On λ-fold Rosa-type Labelings of Bipartite Multigraphs

R. C. Bunge [a], S. I. El-Zanati [a], J. Mudrock [b], C. Vanden Eynden [a], W. Wannasit [c]

[a] *Department of Mathematics, Illinois State University, Normal, IL 61790-4520 USA*

[b] *Engineering, Math & Physical Sciences Division, College of Lake County, Grayslake, IL 60030 USA*

[c] *Department of Mathematics, Chiang Mai University, Chiang Mai 50200, Thailand*

Abstract

It is known that for a given (simple) graph G with n edges, there exits a cyclic G-decomposition of K_{2n+1} if and only if G admits a ρ-labeling. It is also known that if G is bipartite and it admits an ordered ρ-labeling, then there exists a cyclic G-decomposition of K_{2nx+1} for every positive integer x. We extend these concepts to labelings of multigraphs through what we call λ-fold ρ-labelings and ordered λ-fold ρ-labelings. Let ${}^{\lambda}K_m$ denote the λ-fold complete graph of order m. We show that if a subgraph G of ${}^{\lambda}K_{2n/\lambda+1}$ has size n, there exits a cyclic G-decomposition of ${}^{\lambda}K_{2n/\lambda+1}$ if and only if G admits a λ-fold ρ-labeling. If in addition G is bipartite and it admits an ordered λ-fold ρ-labeling, then there exists a cyclic G-decomposition of ${}^{\lambda}K_{2nx/\lambda+1}$ for every positive integer x. We discuss some classes of graphs and multigraphs that admit such labelings.

Keywords: graph labelings, cyclic graph decompositions, λ-fold labelings, ordered labelings

1 Introduction

If a and b are integers we denote $\{a, a+1, \dots, b\}$ by $[a, b]$ (if $a > b$, then $[a, b] = \varnothing$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_m$ the group of integers modulo m.

For a finite set S, we let ${}^{\lambda}S$ denote the multiset that contains every element of S exactly λ times. For example ${}^{2}[a, b]$ is the multiset $\{a, a, a+1, a+1, \dots, b-1, b-1, b, b\}$. Also, let ${}^{\lambda}K_m$ denote the λ-fold complete multigraph of order m (i.e., ${}^{\lambda}K_m$ is the multigraph on m vertices with exactly λ edges between every pair of vertices). Note that a multigraph is not required to contain multiple edges. Thus a graph is a multigraph. If G and K are multigraphs with $V(G) \subseteq V(K)$ and $E(G) \subseteq E(K)$, then we shall refer to G as a *subgraph* of K. (We do this in order to avoid having to use terms such as "submultigraph".)

Let $V({}^{\lambda}K_m) = [0, m-1]$. The *label* of an edge $\{i, j\}$ in ${}^{\lambda}K_m$ is defined to be $|i-j|$. If ℓ is the label of edge $\{i, j\}$ in ${}^{\lambda}K_m$, then let $\ell^* = \min\{\ell, m-\ell\}$. In this case, ℓ^* is called the *length* of edge $\{i, j\}$. Thus if the elements of $V({}^{\lambda}K_m)$ are placed in order as vertices of an equisided m-gon, then the length of edge $\{i, j\}$ is the shortest distance around the polygon between i and j. If L is a set of edge labels, let $L^* = \{\ell^* : \ell \in L\}$ be the corresponding set of edge lengths. An edge is called *wrap-around* if its length does not equal its label. Note that if m is odd, then ${}^{\lambda}K_m$ consists of λm edges of length i for $i \in [1, (m-1)/2]$. If m is even, then ${}^{\lambda}K_m$ consists of λm edges of length i for $i \in [1, m/2-1]$ and $\lambda m/2$ edges of length $m/2$. Let G be a subgraph of ${}^{\lambda}K_m$. By *clicking* G, we mean applying the permutation $i \mapsto i+1$ to $V(G)$ (addition is done modulo n). Moreover in this case, if $j \in \mathbb{N}$, then $G + j$ is the multigraph obtained from G by successively clicking G a total of j times. Note that clicking an edge does not change its length. Also note that $G + j$ is isomorphic to G for every $j \in \mathbb{N}$.

Let K and G be multigraphs with G a subgraph of K. A *G-decomposition* of K is a collection $\Delta = \{G_1, G_2, \dots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that each edge of K appears in exactly one G_i. The elements of Δ are called *G-blocks*. Such a G-decomposition Δ is *cyclic* (*purely cyclic*) if clicking is an automorphism (t-cycle) of Δ. A G-decomposition of ${}^{\lambda}K_m$ is called a *G-design of order m and index λ* or a *$({}^{\lambda}K_m, G)$-design*. A $({}^{\lambda}K_m, G)$-design is *symmetric* if the number of G-blocks in the design equals m. The study of graph decompositions is generally known as the study of graph designs, or G-designs. For recent surveys on G-designs, see [1] and [4].

