On cyclic decompositions of $K_{n+1,n+1} - I$ into a 2-regular graph with at most 2 components*

R. C. Bunge[1], S. I. El-Zanati[1], D. Gibson[2], U. Jongthawonwuth[3], J. Nagel[4], B. Stanley[5], A. Zale[1]

[1]Department of Mathematics
Illinois State University
Normal, IL 61790-4520

[2]Department of Mathematics
Concordia University
Seward, NE 68434

[3]Department of Mathematics
Chulalongkorn University
Bangkok 10330 Thailand

[4]Mathematics Department
Dominican University
River Forest, IL 60305

[5]Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003

Abstract

Let G with n edges be a 2-regular bipartite graph with one or two components. We show that there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$, where I is a 1-factor.

1 Introduction

If m and n are integers with $m \leq n$, we denote $\{m, m+1, \ldots, n\}$ by $[m, n]$. Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. Let K_m have vertex set $\mathbb{Z}_m$ and let G be a subgraph of K_m. By *clicking* G we mean applying the isomorphism $i \mapsto i+1$ to $V(G)$. Likewise, if we let $V(K_{m,m}) = \mathbb{Z}_m \times \mathbb{Z}_2$ with the obvious vertex bipartition, *clicking* a subgraph G of $K_{m,m}$ means to apply the isomorphism $(i, j) \mapsto (i+1, j)$ to $V(G)$.

*Research supported by National Science Foundation Grant No. A1063038

Let $V(K_m) = \{0, 1, \ldots, m-1\}$. The *length* of an edge $e = \{i, j\}$ in K_m is $\min\{|i-j|, m-|i-j|\}$. Note that clicking an edge does not change its length.

Now, let $V(K_{m,m}) = \{0, 1, \ldots, m-1\} \times \mathbb{Z}_2$. The *length* of an edge $e = \{(i, 0), (j, 1)\}$ in $K_{m,m}$ is $j-i$ if $j \geq i$ and $m+j-i$, otherwise. As with K_m, we note that clicking an edge in $K_{m,m}$ does not change its length. Also note that $K_{m,m}$ consists of n edges of length i for $i \in [0, m-1]$. Moreover, the edges of length i for $i \in [0, m-1]$ form a 1-factor in $K_{m,m}$.

Let K and G be graphs with G a subgraph of K. A *G-decomposition* of K is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that the edge sets of the graphs G_i form a partition of the edge set of K. The elements of Δ are called *G-blocks*. Such a G-decomposition is said to be *cyclic* if clicking preserves the G-blocks of Δ. A G-decomposition of K is also called a *(K, G)-design*. The study of (K, G)-designs is known as the study of graph designs or simply of G-designs.

A *G-factor* of a graph K is a set of G-blocks whose vertex sets partition the vertex set of K. A *G-factorization* is a G-decomposition where the G-blocks are partitioned into G-factors. A G-factorization is also called a *resolvable G-decomposition*.

The following is a commonly investigated question in graph designs.

Question 1. *Given a graph G with n edges, for which $2n$-regular graphs K does there exist a (K, G)-design?*

Question 1 is difficult to answer in general. However, it is often the case that (K_{2n+1}, G)-designs do exist. Similarly, $(K_{2n+2} - I, G)$-designs where I is a 1-factor often exist. If G is bipartite, then the following is also asked.

Question 2. *Given a bipartite graph G with n edges, for which n-regular bipartite graphs K does there exist a (K, G)-design?*

In this case, $K_{n,n}$ and $K_{n+1,n+1} - I$, where I is a 1-factor, are the common candidates for K.

Let G be a 2-regular bipartite graph with n edges. It is of interest to learn whether or not G decomposes $K_{n,n}$ and $K_{n+1,n+1} - I$. These questions relate to the complete bipartite graph version of the Oberwolfach problem. In [5], Piotrowski showed that if $n \equiv 0 \pmod 4$, then there exists a G-decomposition (actually a G-factorization) of $K_{n/2,n/2}$. Since $K_{n/2,n/2}$ decomposes $K_{n,n}$, the the existence of a G-decomposition of $K_{n,n}$ follows in this case. We note however that these decompositions need not be cyclic. If $n \equiv 2 \pmod 4$, then little is known about G-decompositions of $K_{n,n}$ or of $K_{n+1,n+1} - I$, except in a few cases. In [6], Sotteau found necessary and sufficient conditions for the existence of a C_n-decomposition of $K_{v,w}$. The corresponding problem for C_n-decompositions of $K_{v,v} - I$

was first investigated in [1] and settled completely in [4]. In [2], cyclic G-decompositions of $K_{n+1,n+1} - I$ are investigated for 2-regular bipartite graphs G of order $n \equiv 0 \pmod 4$, and the following is proved.

