Spectrum for multigraph designs on four vertices and six edges

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1 Introduction

Throughout this paper, we may refer to a multigraph as a graph. However, our graphs contain no loops. If we wish to emphasize that a given graph does not contain parallel edges, then we refer to it as a simple graph. For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set (or multiset) of G, respectively. For a simple graph G and a positive integer λ, we use ${}^{\lambda}G$ to denote the graph obtained from G by replacing each edge in $E(G)$ with λ parallel edges. For edge-disjoint graphs G and H, we use $G \cup H$ to represent the graph with edge set $E(G) \cup E(H)$ and vertex set $V(G) \cup V(H)$. We use $K_{r\times s}$ to denote the complete simple multipartite graph with r parts of size s, and we use $K_{t,\,r\times s}$ to denote the complete simple multipartite graph with one part of size t and r parts of size s. If G is a subgraph of H, we use $H \setminus G$ to denote the graph obtained from H by removing $E(G)$ from $E(H)$.

Let K and G be graphs with G a subgraph of K. A G-*decomposition* of K is a set (or multiset) $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that each edge of K appears in exactly one G_i. Similarly, if G and H are each subgraphs of K, then a $\{G, H\}$-decomposition of K is defined to be a set (or multiset) $\{H_1, H_2, \ldots, H_t\}$ of subgraphs of K each of which is isomorphic to either G or H and such that each edge of K appears in exactly one H_i. If there exists a G-decomposition of K, then we say G *divides* K and write $G \mid K$. A G-decomposition of K

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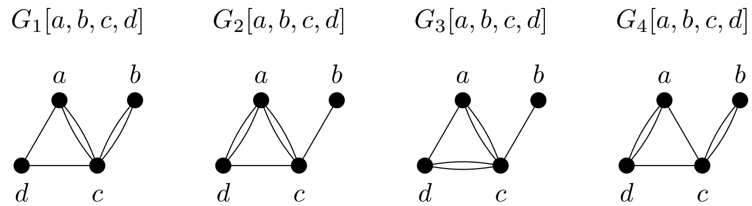

Figure 1: The four multigraphs of a triangle with a pendant edge with two double edges.

is also known as a (K, G)-*design*.

Given a graph G, a classical problem in combinatorics is to find necessary and sufficient conditions for the existence of a G-decomposition of ${}^{\lambda}K_v$. This is known as the *spectrum problem* for G. The set of all such n is called the *spectrum for G-designs of index λ*. The spectrum for G-designs of index 1 has been determined for several classes of graphs including cycles, paths, stars, and simple graphs of order at most 5 (see [2]).

In recent years, there have been some investigations of G-designs of index λ where G is a multigraph with edge multiplicity at least 2. For example, in [4] Carter determined the spectra for G-designs of index λ for all connected cubic multigraphs G of order at most 6. The spectra for G-designs of index λ have been investigated for various multigraphs G of small order (see for example [8], [3], and [9]). In this paper, we consider tripartite multigraphs of small order. In particular, we consider the four multigraphs with 6 edges and edge multiplicity 2 whose underlying simple graph is a triangle with a pendant edge (also called a claw). Henceforth, we refer to these four multigraphs as G_1, G_2, G_3, and G_4 as indexed in Figure 1. We settle the spectrum problem for all four graphs.

Let $G \in \{G_1, G_2, G_3, G_4\}$. Then $G[a, b, c, d]$ denotes the graph with vertex set $\{a, b, c, d\}$ and edge set as represented in Figure 1. For example, $G_1[0, 5, 4, 2]$ denotes the graph with vertex set $\{0, 5, 4, 2\}$ and edge multiset $\big\{\{0, 4\}, \{0, 4\}, \{0, 2\}, \{5, 4\}, \{5, 4\}, \{4, 2\}\big\}$.

The following theorems on decompositions of complete graphs and complete multipartite graphs are used extensively in proving our main results. All of these results can be found in the *Handbook of Combinatorial Designs* [5] (see [1], [6], and [7]).

Theorem 1.1. *If n is an odd positive integer, then there exists a $\{K_3, K_5\}$-decomposition of K_n.*

Theorem 1.2. *If $t \geq 4$, then there exists a K_4-decomposition of $K_{9,\, t\times 6}$.*

Theorem 1.3. *The necessary and sufficient conditions for the existence of* *a K_3-decomposition of $K_{t\times m}$ are (i) $t \geq 3$, (ii) $(t-1)m \equiv 0 \pmod 2$, and (iii) $t(t-1)m^2 \equiv 0 \pmod 6$.*

Theorem 1.4. *If $t \geq 3$ and $t \equiv 0 \pmod 3$, then there exists a K_3-decomposition of $K_{4,\,t\times 2}$.*

Combining the previous two results, we have the following corollary that that is more directly applicable in our general constructions.

Corollary 1.5. *Let $t \geq 3$. There exists a K_3-decomposition of $K_{t\times 2}$ if $t \equiv 0$ or $1 \pmod 3$ and of $K_{4,\,(t-2)\times 2}$ if $t \equiv 2 \pmod 3$.*

The following is a well-known result that is a special case of Wilson's Fundamental Construction (see [7]).

Theorem 1.6. *Let m, n, r, s, and t be positive integers. If there exists a $(K_{t\times m}, K_n)$-design, then there exists a $(K_{t\times ms}, K_{n\times s})$-design. Similarly, if there exists a $(K_{r,\,t\times m}, K_n)$-design, then there exists a $(K_{rs,\,t\times ms}, K_{n\times s})$-design.*

2 Main Results

We now establish some necessary conditions for the existence of a G-decomposition of ${}^{\lambda}K_n$ where $G \in \{G_1, G_2, G_3, G_4\}$.

Lemma 2.1. *Let $\lambda \geq 2$ and $n \geq 4$ be integers and let $G \in \{G_1, G_2, G_3, G_4\}$. If there exists a G-decomposition of ${}^{\lambda}K_n$, then the following hold:*

1. *if $\gcd(\lambda, 6) = 1$, then $n \equiv 0$, 1, 4, or $9 \pmod{12}$;*
2. *if $\gcd(\lambda, 6) = 2$, then $n \equiv 0$ or $1 \pmod 3$;*
3. *if $\gcd(\lambda, 6) = 3$, then $n \equiv 0$ or $1 \pmod 4$;*
4. *if $\gcd(\lambda, 6) = 6$, then $n \geq 4$.*

Proof. Let λ and n be as stated and suppose there exists a G-decomposition of ${}^{\lambda}K_n$. Since the number of edges in G is 6, we must have that $6 \mid \lambda n(n-1)/2$, and thus $12 \mid \lambda n(n-1)$. First, if $\gcd(\lambda, 6) = 1$, then $12 \mid n(n-1)$, and thus $n \equiv 0$, 1, 4, or $9 \pmod{12}$. Second, if $\gcd(\lambda, 6) = 2$, then $6 \mid n(n-1)$, and thus $n \equiv 0$ or $1 \pmod 3$. Third, if $\gcd(\lambda, 6) = 3$, then $4 \mid n(n-1)$, and thus $n \equiv 0$ or $1 \pmod 4$. Finally, if $\gcd(\lambda, 6) = 6$, then $2 \mid n(n-1)$ which is true for all $n \geq 4$. ■

2.1 Nonexistence Results

Before showing what conditions are sufficient for a G-decomposition of ${}^{\lambda}K_n$ (where $G \in \{G_1, G_2, G_3, G_4\}$), we first show some restrictions on the necessary conditions.

Lemma 2.2. *There exists a G_4-decomposition of ${}^{\lambda}K_4$ if and only if* $\lambda \equiv 0 \pmod 3$.

Proof. We first note that Example 3.20 shows that $G_4 \mid {}^{3}K_4$. Therefore, $G_4 \mid {}^{3t}K_4$ for any positive integer t since ${}^{3}K_4 \mid {}^{3t}K_4$. Next, we show the necessary condition.