Let G be a multigraph of size n. A primary question in the study of G-designs is, *"For what values of m and λ does there exist a $({}^{\lambda}K_m, G)$-designs"* If

G does not contain multiple edges, it is often the case that if $m \equiv 1 \pmod{2n}$, then there exists a (K_m, G)-design. A common approach to finding these designs is through the use of graph labelings. We give a brief review of labelings for (simple) graphs and how they relate to cyclic (K_m, G)-designs.

1.1 Graph Labelings

For a graph G, a one-to-one function $f : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [8], Rosa introduced a hierarchy of labelings. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{f}^*(e) = \min\{\bar{f}(e), 2n + 1 - \bar{f}(e)\}$. Let $\bar{f}(E(G)) = \{\bar{f}(e) : e \in E(G)\}$ and let $\bar{f}^*(E(G)) = \{\bar{f}^*(e) : e \in E(G)\}$. Consider the following conditions:

(ℓ1) $f(V(G)) \subseteq [0, 2n]$,

(ℓ2) $f(V(G)) \subseteq [0, n]$,

(ℓ3) $\bar{f}^*(E(G)) = [1, n]$,

(ℓ4) $\bar{f}(E(G)) = [1, n]$.

If in addition G is bipartite with vertex bipartition $\{A, B\}$, consider also the following conditions:

(ℓ5) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$;

(ℓ6) there exists an integer δ such that $f(a) \leq \delta$ for all $a \in A$ and $f(b) > \delta$ for all $b \in B$.

Then a labeling satisfying the conditions:

- (ℓ1) and (ℓ3) is called a ρ-*labeling*;
- (ℓ1) and (ℓ4) is called a σ-*labeling*;
- (ℓ2) and (ℓ4) is called a β-*labeling*.

A β-labeling is necessarily a σ-labeling, which in turn is a ρ-labeling. Suppose G is bipartite. If a ρ-, σ-, or β-labeling of G satisfies condition (ℓ5), then the labeling is *ordered* and is denoted by ρ^+, σ^+, or β^+, respectively. If in addition (ℓ6) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++}, or β^{++}, respectively.

A β-labeling is better known as a *graceful labeling* and a uniformly-ordered β-labeling is an α-*labeling* as introduced in [8]. Labelings of the types above are called *Rosa-type labelings* because of Rosa's original article [8] on the topic. (See [5] for a survey of Rosa-type labelings.) A dynamic survey on general graph labelings is maintained by Gallian [7].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [8].

Theorem 1.1 *Let G be a graph with n edges. There exists a purely cyclic G-decomposition of K_{2n+1} if and only if G admits a ρ-labeling.*

Theorem 1.2 *Let G be a bipartite graph with n edges. If G admits an α-labeling, then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

From a graph decompositions perspective, Theorem 1.2 offers a great advantage over Theorem 1.1. However, there are many classes of bipartite graphs (see [5]) that do not admit α-labelings. Theorem 1.2 was extended to cover graphs that admit ρ^+-labelings in [6].

Theorem 1.3 *Let G be a bipartite graph with n edges. If G admits an ρ^+-labeling, then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

In this work, we introduce the concept of a λ-fold Rosa-type labeling for a multigraph G and obtain results analogous to those in Theorems 1.1 and 1.3. We also give some classes of multigraphs that admit these λ-fold labelings.

2 Extensions of Rosa-Type Labelings to Multigraphs

A ρ-labeling of a graph G with n edges is essentially an embedding of G in K_{2n+1} that induces one edge in G of length i for each $i \in [1, n]$. Clicking this embedding of G in K_{2n+1} produces a purely cyclic (K_{2n+1}, G)-design. If G is bipartite and the ordered property holds in the embedding, then we can obtain a cyclic (K_{2nx+1}, G)-design for every positive integer x. These concepts extend naturally to embeddings in ${}^{\lambda}K_{2n/\lambda+1}$ of a multigraph G with n edges. In such an embedding, G would need to contain the appropriate number of edges of each length. Clicking G would in turn yield a cyclic $({}^{\lambda}K_{2n/\lambda+1}, G)$-design. Again, if the ordered property holds in the embedding, then a cyclic $({}^{\lambda}K_{2nx/\lambda+1}, G)$-design can be obtained for every positive integer x. While every graph G with n edges and no isolated vertices can be embedded in K_{2n+1}, additional restrictions are needed for embeddings of G in ${}^{\lambda}K_{2n/\lambda+1}$. For example, the order of G cannot exceed $2n/\lambda + 1$.