Theorem 1. *Let G be a 2-regular bipartite graph with n edges where $n \equiv 0 \pmod 4$. Then there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$, where I is a 1-factor.*

Finding cyclic G-decomposition of $K_{n+1,n+1} - I$ when $n \equiv 2 \pmod 4$ seems to be far more challenging. In this note, we show that if $n \equiv 2 \pmod 4$ and if G consists of at most two cycles, then there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$.

As is often the case when studying cyclic graph decompositions, graph labelings provide a convenient and powerful tool. We discuss one of these labelings next, and we give some notation.

1.1 Bilabelings

For a bipartite graph G with n edges, the simplest way to obtain a G-decomposition of $K_{n,n}$ is to embed G in $K_{n,n}$ so that there is exactly one edge of G of length i for each $i \in [0, n-1]$. Then clicking G a total of $n-1$ times would yield the desired design cyclically. This result is considered folklore and is used regularly by researchers in the area. In [3], such an embedding of G is called a ρ-bilabeling of G.

Suppose G with n edges has vertex bipartition $\{A \times \{0\}, B \times \{1\}\}$. A *bilabeling* of G is a function $f\colon V(G) \to \mathbb{N}$ such that $f|_{A\times\{0\}}$ and $f|_{B\times\{1\}}$ are injective. Now if $f\colon V(G) \to [0, n-1]$ is a bilabeling of G, we also define $\bar{f}\colon E(G) \to [0, n-1]$ such that if $e = \{(a,0),(b,1)\} \in E(G)$, then $\bar{f}(e) = f\big((b,1)\big) - f\big((a,0)\big)$ if $f\big((b,1)\big) \geq f\big((a,0)\big)$ and $\bar{f}(e) = |E(G)| + f\big((b,1)\big) - f\big((a,0)\big)$, otherwise (i.e., $\bar{f}(e)$ is the length of edge e). Then f is a *ρ-bilabeling* of G if $\{\bar{f}(e)\colon e \in E(G)\} = [0, n-1]$. Thus we have the following.

Theorem 2. *Let G be a bipartite graph of size n. There exists a cyclic G-decomposition of $K_{n,n}$ if and only if G has a ρ-bilabeling.*

It should be noted that not every bipartite graph admits a ρ-bilabeling. The following theorem is stated without proof in [3]. We provide a quick proof here.

Theorem 3. *Let G be a bipartite graph of size n and suppose every vertex of G has even degree. If G admits a ρ-bilabeling then $n \equiv 0 \pmod 4$.*

Proof. Let $\{A \times \{0\}, B \times \{1\}\}$ be a bipartition of $V(G)$. We note first that n must be even since every vertex has even degree and $|E(G)| = \sum_{a\in A} \deg\big((a,0)\big)$. Let f be a ρ-bilabeling of G. Then $\sum_{e\in E(G)} \bar{f}(e) =$

$\sum_{i=0}^{n-1} i = n(n-1)/2$. Moreover, this sum must be even since n is even, $\bar{f}(e) \in \{f((b,1)) - f((a,0)), n + f((b,1)) - f((a,0))\}$ for every edge $e = \{(a,0),(b,1)\}$ in G, and every vertex in G has even degree. Thus 2 divides $n(n-1)/2$. Since n is even, the result follows. ∎

A strategy similar to that of the above proof is used to obtain cyclic G-decompositions of $K_{n+1,n+1} - I$. In this case, we select a length $j \in [0,n]$, and we embed G in $K_{n+1,n+1}$ so that there is exactly one edge of G of length i for each $i \in [0,n] \setminus \{j\}$. The set of all edges of length j forms the 1-factor I. Clicking G a total of n times would yield the desired design cyclically.

