

The spectrum for a multigraph on 4 vertices and 7 edges

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Abstract

Let ${}^\lambda K_n$ denote the multigraph obtained by replacing each edge in the complete graph on n vertices, K_n , with λ repeated edges. Let G denote the multigraph obtained by doubling the edges in a path of length 3 in the paw graph (a triangle with a pendant edge). A G -decomposition of ${}^\lambda K_n$ is a partition of the edges of ${}^\lambda K_n$ into subgraphs each of which is isomorphic to G . In this paper we characterize all pairs (λ, n) for which a G -decomposition of ${}^\lambda K_n$ exists.

1 Introduction

For a graph (multigraph) S , we use $V(S)$ and $E(S)$ to denote the vertex set and the edge set (or multiset) of S , respectively. For a simple graph S and a positive integer λ , we use ${}^\lambda S$ to denote the graph obtained from S by replacing each edge in $E(S)$ with λ repeated edges. For edge-disjoint graphs S and T , we use $S \cup T$ to represent the graph with edge set $E(S) \cup E(T)$ and vertex set $V(S) \cup V(T)$. We use $K_{r \times s}$ to denote the complete multipartite graph with r parts of size s , and we use $K_{r \times s, t}$ to denote the complete multipartite graph with r parts of size s and one part of size t . If S is a subgraph of T , we use $T \setminus S$ to denote the graph obtained from T by removing $E(S)$ from $E(T)$. An S -decomposition of T is a set (or multiset) $\{S_1, S_2, \dots, S_k\}$ of subgraphs of T , each of which is isomorphic to S and such that each edge of T appears in exactly one S_i . Similarly, if S and R

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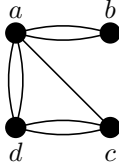


Figure 1: The multigraph G . The given labeling is denoted $G[a, b, c, d]$.

are each subgraphs of T , then a $\{S, R\}$ -decomposition of T is defined to be a set (or multiset) $\{M_1, M_2, \dots, M_t\}$ of subgraphs of T each of which is isomorphic to either S or R and such that each edge of T appears in exactly one M_i .

A central problem in the field of constructive combinatorics consists of finding necessary and sufficient conditions on λ and n for which an S -decomposition of λK_n exists, for a fixed graph S . This is called the *spectrum of S* . In this paper, we characterize the spectrum of the multigraph that is obtained by doubling the edges of a path of length 3 in the paw graph (a triangle with a pendant edge). We denote this multigraph by G (see Figure 1).

The problem pursued in this paper is part of the ongoing efforts to characterize the spectra of all multigraphs of small order. For example, the spectra for all connected cubic multigraphs of order at most 6 was found in [6]. The spectra for various multigraphs on four vertices and six edges were characterized in [5]. Similar projects have been in progress for graphs (see the survey [2]) and directed graphs (for examples see [4] or [3]). Many papers have appeared in support of the efforts for multigraphs, graphs, and digraphs. Here we have only cited a few examples.

Let $G[a, b, c, d]$ denote the graph with vertex set $\{a, b, c, d\}$ and edge set as represented in Figure 1. For example, $G[0, 2, 4, 5]$ denotes the graph with vertex set $\{0, 2, 4, 5\}$ and edge multiset $\{\{0, 5\}, \{0, 5\}, \{0, 4\}, \{5, 4\}, \{5, 4\}, \{0, 2\}, \{0, 2\}\}$.

We use the following results on decompositions of complete graphs and complete multipartite graphs when producing our main constructions. All of these results can be found in the *Handbook of Combinatorial Designs* [7] (see [1], [8], and [9]).

Theorem 1.1. *If n is an odd positive integer, then there exists a $\{K_3, K_5\}$ -decomposition of K_n .*

Theorem 1.2. *The necessary and sufficient conditions for the existence of a K_3 -decomposition of $K_{t \times m}$ are (i) $t \geq 3$, (ii) $(t - 1)m \equiv 0 \pmod{2}$, and (iii) $t(t - 1)m^2 \equiv 0 \pmod{6}$.*

Theorem 1.3. *If $t \geq 3$ and $t \equiv 0 \pmod{3}$, then there exists a K_3 -decomposition of $K_{t \times 2, 4}$.*

Combining the previous two results, we have the following corollary that that is more directly applicable in our general constructions.

