On ordered directed ρ-labelings of bipartite digraphs and cyclic digraph decompositions

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Abstract

It is known that an ordered ρ-labeling of a bipartite graph G with n edges yields a cyclic G-decomposition of K_{2nx+1} for every positive integer x. We extend the concept of an ordered ρ-labeling to bipartite digraphs and show that an ordered directed ρ-labeling of a bipartite digraph D with n arcs yields a cyclic D-decomposition of K^*_{nx+1} for every positive integer x. We also find several classes of bipartite digraphs that admit an ordered directed ρ-labeling.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, then $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_m$ the group of integers modulo m. For a graph (or digraph) H, let $V(H)$ and $E(H)$ denote the vertex set of H and the edge (or arc) set of H, respectively. The *order* and the *size* of a (di-)graph H are $|V(H)|$ and $|E(H)|$, respectively. If we let $V(K_m) = [0, m-1]$, we define the *length* of the edge $\{i, j\}$ in K_m, where $V(K_m) = [0, m-1]$, to be $\min\{|j-i|, m-|j-i|\}$.

Let $V(K_m) = \mathbb{Z}_m$ and let G be a subgraph of K_m. By *rotating* G, we mean applying the permutation $i \mapsto i+1$ to $V(G)$. Let H and G be graphs

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such that G is a subgraph of H. A *G-decomposition* of H is a set $\Delta = \{G_1, G_2, \ldots, G_r\}$ of pairwise edge-disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{r} E(G_i)$. A G-decomposition of K_m is also known as a *(K_m, G)-design*. A (K_m, G)-design Δ is *cyclic* if rotating is an automorphism of Δ. The study of graph decompositions is generally known as the study of graph designs, or G-designs. For recent surveys on G-designs, see [1] and [5].

Let G be a graph with n edges. A primary question in the study of graph designs is, "For what values of m does there exist a (K_m, G)-design?" For most studied graphs G, it is the case that if $m \equiv 1 \pmod{2n}$, then there exists a (K_m, G)-design. A common approach to finding these designs is through the use of graph labelings which are defined in the next section.

Similar concepts to the ones defined above for undirected graphs can be defined for digraphs. First, we introduce additional notation. For an undirected graph G, let G^* denote the digraph obtained from G by replacing each edge $\{u, v\} \in E(G)$ with the arcs (u, v) and (v, u). Let $V(K_m^*) = [0, m-1]$. The *length* of the arc (i, j) is $j - i$ if $j > i$, and it is $m + j - i$, otherwise. Note that $E(K_m^*)$ consists of m arcs of length i for each $i \in [1, m-1]$.

Let $V(K_m^*) = \mathbb{Z}_m$ and let D be a subgraph of K_m^*. By *rotating* D, we mean applying the permutation $i \mapsto i+1$ to $V(D)$. Moreover in this case, if $j \in \mathbb{N}$, then $D + j$ is the digraph obtained from D by successively rotating D a total of j times. Note that rotating an arc does not change its length. Also note that $D + j$ is isomorphic to D for every $j \in \mathbb{N}$.

Let H and D be digraphs such that D is a subgraph of H. A *D-decomposition* of H is a set $\Delta = \{D_1, D_2, \ldots, D_r\}$ of pairwise arc-disjoint subgraphs of H each of which is isomorphic to D and such that $E(H) = \bigcup_{i=1}^{r} E(D_i)$. A D-decomposition of K_m^* is also known as a *(K_m^*, D)-design*. A (K_m^*, D)-design Δ is *cyclic* if rotating is an automorphism of Δ.

Let D be a digraph with n arcs. As with undirected graphs, one can ask, "For what values of m does there exist a (K_m^*, D)-design?" It is the case that if $m \equiv 1 \pmod{n}$, then there exists a (K_m^*, D)-design. Not surprisingly, an approach to finding these designs is through the use of digraph labelings.

