The λ-fold spectrum problem for a multigraph on four vertices and eight edges

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Abstract

Let G be the multigraph obtained by replacing three edges of $K_4 - e$ with double-edges such that there remains a vertex of degree 2. We find necessary and sufficient conditions on n and λ for the existence of a G-decomposition of ${}^{\lambda}K_n$.

1 Introduction

Throughout this paper, we may refer to a multigraph as a graph; however, our graphs contain no loops. If we wish to emphasize that a given graph does not contain parallel edges, then we refer to it as a simple graph. For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set (or multiset) of G, respectively. For a simple graph G and a positive integer λ, we use ${}^{\lambda}G$ to denote the graph obtained from G by replacing each edge in $E(G)$ with λ parallel edges. Alternatively, we let $\lambda \cdot G$ denote the graph consisting of λ vertex-disjoint copies of G. For edge-disjoint graphs G and H, we use $G \cup H$ to represent the graph with edge set $E(G) \cup E(H)$ and vertex set $V(G) \cup V(H)$. We define the *join* of two vertex-disjoint graphs G and H, denoted $G \vee H$, as the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{a, b\} \colon a \in V(G), b \in V(H)\}$. We use $K_{r\times s}$ to denote the complete simple multipartite graph with r parts of size s, and we use $K_{t,\, r\times s}$ to denote the complete simple multipartite graph with one

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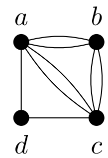

Figure 1: The multigraph $\mathrm{G}_1[a,b,c,d]$ under consideration.

part of size t and r parts of size s. If G is a subgraph of H, we use $H \setminus G$ to denote the graph obtained from H by removing $E(G)$ from $E(H)$.

Let K and G be graphs with G a subgraph of K. A *G-decomposition* of K is a set (or multiset) $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that each edge of K appears in exactly one G_i. Similarly, if G and H are each subgraphs or K, then a $\{G, H\}$-decomposition of K is defined to be a set $\{H_1, H_2, \ldots, H_t\}$ of subgraphs of K each of which is isomorphic to either G or H and such that each edge of K appears in exactly one H_i. If there exists a G-decomposition of K, then we say G *divides* K and write $G \mid K$. A G-decomposition of K is also known as a *(K, G)-design*.

Let G be a graph. A classical problem in the study of graph designs is to find necessary and sufficient conditions for the existence of a $\left({}^{\lambda}K_n, G\right)$-design, also known as a G-design of index λ. This is known as the *spectrum problem* for G. The set of all such n is called the *spectrum for G-designs of index* λ. The spectrum for G-designs of index 1 has been determined for several classes of graphs including cycles, paths, stars, and simple graphs of order at most 5 (see [1]).

In recent years, there have been some investigations of G-designs of index λ where G is a multigraph with edge multiplicity at least 2. For example, in [3] Carter determined the spectra for G-designs of index λ for all connected cubic multigraphs G of order at most 6. The spectra for G-designs of index λ have been investigated for various multigraphs G of small order (see for example [5], [2], and [8]). In this paper we consider a multigraph with 8 edges obtained by replacing a K_3-subgraph of $K_4 - e$ with a ${}^2K_3$ (see Figure 1). In this manuscript we use G_1 to denote this multigraph and settle the spectrum problem for G_1.

We further adopt the convention that $G_1[a,b,c,d]$ denotes the graph with vertex set $\{a,b,c,d\}$ and edge set as represented in Figure 1. For example, $G_1[0,1,2,3]$ denotes the graph with vertex set $\{0,1,2,3\}$ and edge multiset $\big\{\{0,1\},\ \{0,1\},\ \{0,2\},\ \{0,2\},\ \{0,3\},\ \{1,2\},\ \{1,2\},\ \{2,3\}\big\}$.

The following theorems on decompositions of (simple) complete multipartite graphs are used extensively in proving our main results. All of these results can be found in the *Handbook of Combinatorial Designs* [4] ([6] and

[7]).