Corollary 1.4. *Let $t \geq 3$. If $t \equiv 0, 1 \pmod{3}$, then there exists a K_3 -decomposition of $K_{t \times 2}$. If $t \equiv 2 \pmod{3}$, then there exists a K_3 -decomposition of $K_{(t-2) \times 2, 4}$.*

The following is a well-known result that is a special case of Wilson's Fundamental Construction (see [9]).

Theorem 1.5. *Let m, n, r, s , and t be positive integers. If there exists a K_n -decomposition of $K_{t \times m}$, then there exists a $K_{n \times s}$ -decomposition of $K_{t \times ms}$. Similarly, if there exists a K_n -decomposition of $K_{t \times m, r}$, then there exists a $K_{n \times s}$ -decomposition of $K_{t \times ms, rs}$.*

2 List of Small Decompositions

In this section, we present G -decompositions of various graphs that are needed for the general constructions used in Section 3. Recall that we use the notation $G[a, b, c, d]$ to denote a labeled copy of G (see Figure 1). Given some $i \in \mathbb{Z}_n$, define $G[a, b, c, d] + i = G[a + i, b + i, c + i, d + i]$ where all addition is performed in \mathbb{Z}_n . By convention, define $\infty + 1 = \infty$.

Example 2.1. Let $V({}^2K_7) = \mathbb{Z}_6 \cup \{\infty\}$. A G -decomposition of 2K_7 is given by $\bigcup_{i \in \mathbb{Z}_6} \{G[0, \infty, 3, 1] + i\}$.

Example 2.2. Let $V({}^2K_8) = \mathbb{Z}_8$. A G -decomposition of 2K_8 is given by $\bigcup_{i \in \mathbb{Z}_8} \{G[0, 2, 4, 1] + i\}$.

Example 2.3. Let $V({}^2K_{14}) = \mathbb{Z}_{13} \cup \{\infty\}$. A G -decomposition of ${}^2K_{14}$ is given by $\bigcup_{i \in \mathbb{Z}_{13}} \{G[0, \infty, 3, 1] + i, G[0, 5, 10, 4] + i\}$.

Example 2.4. Let $V({}^2K_{15}) = \mathbb{Z}_{15}$. A G -decomposition of ${}^2K_{15}$ is given by $\bigcup_{i \in \mathbb{Z}_{15}} \{G[3, 8, 10, 9] + i, G[0, 2, 7, 4] + i\}$.

Example 2.5. Let $V({}^2K_{28}) = \mathbb{Z}_{27} \cup \{\infty\}$. A G -decomposition of ${}^2K_{28}$ is given by $\bigcup_{i \in \mathbb{Z}_{27}} \{G[0, \infty, 13, 10] + i, G[0, 2, 13, 8] + i, G[0, 6, 11, 12] + i, G[0, 9, 11, 4] + i\}$.

Example 2.6. Let $V({}^2K_{29}) = \mathbb{Z}_{29}$. A G -decomposition of ${}^2K_{29}$ is given by $\bigcup_{i \in \mathbb{Z}_{29}} \{G[8, 20, 15, 12] + i, G[0, 11, 7, 9] + i, G[2, 3, 10, 16] + i, G[14, 24, 22, 27] + i\}$.

Example 2.7. Let $V({}^2K_{3 \times 7}) = \mathbb{Z}_{21}$. A G -decomposition of ${}^2K_{3 \times 7}$ is given by $\bigcup_{i \in \mathbb{Z}_{21}} \{G[0, 5, 10, 2] + i, G[3, 4, 14, 7] + i\}$.

Example 2.8. Let $V({}^2K_{5 \times 7}) = \mathbb{Z}_{35}$. A G -decomposition of ${}^2K_{5 \times 7}$ is given by $\bigcup_{i \in \mathbb{Z}_{35}} \{G[14, 21, 31, 28] + i, G[0, 29, 17, 16] + i, G[10, 22, 19, 6] + i, G[3, 11, 12, 14] + i\}$.

Example 2.9. Let $V({}^7K_4) = \mathbb{Z}_3 \cup \{\infty\}$. A G -decomposition of 7K_4 is given by $\bigcup_{i \in \mathbb{Z}_3} \{G[0, \infty, 1, 2] + i, G[\infty, 0, 1, 2] + i\}$.