2 Labelings

Alex Rosa introduced graph labelings in [15] as a means of tackling graph decomposition problems. In particular, Rosa's labelings of a graph G with n edges yield cyclic G-decompositions of K_{2n+1} and, in one special case, cyclic G-decompositions of K_{2nx+1} for every positive integer x. Others have since introduced variations on Rosa's labelings that lead to additional cyclic graph decompositions. Such labelings have been termed *Rosa-type labelings* because of the influence of Rosa's seminal article on the topic [15]. We summarize some of these labelings in the next subsection and then introduce a Rosa-type labeling for bipartite digraphs.

2.1 Labelings of undirected graphs

A *labeling* of a graph G is an injective function $h\colon V(G) \to \mathbb{N}$. In [15], Rosa introduced a hierarchy of graph labelings. In [7], the concept of ordered labelings was introduced. Because of their relevance to our current work, we emphasize concepts from these two manuscripts. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u)\colon u \in V(G)\}$. Define a function $\bar{f}\colon E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{f}(E(G)) = \{\bar{f}(e)\colon e \in E(G)\}$. Consider the following conditions:

(ℓ1) $f(V(G)) \subseteq [0, 2n]$,

(ℓ2) $f(V(G)) \subseteq [0, n]$,

(ℓ3) $\bar{f}(E(G)) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,

(ℓ4) $\bar{f}(E(G)) = [1, n]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ consider also

(ℓ5) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,

(ℓ6) there exists an integer λ such that $f(a) \le \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying conditions

(ℓ1) and (ℓ3) is called a ρ-*labeling*;

(ℓ1) and (ℓ4) is called a σ-*labeling*;

(ℓ2) and (ℓ4) is called a β-*labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. Suppose G is bipartite. If a ρ-, σ-, or β-labeling of G satisfies condition (ℓ5), then the labeling is *ordered* and is denoted by ρ^+, σ^+, or β^+, respectively. If in addition (ℓ6) is satisfied, the labeling is *uniformly ordered* and is denoted by ρ^{++}, σ^{++}, or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly ordered β-labeling is an α-*labeling* as introduced in [15]. See [6] for a survey of these Rosa-type labelings. A dynamic survey on general graph labelings is maintained by Gallian [9].

As stated earlier, labelings have applications in the study of cyclic graph decompositions. The first results in this area were obtained in [15], which included the following theorems.

Theorem 2.1 (Rosa [15])**.** *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G admits a ρ-labeling.*

Theorem 2.2 (Rosa [15])**.** *Let G be a bipartite graph with n edges that admits an α-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all $x \in \mathbb{Z}^+$.*

Since Theorem 2.2 gives rise to an infinite family of cyclic decompositions, an α-labeling of a graph is preferable to Rosa's other labelings. However, there are many classes of bipartite graphs (see [15]) which do not admit α-labelings. In [7], El-Zanati, Vanden Eynden, and Punnim showed that a ρ^+-labeling of a bipartite graph G with n edges suffices for the existence of cyclic G-decompositions of K_{2nx+1}. In contrast to α-labelings, it is conjectured by El-Zanati and Vanden Eynden (see [6]) that every bipartite graph admits a ρ^+-labeling.

Theorem 2.3 (El-Zanati, Vanden Eynden, and Punnim [7])**.** *Let G be a bipartite graph with n edges that admits a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all $x \in \mathbb{Z}^+$.*

Before we proceed to labelings of digraphs, we note that a ρ-labeling of a graph G with n edges is an embedding of G in K_{2n+1} so as to have one edge of G of each length. Then rotating G a total of $2n$ times yields the cyclic (K_{2n+1}, G)-design. This idea extends in a natural way to embeddings of digraphs with n arcs (and no more than $n+1$ vertices) in K^*_{n+1}. Hence the concept of a directed ρ-labeling extends naturally, too.

2.2 Labelings of digraphs

Let D be a digraph with n arcs and at most $n+1$ vertices. Let $f\colon V(D) \to [0,n]$ be an injective function. Define a function $\bar{f}\colon E(D) \to [1,n]$ as follows: $\bar{f}((a,b)) = f(b) - f(a)$, if $f(b) > f(a)$, and $\bar{f}((a,b)) = n + 1 + f(b) - f(a)$, otherwise. We call f a *directed ρ-labeling of D* if $\{\bar{f}((a,b))\colon (a,b) \in E(D)\} = [1,n]$. Thus, a directed ρ-labeling of D is an embedding of D in K^*_{n+1} such that there is exactly one arc in D of length i for each $i \in [1,n]$.