Theorem 1.1. *The necessary and sufficient conditions for the existence of a K_3-decomposition of $K_{t\times m}$ are (i) $t \geq 3$, (ii) $(t-1)m \equiv 0 \pmod 2$, and (iii) $t(t-1)m^2 \equiv 0 \pmod 6$.*

Theorem 1.2. *If $t \geq 3$ and $t \equiv 0 \pmod 3$, then there exists a K_3-decomposition of $K_{4,\, t\times 2}$.*

Combining the previous two results, we have the following corollary that is more directly applicable in our general constructions.

Corollary 1.3. *Let $t \geq 3$. There exists a K_3-decomposition of $K_{t\times 2}$ if $t \equiv 0$ or $1 \pmod 3$ and of $K_{4,\,(t-2)\times 2}$ if $t \equiv 2 \pmod 3$.*

The following is a well-known result that is a special case of Wilson's Fundamental Construction (see [7]).

Theorem 1.4. *Let m, n, r, s, and t be positive integers. If there exists a $(K_{t\times m},\, K_n)$-design, then there exists a $(K_{t\times ms},\, K_{n\times s})$-design. Similarly, if there exists a $(K_{r,\, t\times m},\, K_n)$-design, then there exists a $(K_{rs,\, t\times ms},\, K_{n\times s})$-design.*

2 Some Small Examples

In this section, we present G_1-decompositions of various graphs that are needed for the constructions used in Section 3. Given some $i \in \mathbb{Z}_n$, we define $\mathrm{G}_1[a,b,c,d]+i = \mathrm{G}_1[a+i, b+i, c+i, d+i]$ where all addition is performed in $\mathbb{Z}_n$. By convention, define $\infty + 1 = \infty$.

2.1 Small designs of index 2

Example 2.1. Let $V({}^2K_8) = \mathbb{Z}_7 \cup \{\infty\}$. Then $\{\mathrm{G}_1[0,1,3,\infty]+i : i \in \mathbb{Z}_7\}$ is a G_1-decomposition of ${}^2K_8$.

Example 2.2. Let $V({}^2K_9) = \mathbb{Z}_9$. Then $\{\mathrm{G}_1[0,1,4,2]+i : i \in \mathbb{Z}_9\}$ is a G_1-decomposition of ${}^2K_9$.

Example 2.3. Let $V({}^2K_{16}) = \mathbb{Z}_{15} \cup \{\infty\}$. Then $\{\mathrm{G}_1[0,6,1,\infty]+i : i \in \mathbb{Z}_{15}\} \cup \{\mathrm{G}_1[0,7,4,2]+i : i \in \mathbb{Z}_{15}\}$ is a G_1-decomposition of ${}^2K_{16}$.

Example 2.4. Let $V({}^2K_{17}) = \mathbb{Z}_{17}$. Then $\{\mathrm{G}_1[0,7,5,11]+i : i \in \mathbb{Z}_{17}\} \cup \{\mathrm{G}_1[0,4,1,9]+i : i \in \mathbb{Z}_{17}\}$ is a G_1-decomposition of ${}^2K_{17}$.

Example 2.5. Let $V({}^2K_{3\times 4}) = \mathbb{Z}_{12}$ with partition $\{V_i : i \in \mathbb{Z}_3\}$, where $V_i = \{j \in \mathbb{Z}_{12} : j \equiv i \pmod 3\}$. Then $\{\mathrm{G}_1[0,5,4,2]+i : i \in \mathbb{Z}_{12}\}$ is a G_1-decomposition of ${}^2K_{3\times 4}$.

2.2 Small designs of index 4

Example 2.6. Let $V\left({}^4K_4\right) = \mathbb{Z}_3 \cup \{\infty\}$. Then $\{\mathrm{G}_1[0, \infty, 1, 2] + i : i \in \mathbb{Z}_3\}$ is a G_1-decomposition of ${}^4K_4$.

Example 2.7. Let $V\left({}^4K_5\right) = \mathbb{Z}_5$. Then $\{\mathrm{G}_1[0, 2, 3, 4] + i : i \in \mathbb{Z}_5\}$ is a G_1-decomposition of ${}^4K_5$.