Example 2.10. Let $V({}^7K_5) = \mathbb{Z}_5$. A G -decomposition of 7K_5 is given by $\bigcup_{i \in \mathbb{Z}_5} \{G[2, 3, 1, 0] + i, G[1, 4, 3, 0] + i\}$.

Example 2.11. Let $V({}^7K_6) = \mathbb{Z}_5 \cup \{\infty\}$. A G -decomposition of 7K_6 is given by $\bigcup_{i \in \mathbb{Z}_5} \{G[0, \infty, 2, 1] + i, G[0, 1, \infty, 2] + i, G[0, \infty, 1, 3] + i\}$.

Example 2.12. Let $V({}^7K_7) = \mathbb{Z}_7$. A G -decomposition of 7K_7 is given by $\bigcup_{i \in \mathbb{Z}_7} \{G[1, 2, 3, 0] + i, G[0, 2, 1, 3] + i, G[3, 6, 0, 1] + i\}$.

Example 2.13. Let $V({}^7K_8) = \mathbb{Z}_7 \cup \{\infty\}$. A G -decomposition of 7K_8 is given by $\bigcup_{i \in \mathbb{Z}_7} \{G[0, \infty, 1, 3] + i, G[1, \infty, 3, 0] + i, G[1, 0, \infty, 3] + i, G[6, 3, 2, 0] + i\}$.

Example 2.14. Let $V({}^7K_9) = \mathbb{Z}_9$. A G -decomposition of 7K_9 is given by $\bigcup_{i \in \mathbb{Z}_9} \{G[8, 4, 6, 5] + i, G[5, 3, 4, 1] + i, G[4, 0, 7, 6] + i, G[8, 7, 4, 1] + i\}$.

Example 2.15. Let $V({}^7K_{10}) = \mathbb{Z}_9 \cup \{\infty\}$. A G -decomposition of ${}^7K_{10}$ is given by $\bigcup_{i \in \mathbb{Z}_9} \{G[0, 1, \infty, 2] + i, G[0, 2, 4, 3] + i, G[0, 2, 3, 4] + i, G[0, 3, 1, \infty] + i, G[0, 4, 2, 6] + i\}$.

Example 2.16. Let $V({}^7K_{11}) = \mathbb{Z}_{11}$. A G -decomposition of ${}^7K_{11}$ is given by $\bigcup_{i \in \mathbb{Z}_{11}} \{G[0, 5, 1, 2] + i, G[0, 2, 3, 6] + i, G[0, 1, 2, 4] + i, G[0, 3, 4, 8] + i, G[0, 4, 5, 10] + i\}$.

Example 2.17. Let $V({}^7K_{16}) = \mathbb{Z}_{15} \cup \{\infty\}$. A G -decomposition of ${}^7K_{16}$ is given by $\bigcup_{i \in \mathbb{Z}_{15}} \{G[0, 5, 1, 3] + i, G[0, 1, 2, 7] + i, G[0, 2, 3, 6] + i, G[0, 3, 4, 8] + i, G[0, 7, 5, 11] + i, G[0, \infty, 6, 11] + i, G[0, \infty, 7, 1] + i, G[0, 2, \infty, 1] + i\}$.

Example 2.18. Let $V({}^7K_{17}) = \mathbb{Z}_{17}$. A G -decomposition of ${}^7K_{17}$ is given by $\bigcup_{i \in \mathbb{Z}_{17}} \{G[0, 7, 1, 8] + i, G[0, 8, 2, 6] + i, G[0, 6, 3, 5] + i, G[0, 5, 4, 7] + i, G[0, 4, 5, 1] + i, G[0, 3, 6, 8] + i, G[0, 2, 7, 6] + i, G[0, 1, 8, 3] + i\}$.

Example 2.19. Let $V({}^7K_{18}) = \mathbb{Z}_{17} \cup \{\infty\}$. A G -decomposition of ${}^7K_{18}$ is given by $\bigcup_{i \in \mathbb{Z}_{17}} \{G[0, 4, 1, 2] + i, G[0, 6, 2, 5] + i, G[0, 5, 3, 4] + i, G[0, 7, 4, 6] + i, G[0, 7, 5, 8] + i, G[0, 8, 6, 7] + i, G[0, \infty, 7, 4] + i, G[0, \infty, 8, 6] + i, G[0, 8, \infty, 5] + i\}$.