The concept of a directed ρ-labeling was first introduced (as an extension of graceful labelings) in 1985 by Bloom and Hsu [3]. They called the labeling a *digraceful labeling*. The concept was investigated further in 2008 by Marr [13] and in 2009 by Kaplan, Lev, and Roditty in [12]. Several other authors have investigated the same concept as graceful labelings of digraphs (see [9] and [8] for summaries of various results on the topic).

As far as the authors are aware, the connection between digraceful labelings and cyclic digraph decompositions was first noted by Kaplan et. al [12], where they proved the directed version of Theorem 2.1. In the following statement of their result, the notation and terminology have been adapted to better suit this paper.

Theorem 2.4 (Kaplan, Lev, and Roditty [12])**.** *Let D be a digraph with n arcs and at most $n+1$ vertices. There exists a cyclic D-decomposition of K_{n+1}^* if and only if D admits a directed ρ-labeling.*

As with Theorem 2.1, Theorem 2.4 yields only one cyclic design. However, the concept of an ordered directed ρ-labeling extends to bipartite digraphs yielding an infinite number of cyclic digraph designs.

Let D be a bipartite digraph with n arcs and at most $n+1$ vertices. Let $\{A, B\}$ be a bipartition of $V(D)$. A directed ρ-labeling f of D is *ordered* if $f(a) < f(b)$ for each arc in $E(D)$ with end vertices $a \in A$ and $b \in B$. An ordered directed ρ-labeling is also called a *directed ρ^+-labeling.* An example of an ordered directed ρ-labeling can be seen in Figure 1. The following theorem extends Theorem 2.3 to digraphs.

Theorem 2.5. *Let D be a bipartite digraph with n arcs that admits a directed ρ^+-labeling. Then there exists a cyclic D-decomposition of K_{nx+1}^* for all $x \in \mathbb{Z}^+$.*

Proof. Let D have vertex bipartition $\{A, B\}$ and let f be a directed ρ^+-labeling of D such that $f(a) < f(b)$ for each arc in $E(D)$ with end vertices $a \in A$ and $b \in B$. Since f is a directed ρ^+-labeling, we have that $\{\bar{f}(e)\colon\ e \in E(D)\} = [1, n]$; hence, each arc in D receives a distinct length.

Let x be a positive integer. If $x = 1$, then Theorem 2.4 applies. Thus we may assume that $x \geq 2$. Let $B_1, B_2, \ldots, B_x$ be x vertex-disjoint copies of B. The vertex in B_i that corresponds to $b \in B$ will be denoted b_i. For each $i \in [1, x]$, let D_i be a copy of D with vertex bipartition $\{A, B_i\}$. Let $D' = D_1 \cup D_2 \cup \cdots \cup D_x$.

Define the labeling $f'\colon\ V(D') \to [0, nx]$ as follows:

$$f'(v) = \begin{cases} f(a) & \text{if } v = a \in A, \\ f(b) + (i-1)n & \text{if } v = b_i \in B_i. \end{cases}$$

The arcs in D' are of two types: either directed away from A or towards A. It is clear that an arc $e \in D$ directed away from A generates the following set of x arc lengths in D': $\{\bar{f}(e), \bar{f}(e) + n, \ldots, \bar{f}(e) + (x-1)n\}$. Notice that the arc lengths in D' generated by an arc $e \in D$ directed towards A form the same set, since for each $i \in [1, x]$, the length of $e = (b_i, a)$ is

$$nx + 1 + f(a) - f(b) - (i-1)n = \bar{f}(e) + (x-i)n.$$

Now, let $z \in [1, nx]$. We will show that some arc in $E(D')$ has length z. There exist unique integers q and r, with $0 \le q \le x-1$ and $1 \le r \le n$, such that $z = nq + r$. Let e be the arc in $E(D)$ such that $\bar{f}(e) = r$. If $e = (u, v)$ with $u \in A$ and $v \in B$, let e_{q+1} denote the arc (u, v_{q+1}) in $E(D')$. Then,

$$\bar{f}'(e_{q+1}) = \bar{f}'(v_{q+1}) - \bar{f}'(u) = f(v) + qn - f(u) = qn + \bar{f}(e) = z.$$