Example 2.8. Let $V\left({}^4K_{3\times 2}\right) = \mathbb{Z}_6$ with partition $\big\{\{0, 3\}, \{1, 4\}, \{2, 5\}\big\}$. Then $\{\mathrm{G}_1[0, 1, 2, 4] + i : i \in \mathbb{Z}_6\}$ is a G_1-decomposition of ${}^4K_{3\times 2}$.

2.3 Small designs of index 5

Example 2.9. Let $V\left({}^5K_{16}\right) = \mathbb{Z}_{15} \cup \{\infty\}$. Then $\{\mathrm{G}_1[\infty, 7, 0, 8] + i : i \in \mathbb{Z}_{15}\} \cup \{\mathrm{G}_1[0, 5, 3, 1] + i : i \in \mathbb{Z}_{15}\} \cup \{\mathrm{G}_1[0, 12, 1, 4] + i : i \in \mathbb{Z}_{15}\} \cup \{\mathrm{G}_1[0, 2, 11, 5] + i : i \in \mathbb{Z}_{15}\} \cup \{\mathrm{G}_1[0, 6, 1, 8] + i : i \in \mathbb{Z}_{15}\}$ is a G_1-decomposition of ${}^5K_{16}$.

Example 2.10. Let $V\left({}^5K_{17}\right) = \mathbb{Z}_{17}$. Then $\{\mathrm{G}_1[0, 11, 3, 1] + i : i \in \mathbb{Z}_{17}\} \cup \{\mathrm{G}_1[0, 8, 13, 16]+i : i \in \mathbb{Z}_{17}\} \cup \{\mathrm{G}_1[0, 6, 5, 1]+i : i \in \mathbb{Z}_{17}\} \cup \{\mathrm{G}_1[0, 4, 2, 7]+i : i \in \mathbb{Z}_{17}\} \cup \{\mathrm{G}_1[0, 7, 14, 6] + i : i \in \mathbb{Z}_{17}\}$ is a G_1-decomposition of ${}^5K_{17}$.

Example 2.11. Let $V\left({}^5K_{32}\right) = \mathbb{Z}_{31} \cup \{\infty\}$. Then $\{\mathrm{G}_1[\infty, 8, 0, 1] + i : i \in \mathbb{Z}_{31}\} \cup \{\mathrm{G}_1[0, 15, 1, 8] + i : i \in \mathbb{Z}_{31}\} \cup \{\mathrm{G}_1[0, 1, 14, 9] + i : i \in \mathbb{Z}_{31}\} \cup \{\mathrm{G}_1[0, 2, 15, 13]+i : i \in \mathbb{Z}_{31}\} \cup \{\mathrm{G}_1[0, 2, 12, 6]+i : i \in \mathbb{Z}_{31}\} \cup \{\mathrm{G}_1[0, 3, 9, 12]+ i : i \in \mathbb{Z}_{31}\} \cup \{\mathrm{G}_1[0, 12, 3, 11] + i : i \in \mathbb{Z}_{31}\} \cup \{\mathrm{G}_1[0, 5, 10, 14] + i : i \in \mathbb{Z}_{31}\} \cup \{\mathrm{G}_1[0, 11, 4, 10] + i : i \in \mathbb{Z}_{31}\} \cup \{\mathrm{G}_1[0, 11, 7, 15] + i : i \in \mathbb{Z}_{31}\}$ is a G_1-decomposition of ${}^5K_{32}$.

Example 2.12. Let $V({}^5K_{33}) = \mathbb{Z}_{33}$. Then $\{\mathrm{G}_1[0, 8, 1, 6] + i : i \in \mathbb{Z}_{33}\} \cup \{\mathrm{G}_1[0, 10, 4, 15]+i : i \in \mathbb{Z}_{33}\} \cup \{\mathrm{G}_1[0, 16, 2, 13]+i : i \in \mathbb{Z}_{33}\} \cup \{\mathrm{G}_1[0, 18, 5, 16]+ i : i \in \mathbb{Z}_{33}\} \cup \{\mathrm{G}_1[0, 12, 3, 14] + i : i \in \mathbb{Z}_{33}\} \cup \{\mathrm{G}_1[0, 1, 26, 25] + i : i \in \mathbb{Z}_{33}\} \cup \{\mathrm{G}_1[0, 29, 6, 9] + i : i \in \mathbb{Z}_{33}\} \cup \{\mathrm{G}_1[0, 2, 16, 4] + i : i \in \mathbb{Z}_{33}\} \cup \{\mathrm{G}_1[0, 5, 18, 11] + i : i \in \mathbb{Z}_{33}\} \cup \{\mathrm{G}_1[0, 3, 12, 10] + i : i \in \mathbb{Z}_{33}\}$ is a G_1-decomposition of ${}^5K_{33}$.