Example 2.20. Let $V({}^7K_{19}) = \mathbb{Z}_{19}$. A G -decomposition of ${}^7K_{19}$ is given by $\bigcup_{i \in \mathbb{Z}_{19}} \{G[0, 2, 1, 3] + i, G[0, 1, 2, 6] + i, G[0, 4, 3, 6] + i, G[0, 7, 4, 5] + i, G[0, 6, 5, 9] + i, G[0, 5, 6, 8] + i, G[0, 8, 7, 14] + i, G[0, 10, 8, 9] + i, G[0, 8, 9, 12] + i\}$.

Example 2.21. Let $V({}^7K_{3 \times 4}) = \mathbb{Z}_{12}$. A G -decomposition of ${}^7K_{3 \times 4}$ is given by $\bigcup_{i \in \mathbb{Z}_{12}} \{G[0, 4, 1, 5] + i, G[0, 5, 2, 1] + i, G[0, 1, 4, 2] + i, G[5, 1, 0, 10] + i\}$.

Example 2.22. Let $V({}^7K_{5 \times 4}) = \mathbb{Z}_{20}$. A G -decomposition of ${}^7K_{5 \times 4}$ is given by $\bigcup_{i \in \mathbb{Z}_{20}} \{G[0, 19, 1, 2] + i, G[0, 18, 2, 6] + i, G[0, 6, 3, 12] + i, G[0, 8, 4, 7] + i, G[0, 4, 6, 8] + i, G[0, 9, 7, 4] + i, G[0, 7, 8, 9] + i, G[0, 7, 9, 6] + i\}$.

Example 2.23. Let $V({}^7K_6 \setminus {}^7K_2) = \mathbb{Z}_6$. A G -decomposition of ${}^7K_6 \setminus {}^7K_2$ is given by $\{G[1, 5, 0, 2], G[2, 5, 0, 3], G[3, 4, 0, 1], G[4, 5, 0, 1], G[1, 0, 2, 5], G[1, 2, 3, 4], G[4, 0, 1, 2], G[1, 4, 5, 3], G[3, 0, 2, 5], G[2, 3, 4, 0], G[2, 0, 5, 4], G[3, 2, 4, 0], G[5, 4, 3, 1], G[4, 2, 5, 3]\}$.

Example 2.24. Let $V({}^7K_7 \setminus {}^7K_3) = \mathbb{Z}_7$. A G -decomposition of ${}^7K_7 \setminus {}^7K_3$ is given by $\{G[0, 4, 3, 2], G[0, 1, 4, 3], G[1, 4, 2, 6], G[0, 3, 1, 4], G[0, 4, 2, 1], G[1, 5, 6, 4], G[2, 0, 3, 5], G[1, 0, 3, 5], G[1, 3, 4, 2], G[1, 6, 5, 3], G[2, 5, 6, 1], G[4, 5, 3, 6], G[3, 6, 5, 1], G[2, 0, 4, 6], G[2, 3, 5, 4], G[3, 0, 6, 2], G[4, 3, 5, 2], G[4, 5, 6, 3]\}$.

Example 2.25. Let $V({}^7K_{10} \setminus {}^7K_2) = \mathbb{Z}_{10}$. A G -decomposition of ${}^7K_{10} \setminus {}^7K_2$ is given by $\{G[1, 6, 0, 2], G[2, 1, 0, 3], G[3, 1, 0, 4], G[4, 1, 0, 5], G[5, 9, 0, 6], G[0, 1, 6, 7], G[0, 1, 7, 8], G[0, 2, 8, 1], G[2, 8, 1, 4], G[3, 8, 1, 5], G[4, 2, 1, 6], G[5, 2, 1, 7], G[6, 4, 1, 9], G[1, 8, 7, 3], G[1, 9, 8, 6], G[1, 7, 9, 4], G[3, 0, 2, 6], G[4, 3, 2, 7], G[5, 3, 2, 8], G[6, 0, 2, 5], G[2, 6, 7, 9], G[2, 7, 8, 0], G[2, 7, 9, 1], G[4, 0, 3, 8], G[5, 1, 3, 9], G[6, 7, 3, 2], G[7, 4, 3, 1], G[3, 9, 5, 8], G[3, 0, 9, 2], G[5, 9, 4, 2], G[6, 5, 4, 3], G[9, 7, 5, 4], G[7, 5, 6, 8], G[8, 7, 6, 3], G[9, 8, 6, 7], G[8, 5, 7, 4], G[9, 8, 7, 3], G[9, 6, 8, 2], G[7, 0, 4, 5], G[8, 0, 4, 9], G[4, 0, 9, 6], G[6, 8, 5, 0], G[7, 3, 5, 0], G[8, 4, 5, 1]\}$.