1.2 Some notation

We denote the directed path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \le i \le k-1$, by $(x_0, x_1, \ldots, x_k)$. The *first vertex* of this path is x_0, the *second vertex* is x_1, and the *last vertex* is x_k. If $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$ are directed paths with $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$.

For the remainder of this manuscript, we consider only subgraphs of a complete bipartite graphs $K_{m,m}$ with vertex set $\{0,1,\ldots,m-1\} \times \mathbb{Z}_2$ and the obvious vertex bipartition. Furthermore, if m, n, and i are integers with $m \le n$, we denote $\{(m,i),(m+1,i),\ldots,(n,i)\}$ by $[(m,i),(n,i)]$

Let $P(k)$ be the path with k edges and $k+1$ vertices given by $((0,0), (k,1),(1,0),(k-1,1),(2,0),(k-2,1),\ldots,(\lceil k/2 \rceil, \lceil k/2 \rceil - \lfloor k/2 \rfloor))$. Note that the set of vertices of this graph is $A \cup B$, where $A = [(0,0),(\lfloor k/2 \rfloor,0)]$, $B = [(\lfloor k/2 \rfloor + 1, 1),(k,1)]$, and every edge joins a vertex of A to one of B. Furthermore, the set of lengths of the edges of $P(k)$ is $[1,k]$.

Now let a and b be nonnegative integers with $a \le b$ and let us add $(a,0)$ to all the vertices of A and $(b,0)$ to all the vertices of B. We denote the resulting graph by $P(a,b,k)$. Note that this graph has the following properties.

- **P1** $P(a,b,k)$ is a path with first vertex $(a,0)$ and second vertex $(b+k,1)$. Its last vertex is $(a+k/2,0)$ if k is even and $(b+(k+1)/2,1)$ if k is odd.
- **P2** Each edge of $P(a,b,k)$ joins a vertex of $A' = [(a,0),(\lfloor k/2 \rfloor + a, 0)]$ to a vertex of $B' = [(\lfloor k/2 \rfloor + 1 + b, 1),(k+b,1)]$.
- **P3** The set of edge lengths of $P(a,b,k)$ is $[b-a+1, b-a+k]$.

Now consider the directed path $Q(k)$ obtained from $P(k)$ replacing each vertex (i,j) with $(k-i, 1-j)$. The new graph is the path $((k,1),(0,0),(k-1,1),(1,0),\ldots,(\lfloor k/2 \rfloor, \lfloor k/2 \rfloor - \lceil k/2 \rceil + 1))$. The set of vertices of $Q(k)$ is

$A \cup B$, where $A = \left[(0,0), (\lceil k/2 \rceil - 1, 0)\right]$ and $B = \left[(\lceil k/2 \rceil, 1), (k, 1)\right]$, and every edge joins a vertex of A to one of B. The set of edge lengths is still $[1, k]$.

We again add $(a, 0)$ to the vertices of A'' and $(b, 0)$ to vertices of B'', where a and b are nonnegative integers with $a \leq b$. We denote the resulting graph by $Q(a, b, k)$. Note that this graph has the following properties.

Q1 $Q(a, b, k)$ is a path with first vertex $(k + b, 1)$. Its last vertex is $(b + k/2, 1)$ if k is even and $(a + (k-1)/2, 0)$ if k is odd.

Q2 Each edge of $Q(a, b, k)$ joins a vertex of $A' = \left[(a, 0), (a + \lceil k/2 \rceil - 1, 0)\right]$ to a vertex of $B = \left[(b + \lceil k/2 \rceil, 1), (b + k, 1)\right]$.

Q3 The set of edge lengths of $Q(a, b, k)$ is $[b - a + 1, b - a + k]$.

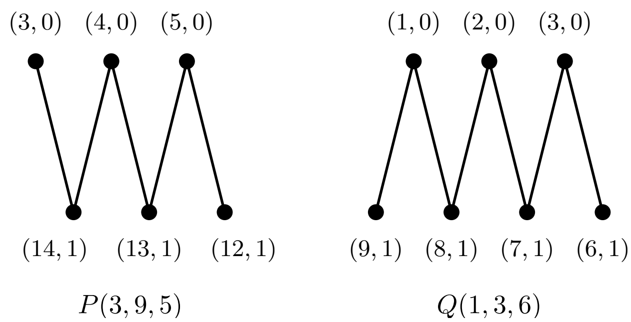

Figure 1: Examples of the $P(a, b, k)$ and $Q(a, b, k)$ notation

For ease of notation, we henceforth use i_0 and i_1 to denote the vertices $(i, 0)$ and $(i, 1)$, respectively.