Example 2.13. Let $V\left({}^5K_{3\times 8}\right) = \mathbb{Z}_{24}$ with partition $\{V_i : i \in \mathbb{Z}_3\}$, where $V_i = \{j \in \mathbb{Z}_{24} : j \equiv i \pmod 3\}$. Then $\{\mathrm{G}_1[0, 1, 11, 13] + i : i \in \mathbb{Z}_{24}\} \cup \{\mathrm{G}_1[0, 11, 1, 5]+i : i \in \mathbb{Z}_{24}\} \cup \{\mathrm{G}_1[0, 2, 7, 8]+i : i \in \mathbb{Z}_{24}\} \cup \{\mathrm{G}_1[0, 2, 7, 14]+i : i \in \mathbb{Z}_{24}\} \cup \{\mathrm{G}_1[0, 4, 8, 16] + i : i \in \mathbb{Z}_{24}\}$ is a G_1-decomposition of ${}^5K_{3\times 8}$.

2.4 Small designs of index 8

Example 2.14. Let $V\left({}^8K_6\right) = \mathbb{Z}_5 \cup \{\infty\}$. Then $\{\mathrm{G}_1[0, \infty, 2, 1] + i : i \in \mathbb{Z}_5\} \cup \{\mathrm{G}_1[0, \infty, 4, 2] + i : i \in \mathbb{Z}_5\} \cup \{\mathrm{G}_1[0, 3, 2, 1] + i : i \in \mathbb{Z}_5\}$ is a G_1-decomposition of ${}^8K_6$.

Example 2.15. Let $V(^8K_7) = \mathbb{Z}_7$. Then $\{G_1[0,1,3,4]+i : i \in \mathbb{Z}_7\} \cup \{G_1[0,1,3,5]+i : i \in \mathbb{Z}_7\} \cup \{G_1[0,1,3,6]+i : i \in \mathbb{Z}_7\}$ is a G_1-decomposition of $^8K_7$.

Example 2.16. Let $V(^8K_{10}) = \mathbb{Z}_9 \cup \{\infty\}$. Then $\{G_1[0,\infty,4,6]+i : i \in \mathbb{Z}_9\} \cup \{G_1[0,\infty,6,1]+i : i \in \mathbb{Z}_9\} \cup \{G_1[0,7,6,1]+i : i \in \mathbb{Z}_9\} \cup \{G_1[0,2,4,1]+i : i \in \mathbb{Z}_9\} \cup \{G_1[0,4,3,2]+i : i \in \mathbb{Z}_9\}$ is a G_1-decomposition of $^8K_{10}$.

Example 2.17. Let $V(^8K_{11}) = \mathbb{Z}_{11}$. Then $\{G_1[0,2,1,5]+i : i \in \mathbb{Z}_{11}\} \cup \{G_1[0,4,2,7]+i : i \in \mathbb{Z}_{11}\} \cup \{G_1[0,6,3,2]+i : i \in \mathbb{Z}_{11}\} \cup \{G_1[0,8,4,1]+i : i \in \mathbb{Z}_{11}\} \cup \{G_1[0,10,5,3]+i : i \in \mathbb{Z}_{11}\}$ is a G_1-decomposition of $^8K_{11}$.