Example 2.26. Let $V({}^7K_{11} \setminus {}^7K_3) = \mathbb{Z}_{11}$. A G -decomposition of ${}^7K_{11} \setminus {}^7K_3$ is given by $\{G[3, 6, 4, 5], G[4, 6, 5, 3], G[5, 6, 3, 4], G[6, 3, 4, 5], G[6, 4, 5, 3], G[7, 10, 8, 9], G[6, 5, 3, 4], G[7, 4, 0, 3], G[9, 10, 7, 8], G[10, 7, 8, 9], G[10, 8, 9, 7], G[10, 9, 7, 8], G[3, 8, 0, 7], G[4, 9, 0, 8], G[5, 10, 0, 9], G[6, 7, 0, 10], G[3, 8, 7, 0], G[4, 9, 8, 0], G[5, 10, 9, 0], G[6, 7, 10, 0], G[8, 10, 9, 7], G[8, 5, 0, 4], G[9, 6, 0, 5], G[10, 3, 0, 6], G[7, 4, 0, 3], G[8, 5, 0, 4], G[9, 6, 0, 5], G[10, 3, 0, 6], G[3, 7, 8, 0], G[4, 8, 9, 0], G[5, 9, 10, 0], G[6, 10, 7, 0], G[3, 2, 1, 9], G[4, 2, 1, 10], G[5, 2, 1, 7], G[6, 2, 1, 8], G[9, 2, 1, 3], G[10, 2, 1, 4], G[7, 2, 1, 5], G[8, 2, 1, 6], G[3, 1, 2, 9], G[4, 1, 2, 10], G[5, 1, 2, 7], G[6, 1, 2, 8], G[1, 8, 4, 7], G[1, 9, 5, 8], G[1, 10, 6, 9], G[1, 7, 3, 10], G[4, 10, 7, 2], G[5, 7, 8, 2], G[6, 8, 9, 2], G[3, 9, 10, 2], G[7, 6, 2, 4], G[8, 3, 2, 5], G[9, 4, 2, 6], G[10, 5, 2, 3]\}$.

3 Main Results

First, we establish the necessary conditions for the existence of a G -decomposition of ${}^\lambda K_n$.

Lemma 3.1. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. If there exists a G -decomposition of ${}^\lambda K_n$, then the following hold:*

1. If $\gcd(7, \lambda) = 1$, then $n \equiv 0, 1 \pmod{7}$.
2. If $\gcd(7, \lambda) = 7$, then $n \geq 4$.

Proof. Suppose there is a G -decomposition of ${}^\lambda K_n$. Since $|V(G)| = 4$, we have that $n \geq 4$. We have that $|E(G)| = 7$ and $|E({}^\lambda K_n)| = \frac{\lambda n(n-1)}{2}$, so 7 divides $\frac{\lambda n(n-1)}{2}$. If $\gcd(\lambda, 7) = 1$, then $n \equiv 0, 1 \pmod{7}$. If the $\gcd(\lambda, 7) = 7$, then this divisibility condition is satisfied for all $n \geq 4$. ■

There are two values of λ for which the necessary conditions fail to be sufficient. This is exhibited in the next lemma.

Lemma 3.2. *There does not exist a G -decomposition of ${}^3 K_n$ or ${}^5 K_n$.*

Proof. Assume that a G -decomposition of ${}^3 K_n$ exists. There are $\frac{3n(n-1)}{14}$ copies of G in this decomposition. Every copy of G has exactly 1 single edge, i.e. an edge of the form $\{a, c\}$ in our standard labeling. In order to use the 3 edges between every pair of vertices in ${}^3 K_n$, we must have at least $\binom{n}{2}$ copies of G in the decomposition. This gives us $\binom{n}{2} \leq \frac{3n(n-1)}{14}$, which is a contradiction. Thus, there cannot exist a G -decomposition of ${}^3 K_n$.