If $e = (v, u)$ with $u \in A$ and $v \in B$, let e_{x-q} denote the arc (v_{x-q}, u) in $E(D')$. Then,

$$\begin{aligned} \bar{f}'(e_{x-q}) &= nx + 1 + [\bar{f}'(u) - \bar{f}'(v_{x-q})] \\ &= nx + 1 + f(u) - [f(v) + (x - q - 1)n] \\ &= n + 1 + f(u) - f(v) + nq \\ &= \bar{f}(e) + nq \\ &= z. \end{aligned}$$

In either case, z is the length of some arc in $E(D')$. Since $|E(D')| = nx$, the labeling f' is a directed ρ-labeling of D'. By Theorem 2.4, there is a cyclic D'-decomposition of K^*_{nx+1}. This gives us the desired cyclic D-decomposition of K^*_{nx+1}. ■

Figure 1 demonstrates how Theorem 2.5 works with a particular digraph with 5 arcs.

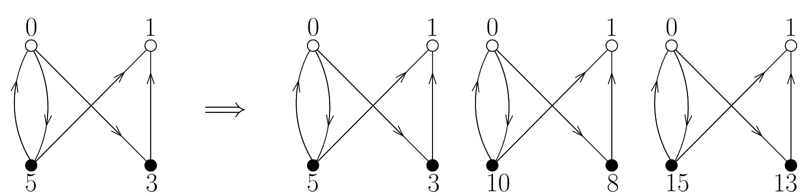


Figure 1: A directed ρ^+-labeling of a bipartite digraph D with 5 arcs and three starters for a cyclic D-decomposition of K^*_{16}.

We now briefly turn our attention to a few collected results on directed ρ- and ρ^+-labelings of some classes of bipartite digraphs. Let P_n denote the path on n vertices. Define the *directed path on n vertices*, denoted DP_n, to be the digraph obtained from an orientation on P_n in which the indegree and outdegree are equal for each non-leaf vertex. Similarly, let DC_n denote the *directed cycle on n vertices* which is obtained from an orientation on a cycle on n vertices such that the indegree and outdegree of every vertex are both 1. Define the *anti-directed cycle on n vertices*, denoted AC_n, to be the digraph obtained from an orientation on a cycle on n vertices which does not have DP_3 as a subgraph. It was shown in [4] that DP_n admits a directed ρ-labeling if and only if n is even. For an even positive integer n, the labeling provided for DP_n with consecutive vertices $a_0, a_1, \ldots, a_{n-1}$ is $\theta(a_i) = (-1)^{i+1} \cdot \frac{i+1}{2} \pmod{n}$. This labeling actually produces a directed ρ^+-labeling of DP_n.

Orientations of the n-star, $K_{1,n}$, are studied in [13], and it is shown that an orientation of $K_{1,n}$ admits a directed ρ-labeling if and only if either (i) n is odd or (ii) n is even and the indegree of the center is even. The labelings constructed use 0 as the label on the center vertex and are therefore directed ρ^+-labelings of the orientation of $K_{1,n}$.

We can use certain labelings of an undirected graph along with a carefully chosen orientation to obtain a directed ρ^+-labeling of the resulting digraph. This is formalized in the following theorem, the proof of which is straightforward and is hence omitted.

Theorem 2.6. *Let G be an undirected bipartite graph with vertex bipartition $\{A, B\}$. If f is a β^+-labeling of G where $f(a) < f(b)$ for every edge $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, then f is a directed ρ^+-labeling of the digraph obtained from G by either orienting every edge from A to B or orienting every edge from B to A.*

It is also easy to see the following.