Let $V({}^{\lambda}K_4) = \{v_0, v_1, v_2, v_3\}$ and let Δ be a G_4-decomposition of ${}^{\lambda}K_4$. Note that $|\Delta| = \big|E({}^{\lambda}K_4)\big|/\big|E(G_4)\big| = \lambda$. Let A be the collection of G_4-blocks in Δ that contain $\{v_0, v_1\}$ as a double edge, B be the collection of G_4-blocks in Δ that contain $\{v_0, v_1\}$ as a single edge, and C be the collection of G_4-blocks in Δ that do not contain the edge $\{v_0, v_1\}$. Note that if $G \in A$, then G contains $\{v_2, v_3\}$ as a double edge; if $G \in C$, then G contains $\{v_2, v_3\}$ as a single edge; and if $G \in B$, then G does not contain the edge $\{v_2, v_3\}$.

Let $a = |A|$, $b = |B|$, and $c = |C|$. Clearly, $\{A, B, C\}$ is a partition of Δ. Therefore,

$$a + b + c = \lambda. \tag{1}$$

Furthermore, since Δ is a decomposition of ${}^{\lambda}K_4$, the blocks must provide λ copies for both edges $\{v_0, v_1\}$ and $\{v_2, v_3\}$. Therefore, $2a + 1b + 0c = \lambda$ and $2a + 0b + 1c = \lambda$. Thus, $b = \lambda - 2a$ and $c = \lambda - 2a$. Therefore, by Equation (1), $a + (\lambda - 2a) + (\lambda - 2a) = \lambda$ and hence $\lambda = 3a$. Since $a \in \mathbb{Z}$, $\lambda \equiv 0 \pmod 3$. ∎

Lemma 2.3. *Let $G \in \{G_1, G_2\}$. There does not exist a G-decomposition of either ${}^{2}K_4$ or ${}^{5}K_4$.*

Proof. Since ${}^{2}K_4 \setminus G_1 = G_2$, it is easy to see why no such G-decomposition of ${}^{2}K_4$ exists. We prove the remaining results via contradiction. Note that since the order of G matches that of the ${}^{5}K_4$, each vertex of ${}^{5}K_4$ must appear in each G-block of such a decomposition.

Case 1: $G_1 \nmid {}^{5}K_4$.
Assume Δ is a G_1-decomposition of ${}^{5}K_4$. For each $v \in V({}^{5}K_4)$, let a_v denote the number of G_1-blocks in Δ where vertex a in $G_1[a, b, c, d]$ identifies with v. Similarly define b_v, c_v, and d_v. Note that a_v, b_v, c_v, and d_v are each nonnegative integers no more than $|\Delta| = 5$. From the degrees of the vertices in G_1 and ${}^{5}K_4$, we have

$$3a_v + 2b_v + 5c_v + 2d_v = 15,$$

and from the cardinality of Δ, we have

$$a_v + b_v + c_v + d_v = 5$$

for each $v \in V({}^{5}K_4)$. All solutions to this system of equations have $a_v \geq 2$. However, $\sum_{v \in V({}^{5}K_4)} a_v = |\Delta| = 5$, and thus a_v cannot be at least 2 for all v in ${}^{5}K_4$. Therefore, we have a contradiction, and no such Δ exists.

CASE 2: $G_2 \nmid {}^5K_4$.
Assume Δ is a G_2-decomposition of ${}^5K_4$. Define a_v, b_v, c_v, and d_v as in the previous case, but now for the number of G_2-blocks in Δ. Again, note that a_v, b_v, c_v, and d_v are each nonnegative integers no more than $|\Delta| = 5$. From the degrees of the vertices in G_2 and ${}^5K_4$, we have

$$4a_v + 1b_v + 4c_v + 3d_v = 15,$$

and from the cardinality of Δ, we have

$$a_v + b_v + c_v + d_v = 5$$

for each $v \in V({}^5K_4)$. All solutions to this system of equations have $d_v \geq 2$. However, $\sum_{v \in V({}^5K_4)} d_v = |\Delta| = 5$, and thus d_v cannot be at least 2 for all v in ${}^5K_4$. Therefore, we have a contradiction, and no such Δ exists. ∎

Lemma 2.4. *Let $G \in \{G_2, G_3\}$. There does not exist a G-decomposition of ${}^2K_6$.*

Proof. We prove the result in two cases, both via contradiction.
CASE 1: $G_2 \nmid {}^2K_6$.
Assume Δ is a G_2-decomposition of ${}^2K_6$. Define a_v, b_v, c_v, and d_v as in the previous case, but now for the number of G_3-blocks in Δ. Again, note that a_v, b_v, c_v, and d_v are each nonnegative integers no more than $|\Delta| = 5$. From the degrees of the vertices in G_3 and ${}^2K_6$, we have

$$3a_v + 1b_v + 5c_v + 3d_v = 10$$

for each $v \in V({}^2K_6)$. All solutions of this equation require $(b_v, c_v) \in \big\{(1,0), (4,0), (2,1), (0,2), (5,1)\big\}$. Let α, β, γ, δ, and ε represent the number of vertices of ${}^2K_6$ with the respectively listed solution types for (b_v, c_v) in Δ. Thus, counting the number of copies of vertices b and c in Δ respectively yields

$$1\alpha + 4\beta + 2\gamma + 0\delta + 5\varepsilon = 5,$$
$$0\alpha + 0\beta + 1\gamma + 2\delta + 1\varepsilon = 5$$

because $|\Delta| = 5$, but counting the number of vertices in ${}^2K_6$ yields

$$\alpha + \beta + \gamma + \delta + \varepsilon = 6.$$

Since α, β, γ, δ, and ε represent nonnegative integers, the above system of three equations has no solution. Therefore, we have a contradiction, and no such Δ exists.
CASE 2: $G_3 \nmid {}^2K_6$.
Assume Δ is a G_3-decomposition of ${}^2K_6$. For each $v \in V({}^2K_6)$, let a_v

denote the number of G_2-blocks in Δ where vertex a in $G_2[a,b,c,d]$ identifies with v. Similarly define b_v, c_v, and d_v. Note that a_v, b_v, c_v, and d_v are each nonnegative integers no more than $|\Delta| = 5$. From the degrees of the vertices in G_2 and ${}^2K_6$, we have

$$4a_v + 1b_v + 4c_v + 3d_v = 10$$

for each $v \in V({}^2K_6)$. All solutions of this equation require $(b_v, d_v) \in \{(2,0), (0,2), (3,1), (1,3), (4,2)\}$. Let α, β, γ, δ, and ε represent the number of vertices of ${}^2K_6$ with the respectively listed solution types for (b_v, d_v) in Δ. Thus, counting the number of copies of vertices b and d in Δ respectively yields

$$2\alpha + 0\beta + 3\gamma + 1\delta + 4\varepsilon = 5,$$
$$0\alpha + 2\beta + 1\gamma + 3\delta + 2\varepsilon = 5$$

because $|\Delta| = 5$, but counting the number of vertices in ${}^2K_6$ yields

$$\alpha + \beta + \gamma + \delta + \varepsilon = 6.$$

Since α, β, γ, δ, and ε represent nonnegative integers, the above system of three equations has only one solution: $(\alpha, \beta, \gamma, \delta, \varepsilon) = (3, 0, 1, 2, 0)$. However, it is easy to verify that no G_3-decomposition of ${}^2K_6$ exists under these conditions. Therefore, we have a contradiction, and no such Δ exists. ■

2.2 Sufficiency Results

In Lemmas 2.5, 2.6, 2.7, and 2.8, we provide sufficient conditions for small values of λ.

Lemma 2.5. *Let $G \in \{G_1, G_2, G_3, G_4\}$ and let $n \geq 4$ be an integer. There exists a G-decomposition of ${}^2K_n$ if and only if $n \equiv 0$ or $1 \pmod 3$ with the exceptions that there exists no G_i-decomposition of ${}^2K_n$ if $(i, n) \in \{(1,4), (2,4), (2,6), (3,6), (4,4)\}$.*

Proof. The necessary conditions are established in Lemma 2.1. For sufficiency, we consider the following cases:

Case 1: $n \equiv 0 \pmod 6$.