A similar counting argument applies to ${}^5 K_n$. The argument would result in the inequality $\binom{n}{2} \leq \frac{5n(n-1)}{14}$, which is a contradiction. ■

Lemma 3.3. *If $n \geq 7$ is an integer with $n \equiv 0, 1 \pmod{7}$, then there exists a G -decomposition of ${}^2 K_n$.*

Proof. First consider the case where $n \equiv 0 \pmod{7}$, say $n = 7x$ for some positive integer x . For $x \in \{1, 2\}$ such a G -decomposition exists by Examples 2.1 and 2.3, respectively. So we can assume that $x \geq 3$.

When x is odd we have a $\{K_3, K_5\}$ -decomposition of K_x by Theorem 1.1. Thus, by Theorem 1.5, there exists a $\{{}^2 K_{3 \times 7}, {}^2 K_{5 \times 7}\}$ -decomposition of ${}^2 K_{x \times 7}$. Examples 2.7 and 2.8 exhibit G -decompositions of ${}^2 K_{3 \times 7}$ and ${}^2 K_{5 \times 7}$, respectively; therefore, G divides ${}^2 K_{x \times 7}$. Notice that ${}^2 K_{7x} = x({}^2 K_7) \cup {}^2 K_{x \times 7}$. Since there exist G -decompositions of ${}^2 K_7$ and ${}^2 K_{x \times 7}$, there exists a G -decomposition of ${}^2 K_{7x}$.

When x is even let $x = 2y$ for some integer $y \geq 4$. Consider the case where $y \equiv 0, 1 \pmod{3}$. By Corollary 1.4, there exists a K_3 -decomposition

of $K_{y \times 2}$. Therefore, by Theorem 1.5, there exists a ${}^2K_{3 \times 7}$ -decomposition of ${}^2K_{y \times 14}$. As in the previous case, the existence of a G -decomposition of ${}^2K_{3 \times 7}$, as exhibited in Example 2.7, implies the existence of a G -decomposition of ${}^2K_{y \times 14}$. We know that ${}^2K_{7x} \cong {}^2K_{14y} \cong y({}^2K_{14}) \cup {}^2K_{y \times 14}$. Since there exist G -decompositions of ${}^2K_{14}$ and ${}^2K_{y \times 14}$, there exists a G -decomposition of ${}^2K_{7x}$.

Next, we consider the case where $y \equiv 2 \pmod{3}$. By Corollary 1.4 there exists a K_3 -decomposition of $K_{(y-2) \times 2, 4}$. Therefore, by Theorem 1.5, there exists a ${}^2K_{3 \times 7}$ -decomposition of ${}^2K_{(y-2) \times 14, 28}$. As in the previous two cases we see that there exists a G -decomposition of ${}^2K_{(y-2) \times 14, 28}$. We know that ${}^2K_{14y} \cong (y-2)({}^2K_{14}) \cup {}^2K_{(y-2) \times 14, 28}$. Since there are G decompositions of ${}^2K_{(y-2) \times 14, 28}$ and ${}^2K_{28}$, by Example 2.5, there exists a G -decomposition of ${}^2K_{7x}$.

Finally, consider the case where $n \equiv 1 \pmod{7}$, say $n = 7x + 1$ for some positive integer $x \geq 3$. We apply the same constructions as in the case where $n \equiv 0 \pmod{7}$ with the following minor adjustments.

When x is odd we notice that ${}^2K_{7x+1} \cong x({}^2K_8) \cup {}^2K_{x \times 7}$. A G -decomposition of 2K_8 exists by Example 2.2; therefore, a G -decomposition of ${}^2K_{7x+1}$ exists.

When $x = 2y$ for some $y \equiv 0, 1 \pmod{3}$ with $y \geq 4$, notice that ${}^2K_{7x+1} \cong {}^2K_{14y+1} \cong y({}^2K_{15}) \cup {}^2K_{y \times 14}$. A G -decomposition of ${}^2K_{15}$ exists by Example 2.4; therefore, a G -decomposition of ${}^2K_{7x+1}$ exists.