Example 2.18. Let $V(^8K_6 \setminus {}^8K_2) = \mathbb{Z}_4 \cup \{\infty_1,\infty_2\}$. Then $\{G_1[\infty_1,0,1,2]$, $G_1[\infty_1,0,1,2]$, $G_1[\infty_1,0,1,2]$, $G_1[\infty_1,0,3,1]$, $G_1[\infty_2,0,1,3]$, $G_1[\infty_2,1,2,0]$, $G_1[\infty_2,1,3,0]$, $G_1[\infty_2,2,0,3]$, $G_1[\infty_2,2,0,3]$, $G_1[\infty_2,3,1,2]$, $G_1[1,2,3,\infty_1]$, $G_1[2,\infty_1,3,\infty_2]$, $G_1[2,\infty_1,3,0]$, $G_1[2,0,3,\infty_1]\}$ is a G_1-decomposition of $^8K_6 \setminus {}^8K_2$, where $\{\infty_1,\infty_2\}$ is the set of vertices incident with the removed edges.

Example 2.19. Let $V(^8K_7 \setminus {}^8K_3) = \mathbb{Z}_4 \cup \{\infty_1,\infty_2,\infty_3\}$. Then

$$\begin{aligned}\{&G_1[\infty_1,0,1,2], G_1[\infty_1,0,1,2], G_1[\infty_1,0,1,2], G_1[\infty_1,0,3,1],\\ &G_1[\infty_2,0,2,1], G_1[\infty_2,0,2,1], G_1[\infty_2,0,3,1], G_1[\infty_2,3,1,2],\\ &G_1[0,\infty_3,1,\infty_2], G_1[0,\infty_3,2,3], G_1[0,\infty_3,2,3], G_1[0,\infty_3,3,\infty_2],\\ &G_1[1,\infty_3,2,\infty_2], G_1[1,\infty_3,3,\infty_1], G_1[1,\infty_3,3,\infty_2], G_1[2,\infty_1,3,\infty_2],\\ &G_1[2,\infty_1,3,\infty_2], G_1[2,\infty_3,3,\infty_1]\}\end{aligned}$$

is a G_1-decomposition of $^8K_7 \setminus {}^8K_3$, where $\{\infty_1,\infty_2,\infty_3\}$ is the set of vertices incident with the removed edges.

3 Main Results

We now establish some necessary conditions for the existence of a G_1-decomposition of $^\lambda K_n$.

Lemma 3.1. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a G_1-decomposition of $^\lambda K_n$ only if the following hold:*

1. *if* $\gcd(\lambda,8)=1$, *then* $n \equiv 0$ *or* $1 \pmod{16}$;
2. *if* $\gcd(\lambda,8)=2$, *then* $n \equiv 0$ *or* $1 \pmod{8}$;
3. *if* $\gcd(\lambda,8)=4$, *then* $n \equiv 0$ *or* $1 \pmod{4}$;
4. *if* $\gcd(\lambda,8)=8$, *then* $n \geq 4$.

Proof. Let $\lambda \geq 2$ and $n \geq 4$ and suppose there exists a G_1-decomposition of ${}^{\lambda}K_n$. Since $E(\mathrm{G}_1) = 8$, we must have that $8 \mid \lambda n(n-1)/2$, and thus $16 \mid \lambda n(n-1)$. First, if $\gcd(\lambda, 8) = 1$, then $16 \mid n(n-1)$, and thus $n \equiv 0$ or $1 \pmod{16}$. Second, if $\gcd(\lambda, 8) = 2$, then $8 \mid n(n-1)$, and thus $n \equiv 0$ or $1 \pmod 8$. Third, if $\gcd(\lambda, 8) = 4$, then $4 \mid n(n-1)$, and thus $n \equiv 0$ or $1 \pmod 4$. Finally, if $\gcd(\lambda, 8) = 8$, then $2 \mid n(n-1)$ which is always true. ■

Before we establish sufficiency for $\left({}^{\lambda}K_n \mathrm{G}_1\right)$-designs, we first give the following negative result regarding designs of index 3.

Lemma 3.2. *There does not exist a* G_1*-decomposition of* ${}^{3}K_n$ *for any integer* $n > 1$.