Theorem 2.7. *Let G be an undirected graph. If f is a β-labeling of G, then f is a directed ρ-labeling of G^*. Moreover, if G is bipartite and f is ordered, then f is a directed ρ^+-labeling of G^*.*

There exist many classes of bipartite graphs that admit α-labelings (see [9]). Since an α-labeling is necessarily a β^+-labeling, the above results apply to a large number of graphs. For example, it is known from [15] that C_n admits an α-labeling if and only if $n \equiv 0 \pmod{4}$. It follows from Theorem 2.6 that if $m \geq 1$, then AC_{4m} admits a directed ρ^+-labeling. It is also known that if G with $n \equiv 0 \pmod{4}$ edges is the vertex-disjoint union of up to three even cycles (but not $C_4 \cup C_4 \cup C_4$), then G admits an α-labeling (see [9]). Furthermore, there exists an α-labeling of C_{4m} that gives rise to a directed ρ^+-labeling of DC_{4m}. This is shown in Section 3. (While the labeling is not identified as an α-labeling, we leave it to the reader to check.) However, it is not true that every α-labeling of C_{4m} gives rise to a directed ρ^+-labeling of DC_{4m}. Figure 2 shows an α-labeling of C_{12} which is not a directed ρ^+-labeling of DC_{12}.

Digraceful labelings of the unions of directed cycles were studied in [3], where the following result was obtained. Let G be a digraph which consists of a vertex-disjoint union of t directed cycles, all of which are on n vertices. Then G admits a directed ρ-labeling if (i) $t = 1$ and n is even, (ii) $t = 2$, or (iii) $n \in \{2, 6\}$. Moreover, G has no directed ρ-labeling if tn is odd.

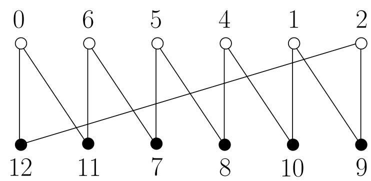


Figure 2: An α-labeling of C_{12} which is not a directed ρ^+-labeling of DC_{12}.

3 Some digraphs that admit directed ρ^+-labelings

In this section, we show that every directed even cycle admits a directed ρ^+-labeling. We also find all bipartite digraphs on up to 4 vertices that admit directed ρ^+-labelings.

3.1 Directed ρ^+-labelings of directed even cycles

Theorem 3.1. *For every even positive integer n, the directed cycle DC_n admits a directed ρ^+-labeling.*

Proof. Let $n = 2k$ where $k \in \mathbb{Z}^+$. Let DC_{2k} have vertex set $\{v_1, v_2, \ldots, v_{2k}\}$ and arc set $\big\{(v_i, v_{i+1})\colon\ 1 \le i \le 2k-1\big\} \cup \big\{(v_{2k}, v_1)\big\}$. Also, let $A = \{v_2, v_4, \ldots, v_{2k}\}$ and $B = \{v_1, v_3, \ldots, v_{2k-1}\}$. Then $\{A, B\}$ is a bipartition of $V(DC_{2k})$.

Consider a labeling $f\colon\ V(DC_{2k}) \to [0, 2k]$ defined by

$$f(v_i) = \begin{cases} i/2 - 1 & \text{if } v_i \in A, \\ 2k - (i-1)/2 & \text{if } v_i \in B \text{ and } i \le k, \\ 2k - (i-1)/2 - 1 & \text{if } v_i \in B \text{ and } i > k. \end{cases}$$

It is easy to check that this function is injective with $f(a) < f(b)$ for any $a \in A$ and $b \in B$, and thus f is ordered.