We break this case into two sub-cases. First let $G \in \{G_1, G_4\}$. Examples 3.2 and 3.6 demonstrate the existence of G-decompositions of both ${}^2K_6$ and ${}^2K_{12}$. Now, let $n = 6t$ for some integer $t \geq 3$. By Corollary 1.5 there exists of K_3-decomposition of either $K_{t\times 2}$ or $K_{4,(t-2)\times 2}$. Thus by Theorem 1.6, either ${}^2K_{3\times 3} \mid {}^2K_{t\times 6}$ or ${}^2K_{3\times 3} \mid {}^2K_{12,(t-2)\times 6}$. By Examples 3.2, 3.6, and 3.15, G divides ${}^2K_6$, ${}^2K_{12}$, and ${}^2K_{3\times 3}$, and therefore $G \mid {}^2K_{6t}$.

Now, let $G \in \{G_2, G_3\}$. Examples 3.6 and 3.11 demonstrate the existence of G-decompositions of both ${}^2K_{12}$ and ${}^2K_{24}$. Let $n = 12t$ for some integer $t \geq 3$. By Corollary 1.5 there exists of K_3-decomposition of either $K_{t\times 2}$ or $K_{4,(t-2)\times 2}$. Thus by Theorem 1.6, either ${}^2K_{3\times 6} \mid {}^2K_{t\times 12}$ or ${}^2K_{3\times 6} \mid {}^2K_{24,(t-2)\times 12}$. Since $K_3 \mid K_{3\times 2}$, we have ${}^2K_{3\times 3} \mid {}^2K_{3\times 6}$. By Examples 3.6, 3.11, and 3.15, G divides ${}^2K_{12}$, ${}^2K_{24}$, and ${}^2K_{3\times 3}$, and therefore $G \mid {}^2K_{12t}$.

Now, Examples 3.10, 3.12, 3.13, and 3.14 demonstrate the existence of G-decompositions of ${}^2K_{18}$, ${}^2K_{30}$, ${}^2K_{42}$, and ${}^2K_{54}$. Let $n = 12t + 6 = 12(t-1) + 18$ for some integer $t \geq 5$. By Lemma 1.2, there exists a K_4-decomposition of $K_{9,(t-1)\times 6}$. Thus by Theorem 1.6, ${}^2K_{4\times 2} \mid {}^2K_{18,(t-1)\times 12}$. By Examples 3.6, 3.10, and 3.16, G divides ${}^2K_{12}$, ${}^2K_{18}$, and ${}^2K_{4\times 2}$, and therefore $G \mid {}^2K_{12t+6}$.

For the remaining case, let $G \in \{G_1, G_2, G_3, G_4\}$. The rest of the proof continues similarly to the prior case, incorporating additional vertices as needed. (For brevity of proof, we note that ${}^2K_7 = {}^2K_{6+1} \setminus {}^2K_1$.)

Case 2: $n \equiv 1$, 3, or 4 (mod 6).
Example 3.1 demonstrates the existence of a G_3-decomposition of ${}^2K_4$. Examples 3.3, 3.4, 3.5, 3.7, 3.8, and 3.9 demonstrate the existence of G-decompositions of ${}^2K_7$, ${}^2K_9$, ${}^2K_{10}$, ${}^2K_{13}$, ${}^2K_{15}$, and ${}^2K_{16}$. Now, let $n = 6t+i$ for some integers $t \geq 3$ and $i \in \{1, 3, 4\}$. By Corollary 1.5 there exists of K_3-decomposition of either $K_{t\times 2}$ or $K_{4,(t-2)\times 2}$. Thus by Theorem 1.6, either ${}^2K_{3\times 3} \mid {}^2K_{t\times 6}$ or ${}^2K_{3\times 3} \mid {}^2K_{12,(t-2)\times 6}$. By Examples 3.3, 3.4, 3.5, 3.7, 3.8, 3.9, 3.18, 3.19, and 3.15, G divides ${}^2K_{6+i}$, ${}^2K_{12+i}$, ${}^2K_{6+i} \setminus {}^2K_i$, and ${}^2K_{3\times 3}$, and the result follows. ■

The following lemma is a direct result of Lemma 2.5, and will be used in the proof of Theorem 2.9.

Lemma 2.6. *Let $G \in \{G_1, G_2, G_3, G_4\}$ and let $n \geq 4$ be an integer. There exists a G-decomposition of ${}^4K_n$ if and only if $n \equiv 0$ or 1 (mod 3) with the exception that $G_4 \nmid {}^4K_4$.*

Proof. The necessary conditions are established in Lemma 2.1. By Examples 3.25 and 3.26, $G_1 \mid {}^4K_4$, $G_2 \mid {}^4K_4$, $G_2 \mid {}^4K_6$, and $G_3 \mid {}^4K_6$. All other G-decompositions of ${}^4K_n$ follows as a result of Lemma 2.5 since ${}^2K_n \mid {}^4K_n$. ■

The next lemma uses a similar proof as Lemma 2.5.

Lemma 2.7. *Let $G \in \{G_1, G_2, G_3, G_4\}$ and let $n \geq 4$ be an integer. There exists an G-decomposition of ${}^3K_n$ if and only if $n \equiv 0$ or 1 (mod 4).*

Proof. The necessary conditions are established in Lemma 2.1. Examples 3.20, 3.21, 3.22, and 3.23 demonstrate the existence of G-decompositions of ${}^3K_4$, ${}^3K_5$, ${}^3K_8$, and ${}^3K_9$. Now, let $n = 4t + i$ for some integers $t \geq 3$ and $i \in \{0, 1\}$. By Corollary 1.5 there exists of K_3-decomposition of either $K_{t\times 2}$ or $K_{4,(t-2)\times 2}$. Thus by Theorem 1.6, either ${}^3K_{3\times 2} \mid {}^3K_{t\times 4}$ or ${}^3K_{3\times 2} \mid {}^3K_{8,(t-2)\times 4}$. By Examples 3.20, 3.21, 3.22, 3.23, and 3.24, G divides ${}^3K_{4+i}$, ${}^3K_{8+i}$, and ${}^3K_{3\times 2}$, and the result follows. ∎

Lemma 2.8. *Let $G \in \{G_1, G_2, G_3, G_4\}$ and let n be a positive integer. There exists a G-decomposition of ${}^6K_n$ if and only if $n \geq 4$.*

Proof. The necessary conditions are established in Lemma 2.1. For sufficiency, we consider the following cases:

Case 1: $n \equiv 0$, 1, 3, 6, 7, 9, or 10 (mod 12).
Example 3.27 demonstrates G_2- and G_3-decompositions of ${}^6K_6$. For all other cases, by Lemma 2.5, G divides ${}^2K_n$ and thus ${}^6K_n$ since ${}^2K_n \mid {}^6K_n$.

Case 2: $n \equiv 2 \pmod{12}$.
Let $n = 12t + 2 = 4(3t) + 2$ for some positive integer t. By Theorem 1.3, a K_3-decomposition of $K_{(3t)\times 2}$ exists, and thus by Theorem 1.6, ${}^6K_{3\times 2} \mid {}^6K_{(3t)\times 4}$. As proved in the above case, G divides ${}^6K_6$, and by Example 3.24, G divides ${}^3K_{3\times 2}$ and thus ${}^6K_{3\times 2}$. Furthermore, Example 3.29 shows a G decomposition of ${}^6K_6 \setminus {}^6K_2$, and so the result follows.

Case 3: $n \equiv 4$, 5, or 8 (mod 12).
By Lemma 2.7, G divides ${}^3K_n$, and thus ${}^6K_n$ since ${}^3K_n \mid {}^6K_n$.