Proof. Assume that a G_1-decomposition of ${}^{3}K_n$ exists. There are $3n(n-1)/16$ copies of G_1 in such a decomposition. We note that every copy of G_1 has exactly 2 edges of multiplicity 1 with the remainder having edge multiplicity 2. We also note that there are an odd number of edges between each of the $\binom{n}{2}$ pairs of vertices in ${}^{3}K_n$. We must then use at least one multiplicity-1 edge between each pair of vertices in ${}^{3}K_n$. With two multiplicity-1 edges per copy of G_1, this requires there to be at least $\binom{n}{2}/2 = n(n-1)/4$ copies. This is a contradiction since $n(n-1)/4$ exceeds $3n(n-1)/16$, the actual number of copies of G_1 in the decomposition. ■

Next, we establish sufficiency of the necessary conditions for certain values of λ.

Lemma 3.3. *There exists a* G_1*-decomposition of* ${}^{2}K_n$ *if* $n \equiv 0$ *or* $1 \pmod 8$.

Proof. If $n \in \{8, 9, 16, 17\}$, then the result follows from Examples 2.1, 2.2, 2.3, and 2.4. Otherwise, let $n = 8t + i$ for some integers $t \geq 3$ and $i \in \{0, 1\}$. By Corollary 1.3, there exists a K_3-decomposition of either $K_{t\times 2}$ or $K_{4,\,(t-2)\times 2}$. A $K_{3\times 4}$-decomposition of either $K_{t\times 8}$ or $K_{16,\,(t-2)\times 8}$ then follows from Theorem 1.4. Since G_1 divides ${}^{2}K_{3\times 4}$ (by Example 2.5), a G_1-decomposition of either ${}^{2}K_{t\times 8}$ or ${}^{2}K_{16,\,(t-2)\times 8}$ then follows.

In the case where $n = 8t$, we note that K_{8t} can be represented as either $(t \cdot K_8) \cup K_{t\times 8}$ or $K_{16} \cup \big((t-2) \cdot K_8\big) \cup K_{16,\,(t-2)\times 8}$. Since there exist G_1-decompositions of ${}^{2}K_8$, ${}^{2}K_{16}$, and ${}^{2}K_{3\times 4}$, there exists a G_1-decomposition of ${}^{2}K_{8t}$.

In the case where $n = 8t + 1$, we note that K_{8t+1} can be represented as either $\big(t \cdot K_8\big) \vee K_1\big) \cup K_{t\times 8} \cong K_{t\times 8} \cup \bigcup_{i=1}^{t} K_9$ or $\Big(\Big(K_{16} \cup \big((t-2) \cdot K_8\big)\Big) \vee K_1\Big) \cup K_{16,\,(t-2)\times 8} \cong K_{16,\,(t-2)\times 8} \cup K_{17} \cup \bigcup_{i=1}^{t-2} K_9$. Since there exist G_1-decompositions of ${}^{2}K_9$, ${}^{2}K_{17}$, and ${}^{2}K_{3\times 4}$, there exists a G_1-decomposition of ${}^{2}K_{8t+1}$. ■

Lemma 3.4. *There exists a* G_1*-decomposition of* ${}^4K_n$ *if* $n \equiv 0$ *or* $1 \pmod 4$.

Proof. If $n \in \{4, 5\}$, then the result follows from Examples 2.6 and 2.7. If $n \in \{8, 9\}$, then the result follows from two copies of the decompositions of index 2 from Examples 2.1 and 2.2. Otherwise, let $n = 4t + i$ for some integers $t \geq 3$ and $i \in \{0, 1\}$. By Corollary 1.3, there exists a K_3-decomposition of either $K_{t\times 2}$ or $K_{4,\,(t-2)\times 2}$. A $K_{3\times 2}$-decomposition of either $K_{t\times 4}$ or $K_{8,\,(t-2)\times 4}$ then follows from Theorem 1.4. Since G_1 divides ${}^4K_{3\times 2}$ (by Example 2.8), a G_1-decomposition of either ${}^4K_{t\times 4}$ or ${}^4K_{8,\,(t-2)\times 4}$ then follows.

In the case where $n = 4t$, we note that K_{4t} can be represented as either $(t \cdot K_4) \cup K_{t\times 4}$ or $K_8 \cup \big((t-2) \cdot K_4\big) \cup K_{8,\,(t-2)\times 4}$. Since there exist G_1-decompositions of ${}^4K_4$, ${}^4K_8$, and ${}^4K_{3\times 2}$, there exists a G_1-decomposition of ${}^4K_{4t}$.