We next consider the set of arc lengths that f induces on $E(DC_{2k})$. For all arcs with a tail in B, we have

$$\begin{aligned}\big\{2k + 1 + \big((i+1)/2 - 1\big) - \big(2k - (i-1)/2\big)\colon\ i \text{ odd}, i \le k\big\} \\ = \{i\colon\ i \text{ odd}, i \le k\} = \begin{cases} \{1, 3, \ldots, k-1\} & \text{if } k \text{ is even}, \\ \{1, 3, \ldots, k\} & \text{if } k \text{ is odd}, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\big\{2k+1+\big((i+1)/2-1\big)-\big(2k-(i-1)/2-1\big)\colon\ i \text{ odd},\ k<i\le 2k-1\big\}\\ =\{i+1\colon\ i \text{ odd},\ k<i\le 2k-1\}\\ =\begin{cases}\{k+2,k+4,\dots,2k\} & \text{if } k \text{ is even},\\ \{k+3,k+5,\dots,2k\} & \text{if } k \text{ is odd}.\end{cases}\end{aligned}$$

For all arcs with a tail in $A\setminus\{v_{2k}\}$, we have

$$\begin{aligned}\big\{\big(2k-\big((i+1)-1\big)/2\big)-\big(i/2-1\big)\colon\ i \text{ even},\ i\le k-1\big\}\\ =\{2k-i+1\colon\ i \text{ even},\ i\le k-1\}\\ =\begin{cases}\{k+3,k+5,\dots,2k-1\} & \text{if } k \text{ is even},\\ \{k+2,k+4,\dots,2k-1\} & \text{if } k \text{ is odd},\end{cases}\end{aligned}$$

and

$$\begin{aligned}\big\{\big(2k-\big((i+1)-1\big)/2-1\big)-\big(i/2-1\big)\colon\ i \text{ even},\ k-1<i\le 2k-2\big\}\\ =\{2k-i\colon\ i \text{ even},\ k-1<i\le 2k-2\}\\ =\begin{cases}\{2,4,\dots,k\} & \text{if } k \text{ is even},\\ \{2,4,\dots,k\} & \text{if } k \text{ is odd}.\end{cases}\end{aligned}$$

Finally, the remaining arc (v_{2k},v_1) has length $(2k)-(k-1)=k+1$, and thus, regardless of the parity of k, we have exactly one arc of length ℓ for each $\ell\in[1,2k]$. ■

Examples of the described labeling in the above proof can be seen in Figure 3. In light of Theorem 2.5, we have the following corollary.

Corollary 3.2. *For every even positive integer n, there exists a cyclic DC_n-decomposition of K^*_{nx+1} for all $x\in\mathbb{Z}^+$.*

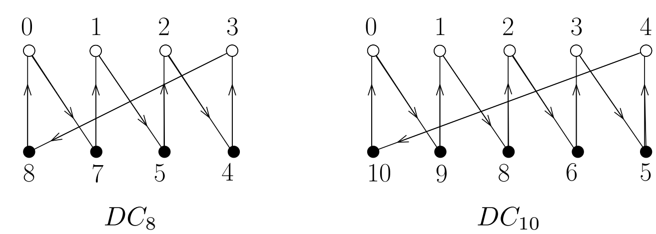
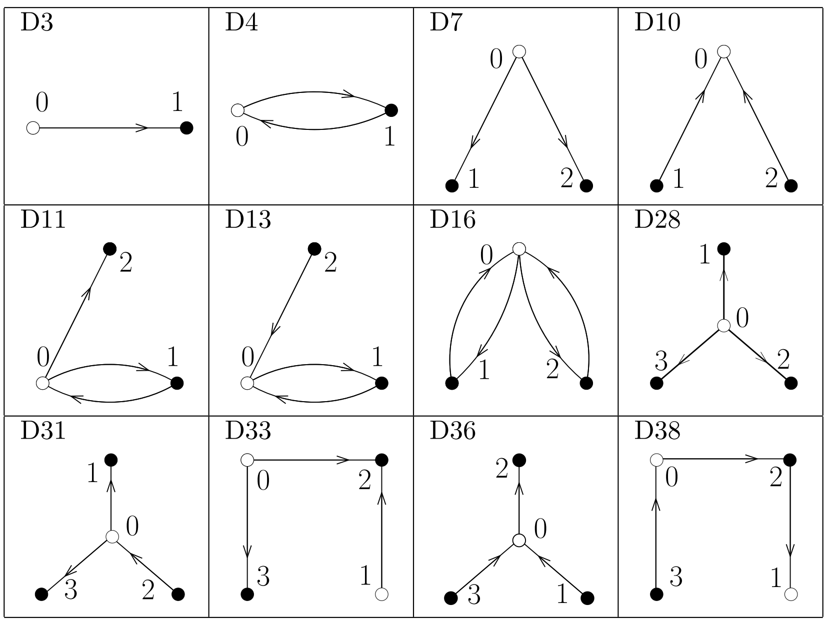