2 Main Results

Lemma 4. *Let G be an even cycle of length n where $n \equiv 2 \pmod 4$ and let I be a 1-factor of $K_{n+1,n+1}$. Then there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$.*

Proof. Let $G = C_{4r+2}$ where $r \in \mathbb{Z}^+$. Let $C_{4r+2} = G_1 + G_2 + ((2r)_0, 0_1, 0_0)$ where

$$G_1 = P(0, 2r + 3, 2r - 2),$$
$$G_2 = P(r - 1, r - 1, 2r + 2).$$

First, we show that $G_1 + G_2 + ((2r)_0, 0_1, 0_0)$ is a cycle of length $4r + 2$. Note that by **P1**, the first vertex of G_1 is 0_0, and the last is $(r - 1)_0$; and the first vertex of G_2 is $(r-1)_0$, and the last is $(2r)_0$. For $1 \le i \le 2$, let A_i and B_i denote the sets labeled A' and B' in **P2** corresponding to the path G_i. Then using **P2**, we compute

$$A_1 = [0_0, (r-1)_0], \qquad B_1 = [(3r+3)_1, (4r+1)_1],$$
$$A_2 = [(r-1)_0, (2r)_0], \qquad B_2 = [(2r+1)_1, (3r1)_1].$$

Note that $V(G_1) \cap V(G_2) = \{(r-1)_0\}$; otherwise, G_1 and G_2 are vertex-disjoint. Therefore, $G_1 + G_2 + ((2r)_0, 0_1, 0_0)$ is a cycle of length $4r+2$.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 2$. By **P3**, we have edge labels

$$E_1 = [2r+4, 4r+1],$$
$$E_2 = [1, 2r+2]$$

yielding edge lengths of the same values. Moreover, the path $((2r)_0, 0_1, 0_0)$ consists of edges with lengths $(-2r)^* = 2r+3$ and 0. Thus, the edge set of G has one edge of each length $i \in [0, 4r+2] \setminus \{4r+2\}$. An example of this labeling is given in Figure 2 with $r = 2$.

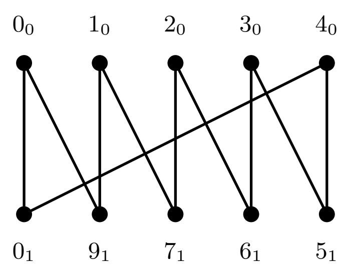

Figure 2: C_{10} with the described labeling

Thus there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$, where I is the 1-factor consisting of all edges of length $4r+2$. ∎

Theorem 5. *Let G be a 2-regular bipartite graph with n edges and at most two components. Then there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$, where I is a 1-factor.*

Proof. If $n \equiv 0 \pmod 4$, then the result follows from Theorem 1. If G is a single cycle of (even) length $n \equiv 2 \pmod 4$, the result is proved in Lemma 4.

Now let $G = C_{4r} \cup C_{4s+2}$ where $r, s \in \mathbb{Z}^+$. We consider four cases.

Case 1: $r < s$.
Let $C_{4r} = G_1 + G_2 + ((2r+1)_0, 0_1, 0_0)$ and $C_{4s+2} = G_3 + G_4 + ((2r+2s+3)_0, 2_1, (2r+2)_0)$ where

$$\begin{aligned} G_1 &= P(0, 2r+4s+3, 2r-1), \\ G_2 &= Q(r+2, r+4s+4, 2r-1), \\ G_3 &= P(2r+2, 4r+2s+4, 2s-2r-1), \\ G_4 &= Q(r+s+3, r+s+3, 2r+2s+1). \end{aligned}$$