In the case where $n = 4t + 1$, we note that K_{4t+1} can be represented as either $\big((t \cdot K_4) \vee K_1\big) \cup K_{t\times 4} \cong K_{t\times 4} \cup \bigcup_{i=1}^{t} K_5$ or $\Big(\Big(K_8 \cup \big((t-2) \cdot K_4\big)\Big) \vee K_1\Big) \cup K_{8,\,(t-2)\times 4} \cong K_{8,\,(t-2)\times 4} \cup K_9 \cup \bigcup_{i=1}^{t-2} K_5$. Since there exist G_1-decompositions of ${}^4K_5$, ${}^4K_9$, and ${}^4K_{3\times 2}$, there exists a G_1-decomposition of ${}^4K_{4t+1}$. ■

Lemma 3.5. *There exists a* G_1*-decomposition of* ${}^5K_n$ *if* $n \equiv 0$ *or* $1 \pmod{16}$.

Proof. If $n \in \{16, 17, 32, 33\}$, then the result follows from Examples 2.9, 2.10, 2.11, and 2.12. Otherwise, let $n = 16t + i$ for some integers $t \geq 3$ and $i \in \{0, 1\}$. By Corollary 1.3, there exists a K_3-decomposition of either $K_{t\times 2}$ or $K_{4,\,(t-2)\times 2}$. A $K_{3\times 8}$-decomposition of either $K_{t\times 16}$ or $K_{32,\,(t-2)\times 16}$ then follows from Theorem 1.4. Since G_1 divides ${}^5K_{3\times 8}$ (by Example 2.13), a G_1-decomposition of either ${}^5K_{t\times 16}$ or ${}^5K_{32,\,(t-2)\times 16}$ then follows.

In the case where $n = 16t$, we note that K_{16t} can be represented as either $(t \cdot K_{16}) \cup K_{t\times 16}$ or $K_{32} \cup \big((t-2) \cdot K_{16}\big) \cup K_{32,\,(t-2)\times 16}$. Since there exist G_1-decompositions of ${}^5K_{16}$, ${}^5K_{32}$, and ${}^5K_{3\times 8}$, there exists a G_1-decomposition of ${}^5K_{16t}$.

In the case where $n = 16t + 1$, we note that K_{16t+1} can be represented as either $\big((t \cdot K_{16}) \vee K_1\big) \cup K_{t\times 16} \cong K_{t\times 16} \cup \bigcup_{i=1}^{t} K_{17}$ or $\Big(\Big(K_{32} \cup \big((t-2) \cdot K_{16}\big)\Big) \vee K_1\Big) \cup K_{32,\,(t-2)\times 16} \cong K_{32,\,(t-2)\times 16} \cup K_{33} \cup \bigcup_{i=1}^{t-2} K_{17}$. Since there exist G_1-decompositions of ${}^5K_{17}$, ${}^5K_{33}$, and ${}^5K_{3\times 8}$, there exists a G_1-decomposition of ${}^5K_{16t+1}$. ■

Lemma 3.6. *There exists a* G_1*-decomposition of* ${}^8K_n$ *for all* $n \geq 4$.

Proof. If $n \in \{6,7,10,11\}$, then the result follows from Examples 2.14, 2.15, 2.16, and 2.17. Otherwise, let $n = 4t+i$ for some integers $t \geq 3$ and $i \in \{2,3\}$. By Corollary 1.3, there exists a K_3-decomposition of either $K_{t\times 2}$ or $K_{4,\,(t-2)\times 2}$. A $K_{3\times 2}$-decomposition of either $K_{t\times 4}$ or $K_{8,\,(t-2)\times 4}$ then follows from Theorem 1.4. A G_1-decomposition of ${}^8K_{3\times 2}$ follows from two copies of the G_1-decomposition of ${}^4K_{3\times 2}$ found in Example 2.8, and a G_1-decomposition of either ${}^5K_{t\times 16}$ or ${}^5K_{32,\,(t-2)\times 16}$ then follows.