Case 4: $n \equiv 11 \pmod{12}$.
Let $n = 12t + 11 = 3(4t + 3) + 2$ for some positive integer t. By Theorem 1.1, a $\{K_3, K_5\}$-decomposition of K_{4t+3} exists, and thus by Theorem 1.6, a $\{{}^6K_{3\times 3}, {}^6K_{5\times 3}\}$-decomposition of ${}^6K_{(4t+3)\times 3}$ exists. By Examples 3.21, 3.15, and 3.17, G divides ${}^3K_5$, ${}^2K_{3\times 3}$, and ${}^2K_{5\times 3}$, and thus G divides ${}^6K_5$, ${}^6K_{3\times 3}$, and ${}^6K_{5\times 3}$. Furthermore, by Example 3.28, G divides ${}^6K_5 \setminus {}^6K_2$, and so the result follows. ∎

We now establish necessary and sufficient conditions for all λ.

Theorem 2.9. *Let $G \in \{G_1, G_2, G_3, G_4\}$ and let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a G-decomposition of ${}^\lambda K_n$ if and only if 12 divides $\lambda n(n-1)$ with the exceptions that there exists no G_i-decomposition of ${}^\lambda K_n$ if $(i, \lambda, n) \in \big\{(1,2,4), (1,5,4), (2,2,4), (2,2,6), (2,5,4), (3,2,6), (4,3j-1,4), (4,3j+1,4)\big\}$ where j is a positive integer.*

Proof. The necessary conditions are established in Lemma 2.1. For sufficiency, let $G \in \{G_1, G_2, G_3, G_4\}$ and let $12 \mid \lambda n(n-1)$. We consider the following cases:

Case 1: $\lambda \equiv 0 \pmod 6$.
Since $\gcd(\lambda, 6) = 6$, we have that $\lambda \geq 6$ and $n \geq 4$. Let $\lambda = 6t$ for some

positive integer t. By Lemma 2.8, G divides ${}^6K_n$, and the result follows because ${}^6K_n \mid {}^\lambda K_n$.

Case 2: $\lambda \equiv 1$ or $5 \pmod 6$.
Since $\gcd(\lambda, 6) = 2$, we have that $n \equiv 0, 1, 4,$ or $9 \pmod{12}$. By Lemma 2.2, $G_4 \nmid {}^\lambda K_4$ since $\lambda \not\equiv 0 \pmod 3$. Furthermore, $G_1 \nmid {}^5K_4$ and $G_4 \nmid {}^5K_4$. For all other cases when $\lambda = 5$, by Lemmas 2.5 and 2.7, $G \mid {}^2K_n$ and $G \mid {}^3K_n$, and thus $G \mid {}^5K_n$.

Let $\lambda = 6(t-1)+3+4(2-i)$ for some integers $t \geq 1$ and $i \in \{0, 1\}$. Now, we split ${}^\lambda K_n$ into one copy of ${}^3K_n$, $(2-i)$ copies of ${}^4K_n$ and $(t-1)$ copies of ${}^6K_n$. By Lemmas 2.7, 2.6, and 2.8, G divides ${}^3K_n$, ${}^4K_n$, and ${}^6K_n$, and the result follows.

Case 3: $\lambda \equiv 2$ or $4 \pmod 6$.
Note that $G_1 \nmid {}^2K_4$, $G_2 \nmid {}^2K_4$, $G_2 \nmid {}^2K_6$, and $G_3 \nmid {}^2K_6$. Furthermore, by Lemma 2.2, $G_4 \nmid {}^\lambda K_4$ since $\lambda \not\equiv 0 \pmod 3$.

Since $\gcd(\lambda, 6) = 2$, we have that $\lambda \geq 2$ and $n \equiv 0$ or $1 \pmod 3$. Lemma 2.5 handles the case $\lambda = 2$. Let $\lambda = 6t + 4i$ for some integers $t \geq 0$ and $i \in \{1, 2\}$. Now, we split ${}^\lambda K_n$ into i copies of ${}^4K_n$ and t copies of ${}^6K_n$. By Lemmas 2.6 and 2.8, G divides ${}^4K_n$ and ${}^6K_n$, and the result follows.

Case 4: $\lambda \equiv 3 \pmod 6$.
Since $\gcd(\lambda, 6) = 3$, we have that $\lambda \geq 3$ and $n \equiv 0$ or $1 \pmod 4$. Let $\lambda = 6t + 3$ for some integer $t \geq 0$. Now, we split ${}^\lambda K_n$ into one copy of ${}^3K_n$ and t copies of ${}^6K_n$. By Lemmas 2.7 and 2.8, G divides ${}^3K_n$ and ${}^6K_n$, and the result follows.

■

3 Appendix of Small Decompositions

In this section we present decompositions of various graphs into our graphs of focus that are needed for the constructions used in Section 2. A G-decomposition of a graph with vertex set V may be written as a pair $(V, \mathcal{C})$, where $\mathcal{C}$ is a collection of copies of G that partition the edge-set of the graph.

Given the graphs represented by the notation $G[a, b, c, d]$ and some $i \in \mathbb{Z}_n$, we define $G[a, b, c, d] + i = G[a + i, b + i, c + i, d + i]$ where all addition is performed in $\mathbb{Z}_n$. By convention, define $\infty + 1 = \infty$.

3.1 $\lambda = 2$

Example 3.1. Let $V = \mathbb{Z}_4$ and let $\mathcal{C} = \{G_3[0, 1, 2, 3], G_3[0, 2, 1, 3]\}$. Then $(V, \mathcal{C})$ is a G_3-decomposition of ${}^2K_4$.

Example 3.2. Let $V = \mathbb{Z}_5 \cup \{\infty\}$.
Let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_5} \{G_1[0, \infty, 2, 1]+i\}$. Then $(V, \mathcal{C})$ is a G_1-decomposition of ${}^2K_6$.

Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_5}\{G_4[0,\infty,1,2]+i\}$. Then $(V,\mathcal{C})$ is a G_4-decomposition of ${}^2K_6$.

Example 3.3. Let $V = \mathbb{Z}_7$.
Let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_7}\{G_1[0,5,4,2]+i\}$. Then $(V,\mathcal{C})$ is a G_1-decomposition of ${}^2K_7$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_7}\{G_2[0,4,1,5]+i\}$. Then $(V,\mathcal{C})$ is a G_2-decomposition of ${}^2K_7$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_7}\{G_3[0,2,3,1]+i\}$. Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^2K_7$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_7}\{G_4[0,1,3,6]+i\}$. Then $(V,\mathcal{C})$ is a G_4-decomposition of ${}^2K_7$.

Example 3.4. Let $V = \mathbb{Z}_3\times\mathbb{Z}_3$.
Let

$$\begin{aligned}\mathcal{C} = \bigcup_{i\in\mathbb{Z}_3}\big\{&G_1[(0,i),(2,2+i),(0,1+i),(1,i)],\\ &G_1[(0,1+i),(2,1+i),(2,i),(1,i)],\\ &G_1[(2,i),(2,2+i),(1,1+i),(1,i)],\\ &G_1[(1,1+i),(2,i),(0,i),(1,i)]\big\}.\end{aligned}$$

Then $(V,\mathcal{C})$ is a G_1-decomposition of ${}^2K_9$.
Now let

$$\begin{aligned}\mathcal{C} = \bigcup_{i\in\mathbb{Z}_3}\big\{&G_2[(0,i),(1,2+i),(1,i),(1,1+i)],\\ &G_2[(0,i),(0,2+i),(2,i),(1,2+i)],\\ &G_2[(0,i),(1,i),(2,2+i),(0,1+i)],\\ &G_2[(2,i),(1,2+i),(2,1+i),(1,i)]\big\}.\end{aligned}$$

Then $(V,\mathcal{C})$ is a G_2-decomposition of ${}^2K_9$.
Now let

$$\begin{aligned}\mathcal{C} = \bigcup_{i\in\mathbb{Z}_3}\big\{&G_3[(0,i),(2,i),(0,1+i),(1,1+i)],\\ &G_3[(0,i),(0,1+i),(1,2+i),(2,2+i)],\\ &G_3[(0,i),(0,2+i),(2,i),(2,1+i)],\\ &G_3[(1,i),(2,i),(1,2+i),(2,1+i)]\big\}.\end{aligned}$$

Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^2K_9$.
Let $V = \mathbb{Z}_2\times\mathbb{Z}_4\cup\{\infty\}$ and let

$$\begin{aligned}\mathcal{C} = \bigcup_{i\in\mathbb{Z}_4}\big\{&G_4[(0,i),\infty,(1,i),(0,1+i)],\\ &G_4[(1,i),\infty,(0,i),(0,2+i)],\\ &G_4[(1,i),(1,3+i),(1,2+i),(0,3+i)]\big\}.\end{aligned}$$

Then $(V,\mathcal{C})$ is a G_4-decomposition of ${}^2K_9$.