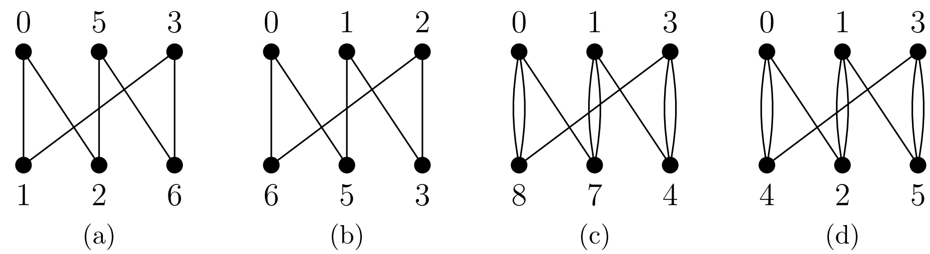


Fig. 1. (a) A 2-fold σ-labeling of C_6, (b) a 2-fold ρ^{++}-labeling of C_6, (c) a 2-fold ρ^{++}-labeling of $\check{C}_6$, and (d) a 3-fold ρ^{+}-labeling of $\check{C}_6$.

2.1 λ-fold Rosa-type Labelings

Let n, k, and λ be positive integers such that n is either λk or $\lambda k + \lambda/2$. Let G be a multigraph of size n, order at most $2n/\lambda + 1$, and edge multiplicity at most λ. Let f be a labeling of G. For $e = \{u, v\} \in E(G)$, let $\bar{f}(e) = |f(u) - f(v)|$ as before, but let $\bar{f}^*(e) = \min\{\bar{f}(e), 2n/\lambda + 1 - \bar{f}(e)\}$. Consider the following conditions:

($\ell'1$) $f(V(G)) \subseteq [0, 2n/\lambda]$,

($\ell'2$) either $n = \lambda k$ and $\bar{f}^*(E(G)) = {}^{\lambda}[1, k]$
or $n = \lambda k + \lambda/2$ and $\bar{f}^*(E(G)) = {}^{\lambda}[1, k] \cup {}^{\lambda/2}\{k + 1\}$,

($\ell'3$) either $n = \lambda k$ and $\bar{f}(E(G)) = {}^{\lambda}[1, k]$
or $n = \lambda k + \lambda/2$ and $\bar{f}(E(G)) = {}^{\lambda}[1, k] \cup {}^{\lambda/2}\{k + 1\}$.

If in addition G is bipartite with vertex bipartition $\{A, B\}$, consider also the following conditions:

($\ell'4$) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$;

($\ell'5$) there exists an integer δ such that $f(a) \leq \delta$ for all $a \in A$ and $f(b) > \delta$ for all $b \in B$.

Then a labeling satisfying the conditions:

($\ell'1$) and ($\ell'2$) is called a *λ-fold ρ-labeling*;

($\ell'1$) and ($\ell'3$) is called a *λ-fold σ-labeling*.

As with Rosa's original labelings, we similarly have that a λ-fold σ-labeling is necessarily a λ-fold ρ-labeling.

If G is bipartite and a λ-fold ρ- or σ-labeling f of G satisfies condition ($\ell'5$), then the labeling is *ordered* and is denoted by ρ^+ or σ^+, respectively. If in addition ($\ell 6$) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++} or σ^{++}, respectively. Figure 1 shows examples of multigraphs with various λ-fold labelings.

In certain cases, the concept of a λ-fold σ-labeling coincides with that of a complete λ-equitable labeling (see [2] and [3]). The two concepts are the same when $\lambda = 2$ and the size of the graph is even.

We will show that a graceful labeling of a graph G with n edges is necessarily a 2-fold ρ-labeling of G. However, there are graphs with size n and order at most $n+1$ that admit a 2-fold ρ^{+}-labeling, but do not admit a graceful labeling (C_6 is one such example).

Lemma 2.1 *If G is a graph and f is a β-labeling of G, then f is necessarily a 2-fold ρ-labeling of G.*

Proof. Let G be a graph with n edges and let f be a β-labeling of G. By the definition of a β-labeling, $\bar{f}(E(G)) = [1, n]$. Now by the definition of a 2-fold ρ-labeling, if $e \in E(G)$, then $\bar{f}^*(e) = \min\{\bar{f}(e), n+1-\bar{f}(e)\}$. Thus, $\bar{f}^*(E(G)) = {}^2[1, n/2]$ if n is even, and $\bar{f}^*(E(G)) = {}^2[1, (n-1)/2] \cup \{(n+1)/2\}$ if n is odd. Therefore, f is a 2-fold ρ-labeling of G. □

Since the 2-fold ρ-labeling of G in the previous result is identical to the (1-fold) β-labeling of G, We necessarily have the following result on ordered labelings.