Figure 3: Directed ρ^+-labelings of directed even cycles.

3.2 Digraphs of small order

According to the reference book *An Atlas of Graphs* [14], there are 51 non-isomorphic bipartite digraphs on up to 4 vertices and no isolated vertices. Of the 51, there are exactly 38 which admit directed ρ^+-labelings (see Table 1). None of the remaining 13 bipartite digraphs admits a directed ρ-labeling (see Table 2). We referenced these digraphs in the same way they are referenced in [14].

Table 1: The bipartite digraphs of order at most 4 that admit a directed ρ^+-labeling.



4 Concluding remarks and acknowledgements

Directed ρ^+-labelings of a digraph D with n arcs lead to cyclic (K^*_{nx+1}, D)-designs for every positive integer x. Other authors have studied (K^*_v, D)-designs although not necessarily cyclic ones. We note three results that are relevant to this work. In [11], Hartman and Mendelsohn find necessary and sufficient conditions for the existence of a (K^*_v, D)-design for each subgraph D of K^*_3. In [10], necessary and sufficient conditions for the existence of a (K^*_v, D)-design for each orientation D of C_4. Necessary and sufficient conditions for the existence of directed cycle decompositions of K^*_v are

Table 1 (continued): The bipartite digraphs of order at most 4 that admit a directed ρ^+-labeling.