Example 3.5. Let $V = \mathbb{Z}_2 \times \mathbb{Z}_5$.
Let

$$\mathcal{C} = \bigcup_{i \in \mathbb{Z}_5} \{G_1[(0,i),(1,1+i),(1,i),(0,1+i)], \\ G_1[(0,1+i),(1,1+i),(0,3+i),(1,i)], \\ G_1[(0,4+i),(1,2+i),(1,i),(0,3+i)]\}.$$

Then $(V, \mathcal{C})$ is a G_1-decomposition of ${}^2K_{10}$.
Now let

$$\mathcal{C} = \bigcup_{i \in \mathbb{Z}_5} \{G_2[(0,i),(1,i),(1,2+i),(1,4+i)], \\ G_2[(0,4+i),(0,i),(0,1+i),(1,2+i)], \\ G_2[(1,2+i),(0,3+i),(0,2+i),(1,3+i)]\}.$$

Then $(V, \mathcal{C})$ is a G_2-decomposition of ${}^2K_{10}$.
Now let

$$\mathcal{C} = \bigcup_{i \in \mathbb{Z}_5} \{G_3[(0,i),(1,2+i),(1,i),(0,1+i)], \\ G_3[(0,4+i),(0,1+i),(0,2+i),(1,i)], \\ G_3[(0,2+i),(1,1+i),(1,4+i),(1,3+i)]\}.$$

Then $(V, \mathcal{C})$ is a G_3-decomposition of ${}^2K_{10}$.
Now let

$$\mathcal{C} = \bigcup_{i \in \mathbb{Z}_5} \{G_4[(0,1+i),(0,2+i),(0,i),(1,i)], \\ G_4[(1,1+i),(1,3+i),(0,1+i),(0,i)], \\ G_4[(1,i),(0,3+i),(1,1+i),(1,2+i)]\}.$$

Then $(V, \mathcal{C})$ is a G_4-decomposition of ${}^2K_{10}$.

Example 3.6. Let $V = \mathbb{Z}_{11} \cup \{\infty\}$.
Let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_{11}} \{G_1[0,10,6,3]+i, G_1[0,\infty,2,1]+i\}$. Then $(V, \mathcal{C})$ is a G_1-decomposition of ${}^2K_{12}$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_{11}} \{G_2[0,\infty,3,2]+i, G_2[0,\infty,5,4]+i\}$. Then $(V, \mathcal{C})$ is a G_2-decomposition of ${}^2K_{12}$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_{11}} \{G_3[0,\infty,3,1]+i, G_3[0,\infty,5,1]+i\}$. Then $(V, \mathcal{C})$ is a G_3-decomposition of ${}^2K_{12}$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_{11}} \{G_4[0,4,8,5]+i, G_4[0,\infty,1,2]+i\}$. Then $(V, \mathcal{C})$ is a G_4-decomposition of ${}^2K_{12}$.

Example 3.7. Let $V = \mathbb{Z}_{13}$.
Let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_{13}} \{G_1[0,1,6,9]+i, G_1[0,1,12,3]+i\}$. Then $(V, \mathcal{C})$ is a G_1-decomposition of ${}^2K_{13}$.

Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{13}}\{G_2[0,10,4,1]+i, G_2[0,11,5,2]+i\}$. Then $(V,\mathcal{C})$ is a G_2-decomposition of ${}^2K_{13}$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{13}}\{G_3[1,6,0,4]+i, G_3[2,6,0,5]+i\}$. Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^2K_{13}$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{13}}\{G_4[0,1,3,12]+i, G_4[0,1,9,6]+i\}$. Then $(V,\mathcal{C})$ is a G_4-decomposition of ${}^2K_{13}$.

Example 3.8. Let $V = \mathbb{Z}_2 \times \mathbb{Z}_7 \cup \{\infty\}$.
Let

$$\begin{aligned}\mathcal{C} = \bigcup_{i\in\mathbb{Z}_7} \big\{ & G_1[(0,2+i),(1,6+i),(0,i),\infty],\\ & G_1[(0,6+i),(0,3+i),(0,i),(1,1+i)],\\ & G_1[(1,3+i),(0,2+i),(1,i),\infty],\\ & G_1[(1,3+i),(0,5+i),(1,2+i),(0,i)],\\ & G_1[(1,6+i),(0,4+i),(1,4+i),(0,3+i)]\big\}.\end{aligned}$$

Then, $(V,\mathcal{C})$ is a G_1-decomposition of ${}^2K_{15}$.
Now let

$$\begin{aligned}\mathcal{C} = \bigcup_{i\in\mathbb{Z}_7} \big\{ & G_2[(0,1+i),(0,i),(1,2+i),(0,2+i)],\\ & G_2[(0,2+i),(1,4+i),(1,6+i),(0,4+i)],\\ & G_2[(1,2+i),(0,1+i),(0,4+i),(1,3+i)],\\ & G_2[(1,3+i),(0,3+i),(0,i),(1,6+i)],\\ & G_2[\infty,(1,2+i),(1,i),(0,i)]\big\}.\end{aligned}$$

Then, $(V,\mathcal{C})$ is a G_2-decomposition of ${}^2K_{15}$.
Now let

$$\begin{aligned}\mathcal{C} = \bigcup_{i\in\mathbb{Z}_7} \big\{ & G_3[(0,i),(1,4+i),(1,3+i),(0,2+i)],\\ & G_3[(0,i),\infty,(1,i),(0,5+i)],\\ & G_3[(0,1+i),(0,3+i),(0,i),(1,5+i)],\\ & G_3[(1,2+i),(0,3+i),(1,i),(1,3+i)],\\ & G_3[\infty,(0,3+i),(0,i),(1,6+i)]\big\}.\end{aligned}$$

Then, $(V,\mathcal{C})$ is a G_3-decomposition of ${}^2K_{15}$.
Now let

$$\begin{aligned}\mathcal{C} = \bigcup_{i\in\mathbb{Z}_7} \big\{ & G_4[(0,1+i),(1,2+i),(1,5+i),(0,3+i)],\\ & G_4[(0,2+i),(0,5+i),(1,3+i),(0,3+i)],\\ & G_4[(0,3+i),(1,6+i),\infty,(0,6+i)],\\ & G_4[(1,i),(1,3+i),(0,i),(1,1+i)],\\ & G_4[(1,2+i),(1,6+i),(0,i),(1,4+i)]\big\}.\end{aligned}$$

Then, $(V, \mathcal{C})$ is a G_4-decomposition of ${}^2K_{15}$.