First, we show that $G_1 + G_2 + ((2r+1)_0, 0_1, 0_0)$ is a cycle of length $4r$, and $G_3 + G_4 + ((2r+2s+3)_0, 2_1, (2r+2)_0)$ is a cycle of length $4s+2$. Note that by **P1** and **Q1**, the first vertex of G_1 is 0_0, and the last is $(3r+4s+3)_1$; the first vertex of G_2 is $(3r+4s+3)_1$, and the last is $(2r+1)_0$; the first vertex of G_3 is $(2r+2)_0$, and the last is $(3r+3s+4)_1$; and the first vertex of G_4 is $(3r+3s+4)_1$, and the last is $(2r+2s+3)_0$. For $1 \le i \le 4$, let A_i and B_i denote the sets labeled A' and B' in **P2** and **Q2** corresponding to the path G_i. Then using **P2** and **Q2**, we compute

$$\begin{aligned} A_1 &= [0_0, (r-1)_0], & B_1 &= [(3r+4s+3)_1, (4r+4s+2)_1], \\ A_2 &= [(r+2)_0, (2r+1)_0], & B_2 &= [(2r+4s+4)_1, (3r+4s+3)_1], \\ A_3 &= [(2r+2)_0, (r+s+1)_0], & B_3 &= [(3r+3s+4)_1, (2r+4s+3)_1], \\ A_4 &= [(r+s+3)_0, (2r+2s+3)_0], & B_4 &= [(2r+2s+4)_1, (3r+3s+4)_1]. \end{aligned}$$

Note that $V(G_1) \cap V(G_2) = \{(3r+4s+3)_1\}$ and $V(G_3) \cap V(G_4) = \{(3r+3s+4)_1\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1 + G_2 + ((2r+1)_0, 0_1, 0_0)$ is a cycle of length $4r$, and $G_3 + G_4 + ((2r+2s+3)_0, 2_1, (2r+2)_0)$ is a cycle of length $4s+2$.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 4$. By **P3** and **Q3**, we have edge lengths

$$\begin{aligned} E_1 &= [2r+4s+4, 4r+4s+2], \\ E_2 &= [4s+3, 2r+4s+1], \\ E_3 &= [2r+2s+3, 4s+1], \\ E_4 &= [1, 2r+2s+1]. \end{aligned}$$

Moreover, the path $((2r+1)_0, 0_1, 0_0)$ consists of edges with lengths $4r+4s+3+(-2r-1) = 2r+4s+2$ and 0, and the path $((2r+2s+3)_0, 2_1, (2r+2)_0)$ consists of edges with lengths $4r+4s+3+(-2r-2s-1) = 2r+2s+2$ and $4r+4s+3+(-2r) = 2r+4s+3$. Thus, the edge set of G has one edge of each length $i \in [0, 4r+4s+2] \setminus \{4s+2\}$. An example of this labeling is given in Figure 3 with $r = 1$ and $s = 2$.

Case 2: $r = s$.
Let $C_{4r} = G_1 + G_2 + ((2r+1)_0, 0_1, 0_0)$ and $C_{4s+2} = G_3 + ((4r+3)_0, 2_1, (2r+

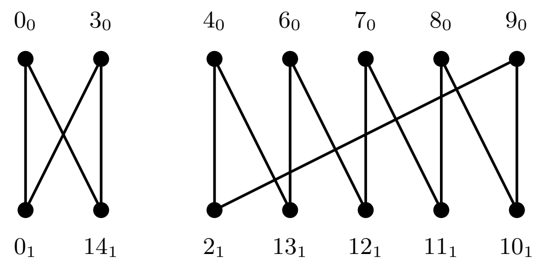

Figure 3: $C_4 \cup C_{10}$ with the described labeling

$2)_0, (6r+3)_1)$ where

$$G_1 = P(0, 6r+3, 2r-1),$$
$$G_2 = Q(r+2, 5r+4, 2r-1),$$
$$G_3 = Q(2r+4, 2r+4, 4r-1).$$