In the case where $n = 4t+2$, we note that K_{4t+2} can be represented as either $\big((t\cdot K_4)\vee K_2\big)\cup K_{t\times 4} \cong K_{t\times 4}\cup K_6\cup\bigcup_{i=1}^{t-1}(K_6\setminus K_2)$ or $\Big(\Big(K_8\cup \big((t-2)\cdot K_4\big)\Big)\vee K_2\Big)\cup K_{8,\,(t-2)\times 4}\cong K_{8,\,(t-2)\times 4}\cup K_{10}\cup\bigcup_{i=1}^{t-2}(K_6\setminus K_2)$. Since there exist G_1-decompositions of ${}^8K_6$, ${}^8K_{10}$, ${}^8K_{3\times 2}$, and ${}^8K_6\setminus{}^8K_2$ (by Example 2.18), there exists a G_1-decomposition of ${}^8K_{4t+2}$.

In the case where $n = 4t+2$, we note that K_{4t+3} can be represented as either $\big((t\cdot K_4)\vee K_3\big)\cup K_{t\times 4} \cong K_{t\times 4}\cup K_7\cup\bigcup_{i=1}^{t-1}(K_7\setminus K_3)$ or $\Big(\Big(K_8\cup \big((t-2)\cdot K_4\big)\Big)\vee K_3\Big)\cup K_{8,\,(t-2)\times 4}\cong K_{8,\,(t-2)\times 4}\cup K_{11}\cup\bigcup_{i=1}^{t-2}(K_7\setminus K_3)$. Since there exist G_1-decompositions of ${}^8K_7$, ${}^8K_{11}$, ${}^8K_{3\times 2}$, and ${}^8K_7\setminus{}^8K_3$ (by Example 2.18), there exists a G_1-decomposition of ${}^8K_{4t+2}$. ■

Finally, we show that the necessary conditions for the existence of a G_1-decomposition of ${}^\lambda K_n$ are sufficient with the only exception being that $\lambda \neq 3$.

Theorem 3.7. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a G_1-decomposition of ${}^\lambda K_n$ if and only if $\lambda \neq 3$ and $16 \mid \lambda n(n-1)$.*

Proof. The necessary conditions are established in Lemmas 3.1 and 3.2. For sufficiency, let $\lambda \neq 3$ and $16 \mid \lambda n(n-1)$. We now consider the following cases:

CASE 1: $\lambda \equiv 0 \pmod 8$.
Since $\gcd(\lambda, 8) = 8$, we have that $\lambda \geq 8$ and $n \geq 4$. Let $\lambda = 8t$ for some positive integer t. By Lemma 3.6, G_1 divides ${}^8K_n$, and the result follows because ${}^8K_n \mid {}^{8t}K_n$.

CASE 2: $\lambda \equiv 1$, 3, 5, or 7 $\pmod 8$.
Since $\gcd(\lambda, 8) = 1$, we have that $\lambda \geq 5$ is odd and $n \equiv 0$ or 1 $\pmod{16}$. Let $\lambda = 2t+5$ for some integer $t \geq 0$. Now, we split ${}^\lambda K_n$ into t copies of ${}^2K_n$ and one copy of ${}^5K_n$. By Lemmas 3.3 and 3.5, G_1 divides ${}^2K_n$ and ${}^5K_n$, and the result follows.

CASE 3: $\lambda \equiv 2$ or 6 $\pmod 8$.
Since $\gcd(\lambda, 8) = 2$, we have that $\lambda \geq 2$ is even and $n \equiv 0$ or 1 $\pmod 8$. Let $\lambda = 2t$ for some positive (odd) integer t. By Lemma 3.3, G_1 divides ${}^2K_n$, and the result follows because ${}^2K_n \mid {}^{2t}K_n$.

Case 4: $\lambda \equiv 4 \pmod 8$.
Since $\gcd(\lambda, 8) = 4$, we have that $\lambda \geq 4$ is a multiple of 4 and $n \equiv 0$ or 1 (mod 4). Let $\lambda = 4t$ for some positive (odd) integer t. By Lemma 3.4, G_1 divides ${}^4K_n$, and the result follows because ${}^4K_n \mid {}^{4t}K_n$. ■

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