Example 3.9. Let $V = \mathbb{Z}_2 \times \mathbb{Z}_8$.
Let

$$\begin{aligned}\mathcal{C} = \bigcup_{i \in \mathbb{Z}_8} \{ & G_1[(0,1+i),(0,4+i),(1,1+i),(0,i)],\\ & G_1[(0,1+i),(0,6+i),(0,4+i),(0,i)],\\ & G_1[(0,7+i),(0,5+i),(1,3+i),(1,2+i)],\\ & G_1[(1,1+i),(0,2+i),(1,4+i),(1,i)],\\ & G_1[(1,1+i),(0,4+i),(1,3+i),(0,i)]\}.\end{aligned}$$

Then, $(V, \mathcal{C})$ is a G_1-decomposition of ${}^2K_{16}$.
Now let

$$\begin{aligned}\mathcal{C} = \bigcup_{i \in \mathbb{Z}_8} \{ & G_2[(1,5+i),(0,2+i),(0,1+i),(0,i)],\\ & G_2[(0,3+i),(0,5+i),(0,1+i),(1,2+i)],\\ & G_2[(0,3+i),(1,2+i),(1,1+i),(0,i)],\\ & G_2[(1,5+i),(1,2+i),(1,3+i),(0,3+i)],\\ & G_2[(1,7+i),(1,i),(1,4+i),(0,4+i)]\}.\end{aligned}$$

Then, $(V, \mathcal{C})$ is a G_2-decomposition of ${}^2K_{16}$.
Now let

$$\begin{aligned}\mathcal{C} = \bigcup_{i \in \mathbb{Z}_8} \{ & G_3[(1,1+i),(0,4+i),(0,3+i),(0,i)],\\ & G_3[(1,2+i),(0,7+i),(0,3+i),(0,1+i)],\\ & G_3[(1,3+i),(1,6+i),(1,5+i),(0,3+i)],\\ & G_3[(1,4+i),(0,1+i),(0,i),(1,5+i)],\\ & G_3[(1,4+i),(1,3+i),(1,7+i),(0,4+i)]\}.\end{aligned}$$

Then, $(V, \mathcal{C})$ is a G_3-decomposition of ${}^2K_{16}$.
Now let

$$\begin{aligned}\mathcal{C} = \bigcup_{i \in \mathbb{Z}_8} \{ & G_4[(0,i),(1,1+i),(1,i),(1,4+i)],\\ & G_4[(0,i),(1,4+i),(1,1+i),(0,2+i)],\\ & G_4[(0,2+i),(0,6+i),(1,1+i),(0,7+i)],\\ & G_4[(0,4+i),(1,6+i),(0,i),(1,1+i)],\\ & G_4[(1,i),(0,1+i),(0,i),(1,2+i)]\}.\end{aligned}$$

Then, $(V, \mathcal{C})$ is a G_4-decomposition of ${}^2K_{16}$.

Example 3.10. Let $V = \mathbb{Z}_{17} \cup \{\infty\}$.
Let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_{17}} \{G_2[0,13,2,8]+i, G_2[0,\infty,5,1]+i, G_2[0,\infty,7,3]+i\}$. Then

$(V,\mathcal{C})$ is a G_2-decomposition of ${}^2K_{18}$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{17}}\{G_3[0,5,9,4]+i, G_3[0,\infty,3,1]+i, G_3[0,\infty,7,1]+i\}$. Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^2K_{18}$.

Example 3.11. Let $V = \mathbb{Z}_{23}\cup\{\infty\}$.
Next, let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{23}}\{G_2[0,1,4,6]+i, G_2[0,2,5,7]+i, G_2[0,\infty,8,9]+i, G_2[0,\infty,10,11]+i\}$. Then $(V,\mathcal{C})$ is a G_2-decomposition of ${}^2K_{24}$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{23}}\{G_3[0,3,6,2]+i, G_3[0,4,7,2]+i, G_3[0,\infty,9,1]+i, G_3[0,\infty,11,1]+i\}$. Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^2K_{24}$.

Example 3.12. Let $V = \mathbb{Z}_{29}\cup\{\infty\}$.
Let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{29}}\{G_2[0,8,5,4]+i, G_2[0,13,12,10]+i, G_2[0,14,11,9]+i, G_2[0,\infty,13,7]+i, G_2[0,\infty,14,8]+i\}$. Then $(V,\mathcal{C})$ is a G_2-decomposition of ${}^2K_{30}$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{29}}\{G_3[0,9,5,2]+i, G_3[0,14,10,1]+i, G_3[0,14,12,1]+i, G_3[0,\infty,13,6]+i, G_3[0,\infty,14,6]+i\}$. Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^2K_{30}$.

Example 3.13. Let $V = \mathbb{Z}_{41}\cup\{\infty\}$.
Let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{41}}\{G_2[0,13,9,6]+i, G_2[0,17,12,8]+i, G_2[0,21,10,7]+i, G_2[0,21,16,14]+i, G_2[0,26,15,13]+i, G_2[0,\infty,18,17]+i, G_2[0,\infty,20,19]+i\}$. Then $(V,\mathcal{C})$ is a G_2-decomposition of ${}^2K_{42}$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{41}}\{G_3[0,20,9,3]+i, G_3[0,21,10,3]+i, G_3[0,21,16,2]+i, G_3[0,23,18,1]+i, G_3[0,24,20,1]+i, G_3[0,\infty,12,4]+i, G_3[0,\infty,15,2]+i\}$. Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^2K_{42}$.

Example 3.14. Let $V = \mathbb{Z}_{53}\cup\{\infty\}$.
Next let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{53}}\{G_2[0,1,6,11]+i, G_2[0,2,9,13]+i, G_2[0,5,12,16]+i, G_2[0,6,14,17]+i, G_2[0,7,15,18]+i, G_2[0,9,19,21]+i, G_2[0,10,20,22]+i, G_2[0,\infty,23,24]+i, G_2[0,\infty,25,26]+i\}$. Then $(V,\mathcal{C})$ is a G_2-decomposition of ${}^2K_{54}$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{53}}\{G_3[0,8,16,4]+i, G_3[0,10,18,3]+i, G_3[0,10,17,3]+i, G_3[0,14,21,2]+i, G_3[0,17,22,2]+i, G_3[0,21,11,5]+i, G_3[0,23,13,4]+i, G_3[0,\infty,24,1]+i, G_3[0,\infty,26,1]+i\}$. Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^2K_{54}$.

Example 3.15. Let $V = \mathbb{Z}_9$.
Let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_9}\{G_1[0,5,4,2]+i\}$. Then $(V,\mathcal{C})$ is a G_1-decomposition of ${}^2K_{3\times 3}$.
Let $V = \mathbb{Z}_3\times\mathbb{Z}_3$.
Let

$$\mathcal{C} = \bigcup_{i\in\mathbb{Z}_3}\{G_2[(0,i),(2,i),(1,1+i),(2,2+i)],\\ G_2[(0,i),(2,1+i),(1,i),(2,i)],\\ G_2[(0,2+i),(2,1+i),(1,1+i),(2,i)]\}.$$

Then, $(V, \mathcal{C})$ is a G_2-decomposition of ${}^2K_{3\times 3}$.
Now let

$$\mathcal{C} = \bigcup_{i\in\mathbb{Z}_3} \big\{G_3[(0,i),(0,2+i),(1,1+i),(2,2+i)], \\ G_3[(1,i),(2,2+i),(0,i),(2,i)], \\ G_3[(1,1+i),(1,i),(2,i),(0,2+i)]\big\}.$$

Then, $(V, \mathcal{C})$ is a G_3-decomposition of of ${}^2K_{3\times 3}$.
Let $V = \mathbb{Z}_9$.
Let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_9}\{G_4[0,5,1,2]+i\}$. Then $(V, \mathcal{C})$ is a G_4-decomposition of ${}^2K_{3\times 3}$.

Example 3.16. Let $V = \mathbb{Z}_8$.
Let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_8}\{G_2[0,5,3,1]+i\}$. Then $(V, \mathcal{C})$ is a G_2-decomposition of ${}^2K_{4\times 2}$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_8}\{G_3[0,5,3,2]+i\}$. Then $(V, \mathcal{C})$ is a G_3-decomposition of ${}^2K_{4\times 2}$.

Example 3.17. Let $V = \mathbb{Z}_{15}$.
Let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{15}}\{G_1[0,6,2,1]+i, G_1[0,13,6,3]+i\}$. Then $(V, \mathcal{C})$ is a G_1-decomposition of ${}^2K_{5\times 3}$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{15}}\{G_2[0,2,4,3]+i, G_2[0,5,7,6]+i\}$. Then $(V, \mathcal{C})$ is a G_2-decomposition of ${}^2K_{5\times 3}$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{15}}\{G_3[0,3,1,7]+i, G_3[0,6,4,7]+i\}$. Then $(V, \mathcal{C})$ is a G_3-decomposition of ${}^2K_{5\times 3}$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_{15}}\{G_4[0,7,3,6]+i, G_4[0,8,1,2]+i\}$. Then $(V, \mathcal{C})$ is a G_4-decomposition of ${}^2K_{5\times 3}$.