First, we show that $G_1 + G_2 + ((2r+1)_0, 0_1, 0_0)$ is a cycle of length $4r$, and $G_3 + ((4r+3)_0, 2_1, (2r+2)_0, (6r+3)_1)$ is a cycle of length $4s+2$. Note that by **P1** and **Q1**, the first vertex of G_1 is 0_0, and the last is $(7r+3)_1$; the first vertex of G_2 is $(7r+3)_1$, and the last is $(2r+1)_0$; and the first vertex of G_3 is $(6r+3)_1$, and the last is $(4r+3)_0$; For $1 \le i \le 3$, let A_i and B_i denote the sets labeled A' and B' in **P2** and **Q2** corresponding to the path G_i. Then using **P2** and **Q2**, we compute

$$A_1 = [0_0, (r-1)_0], \qquad B_1 = [(7r+3)_1, (8r+2)_1],$$
$$A_2 = [(r+2)_0, (2r+1)_0], \qquad B_2 = [(6r+4)_1, (7r+3)_1],$$
$$A_3 = [(2r+4)_0, (4r+3)_0], \qquad B_3 = [(4r+4)_1, (6r+3)_1].$$

Note that $V(G_1) \cap V(G_2) = \{(7r+3)_1\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1 + G_2 + ((2r+1)_0, 0_1, 0_0)$ is a cycle of length $4r$, and $G_3 + ((4r+3)_0, 2_1, (2r+2)_0, (6r+3)_1)$ is a cycle of length $4s+2$.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 4$. By **P3** and **Q3**, we have edge lengths

$$E_1 = [6r+4, 8r+2],$$
$$E_2 = [4r+3, 6r+1],$$
$$E_3 = [1, 4r-1].$$

Moreover, the path $((2r+1)_0, 0_1, 0_0)$ consists of edges with lengths $4r+4s+3+(-2r-1) = 6r+2$ and 0, and the path $((4r+3)_0, 2_1, (2r+2)_0, (6r+3)_1)$ consists of edges with lengths $4r+4s+3+(-4r-1) = 4r+2$, $4r+4s+$

$3 + (-2r) = 6r + 3$, and $4r + 1$. Thus, the edge set of G has one edge of each length $i \in [0, 8r + 2] \setminus \{4r\}$. An example of this labeling is given in Figure 4 with $r = s = 2$.

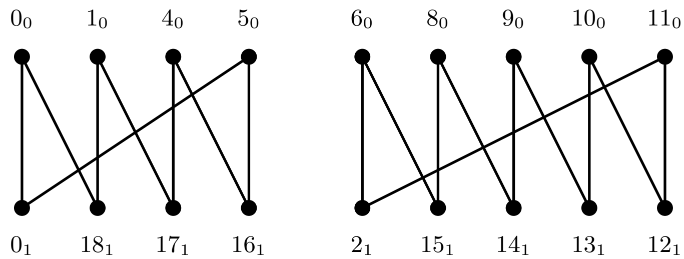

Figure 4: $C_8 \cup C_{10}$ with the described labeling

Case 3: $r = s + 1$.
Let $C_{4s+2} = G_1 + G_2 + ((2s + 2)_0, 1_1, 0_0)$ and $C_{4r} = G_3 + ((2r + 2s + 4)_1, (2r + 2s + 4)_0, 3_1, (2s + 3)_0, (4r + 2s + 2)_1)$ where

$$\begin{aligned} G_1 &= P(0, 4r + 2s + 3, 2s - 1),\\ G_2 &= Q(s + 2, 4r + s + 2, 2s + 1),\\ G_3 &= Q(2s + 5, 2s + 6, 4r - 4), \end{aligned}$$

First, we show that $G_1 + G_2 + ((2s + 2)_0, 1_1, 0_0)$ is a cycle of length $4s + 2$, and $G_3 + ((2r + 2s + 4)_1, (2r + 2s + 4)_0, 3_1, (2s + 3)_0, (4r + 2s + 2)_1)$ is a cycle of length $4r$. Note that by **P1** and **Q1**, the first vertex of G_1 is 0_0, and the last is $(4r + 3s + 3)_1$; the first vertex of G_2 is $(4r + 3s + 3)_1$, and the last is $(2s + 2)_0$; and the first vertex of G_3 is $(4r + 2s + 2)_1$, and the last is $(2r + 2s + 4)_1$. For $1 \le i \le 4$, let A_i and B_i denote the sets labeled A' and B' in **P2** and **Q2** corresponding to the path G_i. Then using **P2** and **Q2**, we compute

$$\begin{aligned} A_1 &= [0_0, (s - 1)_0], & B_1 &= [(4r + 3s + 3)_1, (4r + 4s + 2)_1],\\ A_2 &= [(s + 2)_0, (2s + 2)_0], & B_2 &= [(4r + 2s + 3)_1, (4r + 3s + 3)_1],\\ A_3 &= [(2s + 5)_0, (2r + 2s + 2)_0], & B_3 &= [(2r + 2s + 4)_1, (4r + 2s + 2)_1]. \end{aligned}$$