Corollary 2.2 *If G is a bipartite graph and f is a (uniformly) ordered β-labeling of G, then f is necessarily a (uniformly) ordered 2-fold ρ-labeling of G.*

2.2 Results When $\lambda = 2$

To simplify the proofs of our λ-fold labelings results, we focus first on the case $\lambda = 2$ and show that the following extension of Theorem 1.1 must hold.

Theorem 2.3 *Let G be a subgraph of ${}^2K_{n+1}$ such that $|E(G)| = n$. There exists a purely cyclic $({}^2K_{n+1}, G)$-design if and only if G admits a 2-fold ρ-labeling.*

Proof. Let G be a subgraph of ${}^2K_{n+1}$ such that $|E(G)| = n$. We separate the proof into two cases depending on whether n is even or odd.

Case 1: Suppose n is even.
Let $n = 2m$ and let G admit a 2-fold ρ-labeling. Then for each length $\ell \in [1, m]$, there exist two edges of length ℓ in G. Denote these edges by e'_ℓ and e''_ℓ. Let G' be the subgraph of G with $E(G') = \{e'_\ell : \ell \in [1, m]\}$. Similarly, let G'' be the subgraph with $E(G'') = \{e''_\ell : \ell \in [1, m]\}$. Then G' and G'' are edge

disjoint, and neither of them contains double-edges. Moreover, the 2-fold ρ-labeling of G induces simultaneously a ρ-labeling of G' and a ρ-labeling of G''. Thus $\Delta_{G'} = \{G' + i : i \in [0, 2m]\}$ is a cyclic (K_{2m+1}, G')-design. Similarly, $\Delta_{G''} = \{G'' + i : i \in [0, 2m]\}$ is a cyclic (K_{2m+1}, G'')-design. Since $G = G' \cup G''$ and $2m + 1 = n + 1$, the set $\Delta_G = \{G + i : i \in [0, 2m]\}$ is a cyclic $({}^2K_{n+1}, G')$-design. Conversely, any G-block in a cyclic $({}^2K_{n+1}, G)$-design, induces a 2-fold ρ-labeling of G.

Case 2: Suppose n is odd.
Let $n = 2m + 1$ and let G admit a 2-fold ρ-labeling. As in the previous case, let e'_ℓ and e''_ℓ denote the two edges in G of length ℓ for each $\ell \in [1, m]$. Also, let e_{m+1} be the edge of length $m + 1$. Let G' be the subgraph of G with $E(G') = \{e'_\ell : \ell \in [1, m]\}$, let G'' be the subgraph with $E(G'') = \{e''_\ell : \ell \in [1, m]\}$, and let G_{m+1} be the subgraph with $E(G_{m+1}) = \{e_{m+1}\}$. Let I be the 1-factor in K_{2m+2} induced by the edges of length $m + 1$. Since G' does not contain an edge of length $m + 1$, the set $\Delta_{G'} = \{G' + i : i \in [0, 2m + 1]\}$ is a cyclic $(K_{2m+2} - I, G')$-design. Similarly, the set $\Delta_{G''} = \{G'' + i : i \in [0, 2m + 1]\}$ is a cyclic $(K_{2m+2} - I, G'')$-design. Also, the set $\Delta_{G_{m+1}} = \{G_{m+1} + i : i \in [0, 2m + 1]\}$ is the multigraph obtained by replacing each edge in I with a double edge. Since $G = G' \cup G'' \cup G_{m+1}$ and $2m + 2 = n + 1$, the set $\Delta_G = \{G + i : i \in [0, 2m + 1]\}$ is a cyclic $({}^2K_{n+1}, G')$-design. Conversely, any G-block in a cyclic $({}^2K_{n+1}, G)$-design, induces a 2-fold ρ-labeling of G. □

Similarly, Theorem 1.3 has a corresponding 2-fold counterpart.

Theorem 2.4 *Let G be a bipartite subgraph of ${}^2K_{n+1}$ such that $|E(G)| = n$. If G admits a 2-fold ρ^+-labeling, then a cyclic $({}^2K_{nx+1}, G)$-design exists for every positive integer x.*

Proof. The case $x = 1$ is covered by Theorem 2.3, so we will assume $x \geq 2$. We separate the proof into two cases depending on whether n is even or odd.