When $x = 2y$ for some $y \equiv 2 \pmod{3}$ with $y \geq 4$, notice that ${}^2K_{7x+1} \cong {}^2K_{14y+1} \cong (y-2)({}^2K_{15}) \cup {}^2K_{29} \cup {}^2K_{(y-2) \times 14, 28}$. A G -decomposition of ${}^2K_{29}$ exists by Example 2.6; therefore, a G -decomposition of ${}^2K_{7x+1}$ exists. ■

After making the straight-forward observation that for any positive integer x we have that ${}^{2x}K_n \cong x({}^2K_n)$, the following corollary becomes an obvious consequence of Lemma 3.3.

Corollary 3.4. *If $n \equiv 0, 1 \pmod{7}$ and x is a positive integer, then there exists a G -decomposition of ${}^{2x}K_n$.*

Lemma 3.5. *If n is an integer with $n \geq 4$, then there exists a G -decomposition of 7K_n .*

Proof. We break the proof into four cases.

CASE 1: $n \equiv 0 \pmod{4}$, say $n = 4x$ for some positive integer x .

When $x \in \{1, 2\}$, the results follow from Examples 2.9 and 2.13, respectively. Assume that $x \geq 3$.

Suppose x is odd. The construction is similar to that of the same case in the proof of Lemma 3.3. We know that ${}^7K_{4x} \cong x({}^7K_4) \cup {}^7K_{x \times 4}$. By applying Theorems 1.1 and 1.5 to K_x we obtain the existence of a $\{{}^7K_{3 \times 4}, {}^7K_{5 \times 4}\}$ -decomposition of ${}^7K_{x \times 4}$. There exist G -decompositions of ${}^7K_{3 \times 4}$, ${}^7K_{5 \times 4}$,

and 7K_4 by Examples 2.21, 2.22, and 2.9, respectively. Therefore, a G -decomposition of ${}^7K_{x \times 4}$ exists. So we obtain the desired G -decomposition of ${}^7K_{4x}$.

Suppose x is even, say $x = 2y$ for some positive integer y . When $y \in \{1, 2\}$ the results follow from Examples 2.13 and 2.17, respectively. Assume $y \geq 3$. When $y \equiv 0, 1 \pmod{3}$ note that ${}^7K_{4x} \cong {}^7K_{8y} \cong y({}^7K_8) \cup {}^7K_{y \times 8}$. By applying Theorems 1.3 and 1.5 to $K_{y \times 2}$ we obtain the existence of a ${}^7K_{3 \times 4}$ -decomposition of ${}^7K_{y \times 8}$. There exist G -decompositions of ${}^7K_{3 \times 4}$ and 7K_8 by Examples 2.21 and 2.13, respectively. Therefore, a G -decomposition of ${}^7K_{y \times 8}$ exists, and we obtain the desired G -decomposition of ${}^7K_{4x}$.

When $y \equiv 2 \pmod{3}$ note that ${}^7K_{4x} \cong {}^7K_{8y} \cong (y-2){}^7K_8 \cup {}^7K_{16} \cup {}^7K_{(y-2) \times 8, 16}$. By applying Corollary 1.4, and Theorem 1.5 to ${}^7K_{(y-2) \times 2, 4}$ we obtain the existence of a ${}^7K_{3 \times 4}$ -decomposition of ${}^7K_{(y-2) \times 8, 16}$; hence a G -decomposition of ${}^7K_{(y-2) \times 8, 16}$ exists. There exists a G -decomposition of ${}^7K_{16}$ by Example 2.17. Therefore, we obtain the desired G -decomposition of ${}^7K_{4x}$.

CASE 2: $n \equiv 1 \pmod{4}$, say $n = 4x + 1$ for some $x \geq 1$.

The constructions for this case are similar to the previous case. The ingredients needed are G -decompositions of 7K_5 , 7K_9 , and ${}^7K_{17}$ which are exhibited in Examples 2.10, 2.14, and 2.18, respectively.

CASE 3: $n \equiv 2 \pmod{4}$, say $n = 4x + 2$ for some $x \geq 1$.

The constructions for this case are similar to the previous two cases, but in addition we use G -decompositions of complete multigraphs with holes.

Suppose x be odd. Note that ${}^7K_{4x+2} \cong (x-1)({}^7K_6 \setminus {}^7K_2) \cup {}^7K_6 \cup {}^7K_{x \times 4}$. The required G -decompositions of 7K_6 and ${}^7K_6 \setminus {}^7K_2$ are exhibited in Examples 2.11 and 2.23, respectively. Therefore, we obtain the desired G -decomposition of ${}^7K_{4x+2}$.

