}$ is an S_{k_1,k_2}-decomposition of H with the edges of I as the center edges of the double-stars in the decomposition. ∎

Theorem 16. *Let k_1, k_2 be positive integers and let $G = S_{k_1} \cup S_{k_2}$. Let $n = k_1 + k_2$ and suppose that H is an n-regular bipartite graph. Then G decomposes H.*

Proof. Let (A, B) be a bipartition of $V(H)$, where $A = \{x_1, x_2, \ldots, x_t\}$ and $B = \{y_1, y_2, \ldots, y_t\}$. If H is the complete bipartite graph $K_{n,n}$, then the result is covered by Theorem 14. Otherwise, let I be a 1-factor in $K_{A,B} - H$ and let $H^* = H \cup I$. By Theorem 15, there exists an S_{k_1,k_2}-decomposition Δ of H^* with the edges of I as the center edges of the double-stars in the decomposition. By removing the center edges from the double-stars in Δ, we obtain a G-decomposition of H. ∎

4 Acknowledgement

This research is partially supported by grant number A1063038 from the Division of Mathematical Sciences at the National Science Foundation. Part of this work was done while the second and fourth authors were participants in *REU Site: Mathematics Research Experience for Pre-service and for In-service Teachers* at Illinois State University.

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