Case 1: Suppose n is even.
Let $n = 2m$ and let G admit a 2-fold ρ^+-labeling. Then for each length $\ell \in [1, m]$, there exist two edges of length ℓ in G. Denote these edges by e'_ℓ and e''_ℓ. Let G' be the subgraph of G with $E(G') = \{e'_\ell : \ell \in [1, m]\}$. Similarly, let G'' be the subgraph with $E(G'') = \{e''_\ell : \ell \in [1, m]\}$. Then G' and G'' are edge disjoint, and neither of them contains double-edges. Moreover, the 2-fold ρ^+-labeling of G induces simultaneously a ρ^+-labeling of G' and a ρ^+-labeling of G''. By Theorem 1.3, there exists a cyclic (K_{2mx+1}, G')-design $\Delta_{G'}$ and a cyclic (K_{2mx+1}, G'')-design $\Delta_{G''}$. Moreover, the $2mx + 1$ copies of G' in $\Delta_{G'}$ and the $2mx + 1$ copies of G'' in $\Delta_{G''}$ can be matched so that for each

$i \in \mathbb{Z}_{2mx+1}$ the graph $(G' \cup G'') + i$ is isomorphic to $(G' + i) \cup (G'' + i)$. Thus, $\Delta_G = \{(G' \cup G'') + i : i \in [0, 2mx]\}$ is a cyclic $({}^2K_{nx+1}, G)$-design.

Case 2: Suppose n is odd.
Let $n = 2m + 1$, let f be a 2-fold ρ^+-labeling of G, and let G have vertex bipartition $\{A, B\}$ such that for an edge $e = \{a, b\}$ in $E(G)$, where $a \in A$ and $b \in B$, the label of e is $f(b) - f(a)$. For ease of notation we will denote the label of e as $\bar{f}(e)$.

Let $B_1, B_2, \dots, B_x$ be x vertex-disjoint copies of B. The vertex in B_i that corresponds to $b \in B$ will be denoted b_i. For $i \in [1, x]$, let G_i be a copy of G with vertex bipartition $\{A, B_i\}$. Let $G' = G_1 \cup G_2 \cup \dots \cup G_x$. Thus, G' is a bipartite multigraph with nx edges, edge multiplicity at most 2, and vertex bipartition $\{A, \bigcup_{i=1}^{x} B_i\}$. We use E_e to denote the set of x copies of an edge $e \in E(G)$. Define a labeling $f' \colon V(G') \to [0, nx]$ as follows:

$$f'(v) = \begin{cases} f(a) & \text{if } v = a \in A, \\ f(b) + (i-1)n & \text{if } v = b_i \in B_i. \end{cases}$$

Thus, the labels of the x copies of e are $\bar{f}(e)$, $\bar{f}(e) + n, \dots, \bar{f}(e) + (x-1)n$. These labels are clearly distinct and in the same congruence class modulo n.

First, we show that two edges in G with distinct lengths under f produce disjoint sets of lengths under f'. Let e_1 and e_2 be edges in $E(G)$ such that $\bar{f}^*(e_1) \neq \bar{f}^*(e_2)$. Hence, $\bar{f}(e_1) \neq \bar{f}(e_2)$ and $\bar{f}(e_1) \neq n + 1 - \bar{f}(e_2)$. Since $\bar{f}(e_1), \bar{f}(e_2) \in [1, n]$, we have $\bar{f}(e_1) \not\equiv \bar{f}(e_2)$ and $\bar{f}(e_1) \not\equiv 1 - \bar{f}(e_2)$ modulo n. We now consider

$$\bar{f'}^*(E_{e_1}) = \big\{\min\{\bar{f}(e_1) + (i-1)n, nx + 1 - (\bar{f}(e_1) + (i-1)n)\} : i \in [1, x]\big\}$$

and

$$\bar{f'}^*(E_{e_2}) = \big\{\min\{\bar{f}(e_2) + (i-1)n, nx + 1 - (\bar{f}(e_2) + (i-1)n)\} : i \in [1, x]\big\}.$$

Assume there exists a length $\ell \in \bar{f'}^*(E_{e_1}) \cap \bar{f'}^*(E_{e_2})$. Since $\ell \in (\bar{f}'(E_{e_1}))^*$, ℓ is congruent to $\bar{f}(e_1)$ or $1 - \bar{f}(e_1)$ modulo n. Since $\ell \in (\bar{f}'(E_{e_2}))^*$, ℓ is congruent to $\bar{f}(e_2)$ or $1 - \bar{f}(e_2)$ modulo n. Thus, either $\bar{f}(e_1) \equiv \bar{f}(e_2)$ or $\bar{f}(e_1) \equiv n + 1 - \bar{f}(e_2)$ modulo n; however, this contradicts the assumption that $(\bar{f}(e_1))^* \neq (\bar{f}(e_2))^*$. Therefore, $(\bar{f}'(E_{e_1}))^* \cap (\bar{f}'(E_{e_2}))^* = \varnothing$.