Example 3.18. Let $V = \mathbb{Z}_6 \cup \{\infty_1, \infty_2, \infty_3\}$.
Let

$$\mathcal{C} = \{G_1[0,5,\infty_1,1], G_1[\infty_1,2,4,3], G_1[3,4,1,\infty_1], G_1[3,\infty_1,2,\infty_2], \\ G_1[\infty_2,4,0,3], G_1[\infty_2,5,1,2], G_1[0,\infty_2,5,\infty_3], G_1[\infty_3,\infty_2,4,5], \\ G_1[3,1,\infty_3,0], G_1[0,\infty_3,2,1], G_1[3,2,5,4]\}.$$

Then $(V, \mathcal{C})$ is a G_1-decomposition of ${}^2K_9 \setminus {}^2K_3$.
Now let

$$\mathcal{C} = \{G_2[\infty_1,1,4,3], G_2[\infty_1,3,1,2], G_2[\infty_1,3,5,0], G_2[\infty_2,1,3,2], \\ G_2[\infty_2,2,0,1], G_2[\infty_2,3,5,4], G_2[\infty_3,0,5,4], G_2[\infty_3,3,0,1], \\ G_2[\infty_3,4,3,2], G_2[4,3,0,2], G_2[5,4,1,2]\}.$$

Then $(V, \mathcal{C})$ is a G_2-decomposition of ${}^2K_9 \setminus {}^2K_3$.
Now let

$$\mathcal{C} = \{G_3[0,5,\infty_1,2], G_3[\infty_1,5,1,3], G_3[\infty_1,3,4,5], G_3[0,\infty_1,3,5], \\ G_3[3,2,\infty_2,4], G_3[\infty_2,0,5,2], G_3[2,\infty_2,1,4], G_3[\infty_2,\infty_3,0,1], \\ G_3[5,3,\infty_3,1], G_3[\infty_3,2,4,0], G_3[\infty_3,0,2,3]\}.$$

Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^2K_9 \setminus {}^2K_3$.
Now let

$$\mathcal{C} = \{G_4[\infty_1,4,1,0], G_4[\infty_1,0,4,3], G_4[\infty_1,2,4,5], G_4[\infty_1,\infty_2,1,2], G_4[\infty_2,1,3,2], G_4[\infty_2,0,3,4], G_4[5,\infty_2,0,1], G_4[\infty_3,\infty_2,5,0], G_4[\infty_3,0,2,1], G_4[\infty_3,5,2,3], G_4[\infty_3,3,5,4]\}.$$

Then $(V,\mathcal{C})$ is a G_4-decomposition of ${}^2K_9 \setminus {}^2K_3$.

Example 3.19. Let $V = \mathbb{Z}_6 \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}$.
Let

$$\mathcal{C} = \{G_1[0,5,\infty_1,1], G_1[\infty_1,5,3,2], G_1[2,3,1,\infty_1], G_1[2,\infty_1,4,\infty_2], G_1[0,3,\infty_2,2], G_1[1,5,\infty_2,4], G_1[2,1,\infty_3,0], G_1[4,3,\infty_3,5], G_1[0,1,5,\infty_3], G_1[5,1,\infty_4,4], G_1[\infty_4,5,2,3], G_1[4,0,3,\infty_4], G_1[4,\infty_4,0,1]\}.$$

Then $(V,\mathcal{C})$ is a G_1-decomposition of ${}^2K_{10} \setminus {}^2K_4$.
Now let

$$\mathcal{C} = \{G_2[\infty_1,0,2,1], G_2[\infty_1,1,3,4], G_2[\infty_1,2,5,0], G_2[\infty_2,0,2,1], G_2[\infty_2,1,4,3], G_2[\infty_2,2,5,0], G_2[\infty_3,0,4,5], G_2[\infty_3,3,0,1], G_2[\infty_3,4,2,3], G_2[\infty_4,0,4,5], G_2[\infty_4,4,1,0], G_2[\infty_4,4,2,3], G_2[5,0,3,1]\}.$$

Then $(V,\mathcal{C})$ is a G_2-decomposition of ${}^2K_{10} \setminus {}^2K_4$.
Now let

$$\mathcal{C} = \{G_3[2,3,\infty_1,4], G_3[\infty_1,1,0,5], G_3[\infty_1,4,1,5], G_3[4,\infty_1,3,\infty_4], G_3[2,5,\infty_2,4], G_3[\infty_2,4,0,3], G_3[\infty_2,4,1,3], G_3[\infty_4,\infty_2,5,4], G_3[4,1,\infty_3,0], G_3[2,5,\infty_3,3], G_3[2,\infty_3,5,3], G_3[\infty_4,\infty_3,1,2], G_3[2,1,0,\infty_4]\}.$$

Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^2K_{10} \setminus {}^2K_4$.
Now let

$$\mathcal{C} = \{G_4[\infty_1,4,0,2], G_4[\infty_1,0,3,4], G_4[\infty_1,2,3,5], G_4[\infty_1,\infty_2,0,1], G_4[1,3,\infty_2,5], G_4[\infty_2,3,1,4], G_4[\infty_2,\infty_3,5,2], G_4[\infty_3,5,0,2], G_4[5,\infty_3,3,4], G_4[\infty_3,\infty_4,0,1], G_4[\infty_4,\infty_3,4,1], G_4[4,3,\infty_4,2], G_4[\infty_4,1,2,5]\}.$$

Then $(V,\mathcal{C})$ is a G_4-decomposition of ${}^2K_{10} \setminus {}^2K_4$.

3.2 $\lambda = 3$

Example 3.20. Let $V = \mathbb{Z}_3 \cup \{\infty\}$.
Let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_3} \{G_1[\infty, 2, 1, 0] + i\}$. Then $(V, \mathcal{C})$ is a G_1-decomposition of ${}^3K_4$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_3} \{G_2[0, 2, 1, \infty] + i\}$. Then $(V, \mathcal{C})$ is a G_2-decomposition of ${}^3K_4$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_3} \{G_3[0, 2, 1, \infty] + i\}$. Then $(V, \mathcal{C})$ is a G_3-decomposition of ${}^3K_4$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_3} \{G_4[0, 2, 1, \infty] + i\}$. Then $(V, \mathcal{C})$ is a G_4-decomposition of ${}^3K_4$.

Example 3.21. Let $V = \mathbb{Z}_5$.
Let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_5} \{G_1[0, 1, 2, 4] + i\}$. Then $(V, \mathcal{C})$ is a G_1-decomposition of ${}^3K_5$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_5} \{G_2[0, 3, 4, 2] + i\}$. Then $(V, \mathcal{C})$ is a G_2-decomposition of ${}^3K_5$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_5} \{G_3[0, 3, 4, 2] + i\}$. Then $(V, \mathcal{C})$ is a G_3-decomposition of ${}^3K_5$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_5} \{G_4[0, 3, 4, 2] + i\}$. Then $(V, \mathcal{C})$ is a G_4-decomposition of ${}^3K_5$.

Example 3.22. Let $V = \mathbb{Z}_7 \cup \{\infty\}$.
Let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_7} \{G_1[2, 1, 4, 0] + i, G_1[\infty, 2, 1, 0] + i\}$. Then $(V, \mathcal{C})$ is a G_1-decomposition of ${}^3K_8$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_7} \{G_2[0, 1, 5, 3] + i, G_2[0, 2, 1, \infty] + i\}$. Then $(V, \mathcal{C})$ is a G_2-decomposition of ${}^3K_8$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_7} \{G_3[0, 1, 4, 2] + i, G_3[0, 2, 1, \infty] + i\}$. Then $(V, \mathcal{C})$ is a G_3-decomposition of ${}^3K_8$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_7} \{G_4[0, 1, 4, 2] + i, G_4[0, 2, 1, \infty] + i\}$. Then $(V, \mathcal{C})$ is a G_4-decomposition of ${}^3K_8$.