Note that $V(G_1) \cap V(G_2) = \{(4r + 3s + 3)_1\}$; otherwise, G_i and G_j are vertex-disjoint for $i \ne j$. Therefore, $G_1 + G_2 + ((2s + 2)_0, 1_1, 0_0)$ is a cycle of length $4s+2$, and $G_3 + ((2r+2s+4)_1, (2r+2s+4)_0, 3_1, (2s+3)_0, (4r+2s+2)_1)$ is a cycle of length $4r$.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 3$. By **P3** and **Q3**, we have edge lengths

$$E_1 = [4r + 2s + 4, 4r + 4s + 2],$$
$$E_2 = [4r + 1, 4r + 2s + 1],$$
$$E_3 = [2, 4r - 3].$$

Moreover, the path $((2s + 2)_0, 1_1, 0_0)$ consists of edges with lengths $4r + 4s + 3 + (-2s - 1) = 4r + 2s + 2$ and 1, and the path $((2r + 2s + 4)_1, (2r + 2s + 4)_0, 3_1, (2s + 3)_0, (4r + 2s + 2)_1)$ consists of edges with lengths 0, $4r + 4s + 3 + (-2r - 2s - 1) = 2r + 2s + 2 = 4r$, $4r + 4s + 3 + (-2s) = 4r + 2s + 3$, and $4r - 1$. Thus, the edge set of G has one edge of each length $i \in [0, 4r + 4s + 2] \setminus \{4r - 2\}$. An example of this labeling is given in Figure 5 with $r = 2$ and $s = 1$.

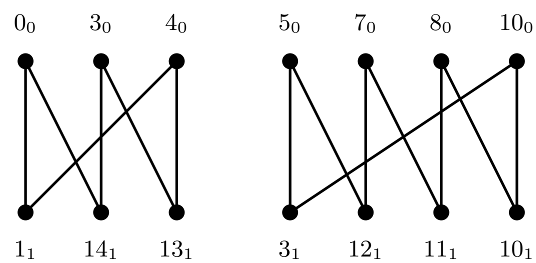

Figure 5: $C_6 \cup C_8$ with the described labeling

Case 4: $r > s + 1$.
Let $C_{4s+2} = G_1 + G_2 + ((2s + 2)_0, 1_1, 0_0)$ and $C_{4r} = G_3 + G_4 + ((2s + 2r + 4)_1, (2s + 2r + 4)_0, 3_1, (2s + 3)_0)$ where

$$G_1 = P(0, 4r + 2s + 3, 2s - 1),$$
$$G_2 = Q(s + 2, 4r + s + 2, 2s + 1),$$
$$G_3 = P(2s + 3, 2r + 4s + 5, 2r - 2s - 3),$$
$$G_4 = Q(r + s + 3, r + s + 4, 2s + 2r).$$

First, we show that $G_1 + G_2 + ((2s + 2)_0, 1_1, 0_0)$ is a cycle of length $4s + 2$, and $G_3 + G_4 + ((2s + 2r + 4)_1, (2s + 2r + 4)_0, 3_1, (2s + 3)_0)$ is a cycle of length $4r$. Note that by **P1** and **Q1**, the first vertex of G_1 is 0_0, and the last is $(4r + 3s + 3)_1$; the first vertex of G_2 is $(4r + 3s + 3)_1$, and the last is $(2s + 2)_0$; the first vertex of G_3 is $(2s + 3)_0$, and the last is $(3r + 3s + 4)_1$; and the first vertex of G_4 is $(3r + 3s + 4)_1$, and the last is $(2r + 2s + 4)_1$. For $1 \leq i \leq 4$, let A_i and B_i denote the sets labeled A' and B' in **P2** and