Next, we show that two edges in G with the same length under f produce identical sets of lengths under f'. Let e_1 and e_2 be edges in $E(G)$ such that $(\bar{f}(e_1))^* = (\bar{f}(e_2))^* = \ell$. If e_1 and e_2 have the same label under f, then the sets of edge lengths $(\bar{f}'(E_{e_1}))^*$ and $(\bar{f}'(E_{e_2}))^*$ are clearly identical. We then consider the case where $\bar{f}(e_1) \neq \bar{f}(e_2)$. Hence, $\bar{f}(e_1) = n+1-\bar{f}(e_2)$, and we have

$$\begin{aligned}
(\bar{f}'(E_{e_1}))^* &= \big\{\min\{\bar{f}(e_1)+(i-1)n, nx+1-(\bar{f}(e_1)+(i-1)n)\} : i \in [1,x]\big\} \\
&= \big\{\min\{\bar{f}(e_1)+(x-i)n, nx+1-(\bar{f}(e_1)+(x-i)n)\} : i \in [1,x]\big\} \\
&= \big\{\min\{(n+1-\bar{f}(e_2))+nx-in, \\
&\qquad\qquad nx+1-((n+1-\bar{f}(e_2))+nx-in)\} : i \in [1,x]\big\} \\
&= \big\{\min\{nx+1-(\bar{f}(e_2)+(i-1)n), \bar{f}(e_2)+(i-1)n\} : i \in [1,x]\big\} \\
&= (\bar{f}'(E_{e_2}))^*.
\end{aligned}$$

Second, we show that two of the x copies produced by an edge e in G have the same length in ${}^2K_{nx+1}$ under f' only when the label of e is $(n+1)/2$. Let i and j be distinct integers in $[1,x]$. Suppose the i^{th} copy of e and the j^{th} copy of e have the same length. Since it was shown that they cannot have the same label, $\bar{f}(e)+(i-1)n = nx+1-[\bar{f}(e)+(j-1)n]$. Hence, $\bar{f}(e) \equiv 1-\bar{f}(e) \pmod{n}$ which implies that $2\bar{f}(e) \equiv 1 \pmod{n}$. Thus, $2\bar{f}(e) = nt+1$ for some integer t. Since $\bar{f}(e) \in [1,n]$, we must have $t=1$, and thus $\bar{f}(e) = (n+1)/2$. For the remainder of this proof, let $\hat{e}$ denote the single edge in G such that $\bar{f}(\hat{e}) = (n+1)/2$.

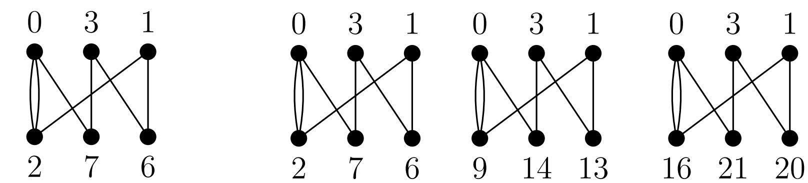
Third, we examine the lengths of the edges in $E_{\hat{e}}$ under f'. It was proven above that the lengths in $(\bar{f}'(E_{\hat{e}}))^*$ are distinct from all others in $(\bar{f}'(E(G')))^*$. To determine the lengths in question, we first compute the following edge labels:

$$\begin{aligned}
\bar{f}'(E_{\hat{e}}) &= \left\{\tfrac{n+1}{2}+(i-1)n : i \in \left[1, \left\lfloor\tfrac{x+1}{2}\right\rfloor\right]\right\} \\
&\quad \cup \left\{\tfrac{n+1}{2}+(i-1)n : i \in \left[\left\lfloor\tfrac{x+1}{2}\right\rfloor+1, x\right]\right\} \\
&= \left\{\tfrac{n+1}{2}+(i-1)n : i \in \left[1, \left\lfloor\tfrac{x+1}{2}\right\rfloor\right]\right\} \\
&\quad \cup \left\{\tfrac{n+1}{2}+(x-i)n : i \in \left[1, x-\left\lfloor\tfrac{x+1}{2}\right\rfloor\right]\right\} \\
&= \left\{\tfrac{n+1}{2}+(i-1)n : i \in \left[1, \left\lfloor\tfrac{x+1}{2}\right\rfloor\right]\right\} \\
&\quad \cup \left\{nx+1-\left(\tfrac{n+1}{2}+(i-1)n\right) : i \in \left[1, x-\left\lfloor\tfrac{x+1}{2}\right\rfloor\right]\right\}. \qquad (1)
\end{aligned}$$

Since n is odd, the longest possible length in ${}^2K_{nx+1}$ is

$$\left\lfloor\frac{nx+1}{2}\right\rfloor = \begin{cases}(nx+1)/2 & \text{if } x \text{ is odd,}\\ nx/2 & \text{if } x \text{ is even.}\end{cases}$$