Suppose x is even, say $x = 2y$ for some positive integer y . If $y \equiv 0, 1 \pmod{3}$, then note that ${}^7K_{4x+2} \cong {}^7K_{8y+2} \cong (y-1)({}^7K_{10} \setminus {}^7K_2) \cup {}^7K_{10} \cup {}^7K_{y \times 8}$. The required G -decompositions of ${}^7K_{10}$ and ${}^7K_{10} \setminus {}^7K_2$ are exhibited in Examples 2.15 and 2.25, respectively. Therefore, we obtain the desired G -decomposition of ${}^7K_{4x+2}$.

If $y \equiv 2 \pmod{3}$, then note that ${}^7K_{4x+2} \cong {}^7K_{8y+2} \cong (y-2)({}^7K_{10} \setminus {}^7K_2) \cup {}^7K_{18} \cup {}^7K_{(y-2) \times 8, 16}$. The required G -decomposition of ${}^7K_{18}$ is exhibited in Example 2.19. Therefore, we obtain the desired G -decomposition of ${}^7K_{4x+2}$.

CASE 4: $n \equiv 3 \pmod{4}$, say $n = 4x + 3$ for some $x \geq 1$.

The constructions for this case are similar to the previous case.

Suppose x be odd. Note that ${}^7K_{4x+3} \cong (x-1)({}^7K_7 \setminus {}^7K_3) \cup {}^7K_7 \cup {}^7K_{x \times 4}$. The required G -decompositions of 7K_7 and ${}^7K_7 \setminus {}^7K_3$ are exhibited in Examples 2.12 and 2.24, respectively. Therefore, we obtain the desired G -decomposition of ${}^7K_{4x+3}$.

Suppose x is even, say $x = 2y$ for some positive integer y . If $y \equiv 0, 1$

(mod 3), then note that ${}^7K_{8y+3} \cong (y-1)({}^7K_{11} \setminus {}^7K_3) \cup {}^7K_{11} \cup {}^7K_{y \times 8}$. The required G -decompositions of ${}^7K_{11}$ and ${}^7K_{11} \setminus {}^7K_3$ are exhibited in Examples 2.16 and 2.26, respectively. Therefore, we obtain the desired G -decomposition of ${}^7K_{4x+3}$.

If $y \equiv 2 \pmod{3}$, then note that ${}^7K_{8y+3} \cong (y-2)({}^7K_{11} \setminus {}^7K_3) \cup {}^7K_{19} \cup {}^7K_{(y-2) \times 8,16}$. The required G -decomposition of ${}^7K_{19}$ is exhibited in Example 2.20. Therefore, we obtain the desired G -decomposition of ${}^7K_{4x+3}$. ■

Corollary 3.6. *If $n \geq 4$ and x is a positive integer, then exists a G -decomposition of 7K_n .*

Combining Corollaries 3.4 and 3.6 and Lemma 3.2 we obtain our main result that characterizes the spectrum of G .

Theorem 3.7. *Let $n \geq 4$ and $\lambda \geq 2$ be integers. A G -decomposition of ${}^\lambda K_n$ exists if and only if $\gcd(\lambda, 7) = 7$, or $\gcd(\lambda, 7) = 1$ and $n \equiv 0, 1 \pmod{7}$, with the exceptions that G -decompositions of 3K_n and 5K_n do not exist.*

Proof. Let n and λ be positive integers with $n \geq 4$. When $\lambda \equiv 0 \pmod{7}$ the result follows directly from Corollary 3.6.

Now suppose that $n \equiv 0, 1 \pmod{7}$. When λ is even, the result follows directly from Corollary 3.4. When $\lambda \in \{3, 5, 7\}$ the results follow from Lemma 3.2 and the previous case. When $\lambda \geq 9$ is odd we have that $\lambda = 7s + 2t$ for some integers $s, t \geq 1$. Note that ${}^\lambda K_n \cong s{}^7K_n \cup t{}^2K_n$. By Lemmas 3.5 and 3.3 we know that G -decompositions of 7K_n and 2K_n exist. Thus, the desired G -decomposition of ${}^\lambda K_n$ exists. ■

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