Q2 corresponding to the path G_i. Then using **P2** and **Q2**, we compute

$$
\begin{aligned}
A_1 &= [0_0, (s-1)_0], & B_1 &= [(4r+3s+3)_1, (4r+4s+2)_1],\\
A_2 &= [(s+2)_0, (2s+2)_0], & B_2 &= [(4r+2s+3)_1, (4r+3s+3)_1],\\
A_3 &= [(2s+3)_0, (r+s+1)_0], & B_3 &= [(3r+3s+4)_1, (4r+2s+2)_1],\\
A_4 &= [(r+s+3)_0, (2s+2r+2)_0], & B_4 &= [(2r+2s+4)_1, (3r+3s+4)_1].
\end{aligned}
$$

Note that $V(G_1) \cap V(G_2) = \{(4r+3s+3)_1\}$ and $V(G_3) \cap V(G_4) = \{(3r+3s+4)_1\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1 + G_2 + ((2s+2)_0, 1_1, 0_0)$ is a cycle of length $4s+2$, and $G_3 + G_4 + ((2s+2r+4)_1, (2s+2r+4)_0, 3_1, (2s+3)_0)$ is a cycle of length $4r$.

Next, let E_i denote the set of edge labels in G_i for $1 \leq i \leq 4$. By **P3** and **Q3**, we have edge lengths

$$
\begin{aligned}
E_1 &= [4r+2s+4, 4r+4s+2],\\
E_2 &= [4r+1, 4r+2s+1],\\
E_3 &= [2s+2r+3, 4r-1],\\
E_4 &= [2, 2s+2r+1].
\end{aligned}
$$

Moreover, the path $((2s+2)_0, 1_1, 0_0)$ consists of edges with lengths $4r+4s+3+(-2s-1) = 4r+2s+2$ and 1, and the path $((2s+2r+4)_1, (2s+2r+4)_0, 3_1, (2s+3)_0)$ consists of edges with lengths 0, $4r+4s+3+(-2r-2s-1) = 2r+2s+2$, and $4r+4s+3+(-2s) = 4r+2s+3$. Thus, the edge set of G has one edge of each length $i \in [0, 4r+4s+2] \setminus \{4r\}$. An example of this labeling is given in Figure 6 with $r=3$ and $s=1$.

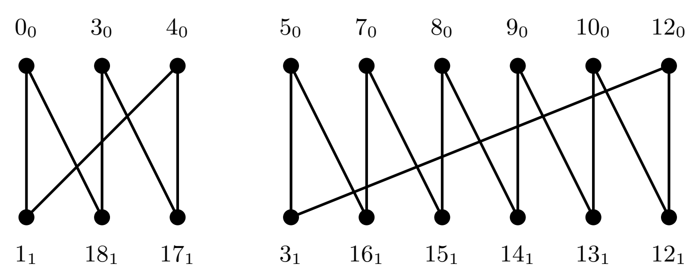

Figure 6: $C_6 \cup C_{12}$ with the described labeling

Thus in each of the four cases, there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$, where I is a 1-factor. ∎

3 Acknowledgement and Final Note

This research is partially supported by grant number A1063038 from the Division of Mathematical Sciences at the National Science Foundation. This work was done while the third and the fifth through seventh authors were participants in *REU Site: Mathematics Research Experience for Pre-service and for In-service Teachers* at Illinois State University.

References

[1] A. Archdeacon, M. Dembowsky, J. Dinitz, and H. Gavlas, Cycle systems in the complete graph minus a one-factor, *Discrete Math.* **284** (2004), 37–43.

[2] R. C. Bunge, S. I. El-Zanati, D. Gibson, J. Nagel, B. Stanley, and A. Zale, On cyclic decompositions of $K_{n+1,n+1} - I$ into 2-regular graphs of size n, preprint.

[3] S. I. El-Zanati and C. Vanden Eynden, On Rosa-type labelings and cyclic graph decompositions, *Mathematica Slovaca* **59** (2009), 1–18.

[4] J. Ma, L. Pu, and H. Shen, Cycle decompositions of $K_{n,n} - I$, *SIAM J. Discrete Math.* **20**, 603–609.

[5] W.-L. Piotrowski, The solution of the bipartite analogue of the Oberwolfach problem, *Discrete Math.* **97** (1991), 339–356.

[6] D. Sotteau, Decompositions of $K_{m,n}(K^*_{m,n})$ into cycles (circuits) of length $2k$, *J. Combin. Theory Ser. B* **29** (1981), 75–81.