Clearly, the first set of labels in the union of (1) would yield $\lfloor(x+1)/2\rfloor$ distinct



(a) G with a 2-fold ρ^+-labeling

(b) Three starters for a cyclic $({}^2K_{22}, G)$-design

Fig. 2. A 2-fold ρ^+-labeling of a multigraph G with 7 edges and the 3 starters obtained using Theorem 2.4 for a cyclic G-decomposition of ${}^2K_{22}$.

lengths including the longest length if x is odd, while the second set of labels in the union of (1) would yield $x - \lfloor (x+1)/2 \rfloor$ distinct lengths. Therefore,

$$(\bar{f}'(E_{\hat{e}}))^* = \begin{cases} {}^2\left\{\frac{n+1}{2}, \frac{n+1}{2}+n, \ldots, \frac{nx+1}{2}-n\right\} \cup \left\{\frac{nx+1}{2}\right\} & \text{if } x \text{ is odd,} \\ {}^2\left\{\frac{n+1}{2}, \frac{n+1}{2}+n, \ldots, \frac{nx+1-n}{2}\right\} & \text{if } x \text{ is even.} \end{cases} \quad (2)$$

Finally, we use a counting argument to show that we must have the correct multiplicity of each length in ${}^2K_{nx+1}$. Consider two subgraphs of G: H_1 and H_2 where each has exactly one edge of each length $\ell \in [1, (n-1)/2]$. Similarly, let H_1' and H_2' be the corresponding subgraphs of G' that together contain all edges of G' except those in $E_{\hat{e}}$. Then for $i \in \{1, 2\}$, $(\bar{f}'(E(H_i')))^*$ is composed of $(n-1)/2$ disjoint sets of lengths in $[1, \lfloor (nx+1)/2 \rfloor]$ each containing x distinct lengths and none containing a length that is congruent to $(n+1)/2 \pmod{n}$. Since there are exactly $(nx-x)/2$ such lengths, $(\bar{f}'(E(H_1' \cup H_2')))^*$ must contain exactly 2 edges of each length in $[1, \lfloor (nx+1)/2 \rfloor]$ not congruent to $(n+1)/2 \pmod{n}$. Therefore, $(\bar{f}'(E(G')))^*$ contains the necessary multiplicity of each length in $[1, \lfloor (nx+1)/2 \rfloor]$, and thus f' is a 2-fold ρ-labeling of G'. Hence by Theorem 2.3, there exists a cyclic $({}^2K_{nx+1}, G')$-design. Since there exists a (G', G)-design, the result follows. □

Figure 2 shows a 2-fold ρ^+-labeling of a multigraph G with 7 edges and the 3 starters obtained using Theorem 2.4 for a cyclic G-decomposition of ${}^2K_{22}$.

2.3 Results When $\lambda > 2$

The following is an extension of Theorem 2.3 to λ-fold. The proof is similar to that of Theorem 2.3.

Theorem 2.5 *Let n and λ be positive integers such that $n \equiv 0$ or $\lambda/2 \pmod{\lambda}$. Let G be a subgraph of ${}^\lambda K_{2n/\lambda+1}$ such that $|E(G)| = n$. There* *exists a purely cyclic* $({}^{\lambda}K_{2n/\lambda+1}, G)$*-design if and only if* G *admits a* λ*-fold* ρ*-labeling.*

The following is an extension of Theorem 2.4 to λ-fold.

Theorem 2.6 *Let* n *and* λ *be positive integers such that* $n \equiv 0$ *or* $\lambda/2 \pmod{\lambda}$*. Let* G *be a subgraph of* ${}^{\lambda}K_{2n/\lambda+1}$ *such that* $|E(G)| = n$*. If* G *admits a* λ*-fold* ρ^{+}*-labeling, then a cyclic* $({}^{\lambda}K_{2nx/\lambda+1}, G)$*-design exists for every positive integer* x*.*

Proof. The case $x = 1$ is covered by Theorem 2.5. So we will assume $x \geq 2$. We separate the proof into two cases depending on the value of n.