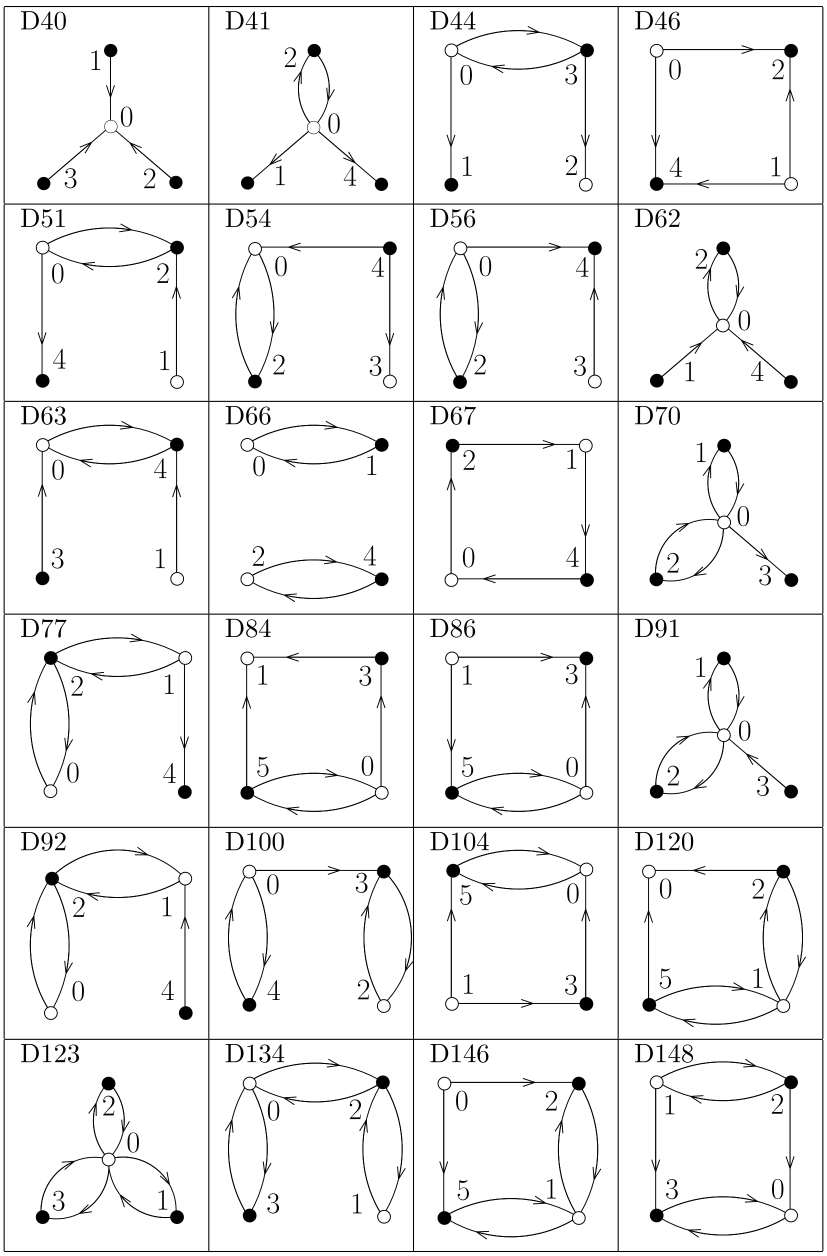


Table 1 (continued): The bipartite digraphs of order at most 4 that admit a directed ρ^+-labeling.

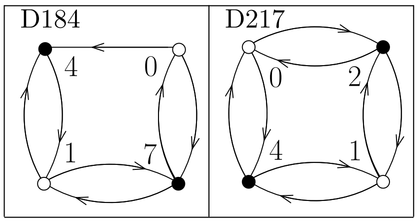
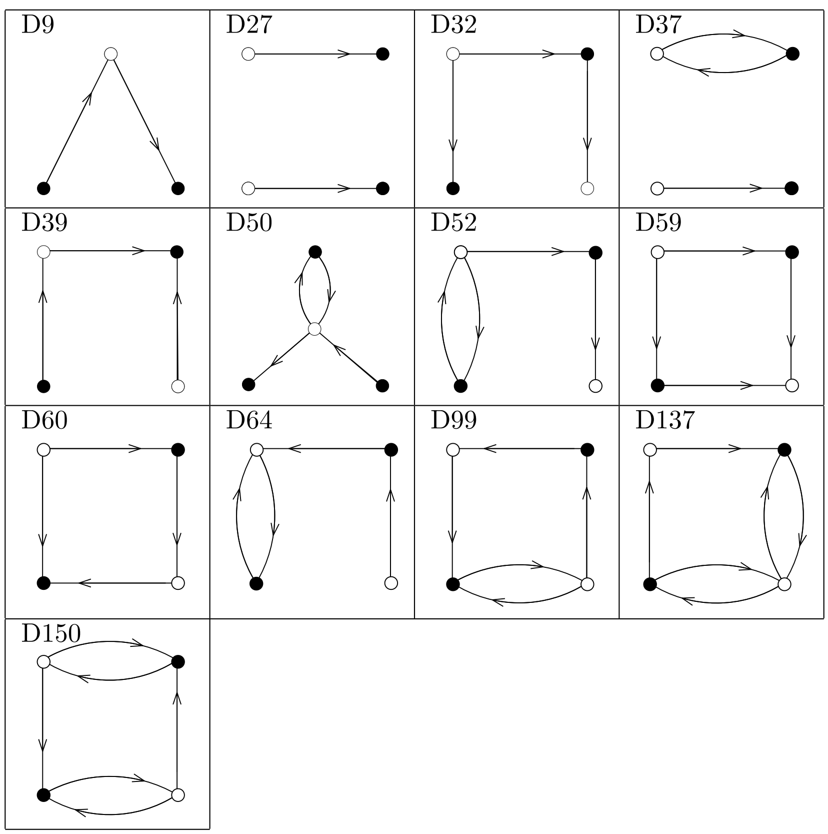


Table 2: The bipartite digraphs of order at most 4 that do not admit a directed ρ-labeling.



found in [2].

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