Example 3.23. Let $V = \mathbb{Z}_9$.
Let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_9} \{G_1[0, 1, 2, 8] + i, G_1[0, 1, 6, 4] + i\}$. Then $(V, \mathcal{C})$ is a G_1-decomposition of ${}^3K_9$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_9} \{G_2[0, 3, 4, 6] + i, G_2[0, 3, 8, 2] + i\}$. Then $(V, \mathcal{C})$ is a G_2-decomposition of ${}^3K_9$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_9} \{G_3[0, 3, 4, 6] + i, G_3[0, 3, 8, 2] + i\}$. Then $(V, \mathcal{C})$ is a G_3-decomposition of ${}^3K_9$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_9} \{G_4[0, 3, 4, 6] + i, G_4[0, 3, 8, 2] + i\}$. Then $(V, \mathcal{C})$ is a G_4-decomposition of ${}^3K_9$.

Example 3.24. Let $V = \mathbb{Z}_6$.
Let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_6} \{G_1[0, 3, 5, 4] + i\}$. Then $(V, \mathcal{C})$ is a G_1-decomposition of ${}^3K_{3 \times 2}$.
Now let $\mathcal{C} = \bigcup_{i \in \mathbb{Z}_6} \{G_2[0, 3, 5, 4] + i\}$. Then $(V, \mathcal{C})$ is a G_2-decomposition

of ${}^3K_{3\times 2}$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_6}\{G_3[0,4,2,1]+i\}$. Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^3K_{3\times 2}$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_6}\{G_4[0,4,2,1]+i\}$. Then $(V,\mathcal{C})$ is a G_4-decomposition of ${}^3K_{3\times 2}$.

3.3 $\lambda = 4$

Example 3.25. Let $V = \mathbb{Z}_4$.
Let $\mathcal{C} = \{G_1[0,1,2,3], G_1[1,0,3,2], G_1[2,1,0,3], G_1[3,0,1,2]\}$. Then $(V,\mathcal{C})$ is a G_1-decomposition of ${}^4K_4$.
Now let $\mathcal{C} = \{G_2[0,1,2,3], G_2[0,2,3,1], G_2[2,3,1,0], G_2[3,0,1,2]\}$. Then $(V,\mathcal{C})$ is a G_2-decomposition of ${}^4K_4$.

Example 3.26. Let $V = \mathbb{Z}_5 \cup \{\infty\}$.
Let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_5}\{G_2[\infty,4,2,0]+i, G_2[0,2,4,1]+i\}$. Then $(V,\mathcal{C})$ is a G_2-decomposition of ${}^4K_6$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_5}\{G_3[0,\infty,1,2]+i, G_3[0,1,3,\infty]+i\}$. Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^4K_6$

3.4 $\lambda = 6$

Example 3.27. Let $V = \mathbb{Z}_5 \cup \{\infty\}$.
Let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_5}\{G_2[0,3,2,1]+i, G_2[0,1,2,\infty]+i, G_2[0,1,2,\infty]+i\}$. Then $(V,\mathcal{C})$ is a G_2-decomposition of ${}^6K_6$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_5}\{G_3[0,3,2,1]+i, G_3[0,1,2,\infty]+i, G_3[0,1,2,\infty]+i\}$. Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^6K_6$.

Example 3.28. Let $V = \mathbb{Z}_3 \cup \{\infty_1,\infty_2\}$.
Let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_3}\{G_1[0,\infty_1,1,\infty_2]+i\}$. Then $(V,\mathcal{C})$ is a G_1-decomposition of ${}^2K_5 \setminus {}^2K_2$. Note that by taking 3 copies of ${}^2K_5 \setminus {}^2K_2$ we have ${}^6K_5 \setminus {}^6K_2$, so there is a G_1-decomposition of ${}^6K_5 \setminus {}^6K_2$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_3}\{G_2[0,\infty_1,1,\infty_2]+i, G_2[0,\infty_1,1,\infty_2]+i, G_2[\infty_1,0,1,2]+i\}$. Then $(V,\mathcal{C})$ is a G_2-decomposition of ${}^6K_5 \setminus {}^6K_2$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_3}\{G_3[0,1,\infty_1,2]+i, G_3[0,\infty_1,1,\infty_2]+i, G_3[\infty_2,0,1,2]+i\}$. Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^6K_5 \setminus {}^6K_2$.
Now let $\mathcal{C} = \bigcup_{i\in\mathbb{Z}_3}\{G_4[0,\infty_2,1,2]+i, G_4[\infty_1,\infty_2,0,1]+i, G_4[\infty_1,\infty_2,0,1]+i\}$. Then $(V,\mathcal{C})$ is a G_4-decomposition of ${}^6K_5 \setminus {}^6K_2$.

Example 3.29. Let $V = \mathbb{Z}_4 \cup \{\infty_1, \infty_2\}$.
Let

$$\begin{aligned}\mathcal{C} = \{&G_1[0,2,\infty_1,3], G_1[2,0,\infty_1,1], G_1[1,0,3,\infty_1], G_1[3,2,1,\infty_1],\\ &G_1[1,\infty_1,0,2], G_1[3,\infty_1,2,0], G_1[\infty_1,\infty_2,1,3], G_1[\infty_1,\infty_2,3,1],\\ &G_1[1,2,\infty_2,0], G_1[3,0,\infty_2,2], G_1[0,\infty_2,3,2], G_1[2,\infty_2,1,0],\\ &G_1[\infty_2,1,0,2], G_1[\infty_2,3,2,0]\}.\end{aligned}$$

Then $(V,\mathcal{C})$ is a G_1-decomposition of ${}^6K_6 \setminus {}^6K_2$.
Now let

$$\begin{aligned}\mathcal{C} = \{&G_2[\infty_1,1,2,3], G_2[\infty_1,3,0,1], G_2[\infty_1,\infty_2,0,2], G_2[\infty_1,\infty_2,2,0],\\ &G_2[\infty_1,\infty_2,1,3], G_2[\infty_1,\infty_2,3,1], G_2[\infty_2,3,0,2], G_2[\infty_2,1,2,0],\\ &G_2[\infty_2,0,1,3], G_2[\infty_2,2,3,1], G_2[1,\infty_2,2,0], G_2[3,\infty_2,0,2],\\ &G_2[0,\infty_2,3,1], G_2[2,\infty_2,1,3]\}.\end{aligned}$$

Then $(V,\mathcal{C})$ is a G_2-decomposition of ${}^6K_6 \setminus {}^6K_2$.
Now let

$$\begin{aligned}\mathcal{C} = \{&G_3[0,3,\infty_1,2], G_3[2,1,\infty_1,0], G_3[\infty_2,\infty_1,0,1], G_3[\infty_2,\infty_1,2,3],\\ &G_3[1,0,\infty_1,3], G_3[3,2,\infty_1,1], G_3[\infty_2,\infty_1,1,0], G_3[\infty_2,\infty_1,3,2],\\ &G_3[\infty_2,1,3,0], G_3[\infty_2,3,1,2], G_3[\infty_2,0,2,1], G_3[\infty_2,2,0,3],\\ &G_3[0,1,3,2], G_3[2,3,1,0]\}.\end{aligned}$$

Then $(V,\mathcal{C})$ is a G_3-decomposition of ${}^6K_6 \setminus {}^6K_2$.
Now let

$$\begin{aligned}\mathcal{C} = \{&G_4[0,2,\infty_1,1], G_4[2,0,\infty_1,3], G_4[\infty_1,1,0,2], G_4[\infty_1,3,2,0],\\ &G_4[\infty_1,0,3,1], G_4[\infty_1,2,1,3], G_4[\infty_2,\infty_1,1,3], G_4[\infty_2,\infty_1,3,1],\\ &G_4[\infty_2,1,0,2], G_4[\infty_2,3,2,0], G_4[\infty_2,0,3,1], G_4[\infty_2,2,1,3],\\ &G_4[\infty_2,1,2,0], G_4[\infty_2,3,0,2]\}.\end{aligned}$$

Then $(V,\mathcal{C})$ is a G_4-decomposition of ${}^6K_6 \setminus {}^6K_2$.

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