Case 1: Suppose $n \equiv 0 \pmod{\lambda}$.
Let $n = \lambda k$ and let G admit a λ-fold ρ^{+}-labeling. Then for each length $\ell \in [1, k]$, there exist λ edges of length ℓ in G. Denote these edges by $e_{\ell,1}, e_{\ell,2}, \ldots, e_{\ell,\lambda}$. For $i \in [1, \lambda]$, let G_i be the subgraph of G with $E(G_i) = \{e_{\ell,i} : \ell \in [1, k]\}$. Then $G_1, G_2, \ldots, G_\lambda$ are edge disjoint and none of them contains double-edges. Moreover, the λ-fold ρ^{+}-labeling of G induces a ρ^{+}-labeling of G_i simultaneously for all $i \in [1, \lambda]$. By Theorem 1.3, there exists a cyclic (K_{2kx+1}, G_i)-design Δ_{G_i} for all $i \in [1, \lambda]$. Moreover, the $2kx + 1$ copies of each G_i in their respective Δ_{G_i} can be matched so that for each $j \in \mathbb{Z}_{2kx+1}$ the graph $(\bigcup_{i=1}^{\lambda} G_i) + j$ is isomorphic to $\bigcup_{i=1}^{\lambda}(G_i + j)$. Thus, $\Delta_G = \{(G_1 \cup G_2 \cup \cdots \cup G_\lambda) + j : j \in [0, 2kx]\}$ is a cyclic $({}^{\lambda}K_{2nx/\lambda+1}, G)$-design.

Case 2: Suppose $n \equiv \lambda/2 \pmod{\lambda}$.
Then λ is necessarily even. Let $\lambda = 2m$, let $n = \lambda k + m$ (hence $2n/\lambda = 2k+1$), and let G admit a λ-fold ρ^{+}-labeling. Then for each length $\ell \in [1, k]$, there exist $2m$ edges of length ℓ in G. Furthermore, there exist m edges of length $k + 1$ in G. Partition $E(G)$ into sets $E_1, E_2, \ldots, E_m$ such that each contains exactly 2 edges of each length $\ell \in [1, k]$ and 1 edge of length $k+1$. For $i \in [1, m]$, let G_i be the subgraph of G with $E(G_i) = E_i$. Then $G_1, G_2, \ldots, G_m$ are edge-disjoint and each has edge multiplicity at most 2. Moreover, the λ-fold ρ^{+}-labeling of G induces a 2-fold ρ^{+}-labeling of G_i simultaneously for all $i \in [1, m]$. By Theorem 2.4, there exists a cyclic $(K_{(2k+1)x+1}, G_i)$-design Δ_{G_i} for all $i \in [1, m]$. Moreover, the $(2k + 1)x + 1$ copies of each G_i in their respective Δ_{G_i} can be matched so that for each $j \in \mathbb{Z}_{(2k+1)x+1}$ the graph $(\bigcup_{i=1}^{m} G_i) + j$ is isomorphic to $\bigcup_{i=1}^{m}(G_i + j)$. Thus, $\Delta_G = \{(G_1 \cup G_2 \cup \cdots \cup G_m) + j : j \in [0, (2k + 1)x]\}$ is a cyclic $({}^{\lambda}K_{2nx/\lambda+1}, G)$-design. □

3 Labelings of Some Classes of Graphs and Multigraphs

By Lemma 2.1, a β-labeling of a graph G is necessarily a 2-fold ρ-labeling of G. Moreover, an α-labeling of G is necessarily a 2-fold ρ^{++}-labeling of G. We direct the interested reader to Gallian's dynamic survey on graph labelings [7] for updated results on these topics. Moreover, some of the results on k-labelings (see [2] and [3]) can be stated as k-fold σ-labelings results. Although not all even cycles are graceful, they all admit uniformly-ordered 2-fold ρ-labelings.

Theorem 3.1 *The cycle C_{2m} admits a 2-fold ρ^{++}-labeling for every $m \geq 2$.*

Proof. If m is even, then C_{2m} admits an α-labeling which in turn is a 2-fold ρ^{++}-labeling. So assume m is odd and let $m = 2r+1$ for some nonnegative integer r. Let C_{4r+2} have vertices $v_1, v_2, \ldots, v_{4r+2}$, where v_i is adjacent to both v_{i-1} and v_{i+1} for $i \in [2, 4r+1]$ and v_1 is adjacent to v_2 and to v_{4r+2}. Define a labeling f of G by

$$f(v_j) = \begin{cases} (j-1)/2 & \text{if } j \text{ is odd,} \\ 4r+2-j/2 & \text{if } j/2 \in [1, r], \\ 4r+1-j/2 & \text{if } j/2 \in [r+1, 2r], \\ 4r+2 & \text{if } j = 4r+2. \end{cases}$$

It is easy to verify that f is a 2-fold ρ^{++}-labeling of C_{4r+2}. □

There have also been some investigations of λ-fold labelings of some classes of multigraphs. For example, let $\check{C}_{2n}$ denote the 3-regular multigraph obtained by replacing each edge from a 1-factor in C_{2n} with a pair of parallel edges. It is shown in [9] that $\check{C}_{2n}$ admits a 2-fold ρ^{++}-labeling for all integers $n \geq 2$.

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