

ON DECOMPOSING THE COMPLETE SYMMETRIC
DIGRAPH INTO ORIENTATIONS OF $K_4 - e$

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Abstract

Let D be any of the 10 digraphs obtained by orienting the edges of $K_4 - e$. We establish necessary and sufficient conditions for the existence of a (K_n^*, D) -design for 8 of these digraphs. Partial results as well as some nonexistence results are established for the remaining 2 digraphs.

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1. INTRODUCTION

Let \mathbb{Z}_m denote the group of integers modulo m . For integers a and b with $a \leq b$, let $[a, b] = \{a, a + 1, \dots, b\}$. For a graph (or digraph) H , let $V(H)$ and $E(H)$ denote the vertex set of H and the edge (or arc) set of H , respectively. The *order* and the *size* of a graph (or digraph) H are $|V(H)|$ and $|E(H)|$, respectively.

We denote the complete multipartite graph with parts of sizes a_i for $1 \leq i \leq m$ by K_{a_1, a_2, \dots, a_m} . If $a_i = a$ for all $i \in \{1, 2, \dots, m\}$, then we use the notation $K_{m \times a}$. Additionally, $K_{m \times a, b}$ denotes the complete multipartite graph with m parts of size a and one part of size b .

Let H be a graph and let \mathcal{G} be a set of subgraphs of H . We will refer to a graph $G \in \mathcal{G}$ as a G -block. A \mathcal{G} -decomposition of H is a set $\Delta = \{G_1, G_2, \dots, G_r\}$

of pairwise edge-disjoint subgraphs of H such that for every $i \in [1, r]$, $G_i \cong G$ for some $G \in \mathcal{G}$ and such that $E(H) = \bigcup_{i=1}^r E(G_i)$. Of particular importance is when $\mathcal{G} = \{G\}$, in which case we write “ G -decomposition of H ” instead of “ $\{G\}$ -decomposition of H .” A G -decomposition of H is also known as an (H, G) -*design*. The set of all n for which K_n admits a G -decomposition is called the *spectrum of G* . The spectrum has been determined for many classes of graphs, including for all graphs on at most 4 vertices [4] and all graphs on 5 vertices (see [3] and [10]). We direct the reader to [2] and [5] for recent surveys on graph decompositions.

By *blowing up* the vertices of a graph G by some positive integer t , we mean replacing every vertex of G with t independent vertices and replacing every edge in G by a $K_{t,t}$. For example, assume we have a $(K_{x \times 2}, K_3)$ -design. After blowing up the vertices of $K_{x \times 2}$ by 5, our corresponding $(K_{x \times 2}, K_3)$ -design becomes a $(K_{x \times 10}, K_{3 \times 5})$ -design.

Similar concepts to the ones defined above for undirected graphs can be defined for digraphs. First, we introduce additional notation. For an undirected graph G , let G^* denote the digraph obtained from G by replacing each edge $\{u, v\} \in E(G)$ with the arcs (u, v) and (v, u) . Thus K_n^* , the *complete symmetric digraph of order n* , is the digraph on n vertices with the arcs (u, v) and (v, u) between every pair of distinct vertices u and v .

Let D and H be digraphs such that D is a subgraph of H . The *reverse orientation of D* , denoted $\text{Rev}(D)$, is the digraph with vertex set $V(D)$ and arc set $\{(v, u) : (u, v) \in E(D)\}$. A D -*decomposition* of H is a set $\Delta = \{D_1, D_2, \dots, D_r\}$ of pairwise arc-disjoint subgraphs of H each of which is isomorphic to D and such that $E(H) = \bigcup_{i=1}^r E(D_i)$. As with the undirected case, a D -decomposition of H is also known as an (H, D) -*design*, and the set of all n for which K_n^* admits a D -decomposition is called the *spectrum of D* . Furthermore, we say D is *self-complementary in H* if D is isomorphic to the digraph with arc set $E(H) \setminus E(D)$. That is, D is self-complementary in H if H has size $2 \cdot |E(D)|$ and there exists an (H, D) -design.

The spectra for several digraphs of small order have been determined. This includes the spectra for all digraphs on at most 3 vertices [11] and all bipartite digraphs on 4 vertices with up to 5 arcs [7].

In this paper, we extend the known results on small digraphs by determining the spectrum for 8 of the 10 digraphs obtained by orienting the edges of $K_4 - e$, the graph obtained from removing a single edge from K_4 . Some nonexistence results are proven for the remaining 2 such digraphs. We use the naming convention found in *An Atlas of Graphs* [13] by Read and Wilson. The digraphs under investigation are shown in Figures 1 and 2 with a key that denotes a labeled copy for each of the 10 digraphs of interest. For example, $D75[w, x, y, z]$ refers to the digraph with vertex set $\{w, x, y, z\}$ and arc set $\{(w, x), (w, y), (w, z), (x, y), (z, y)\}$.

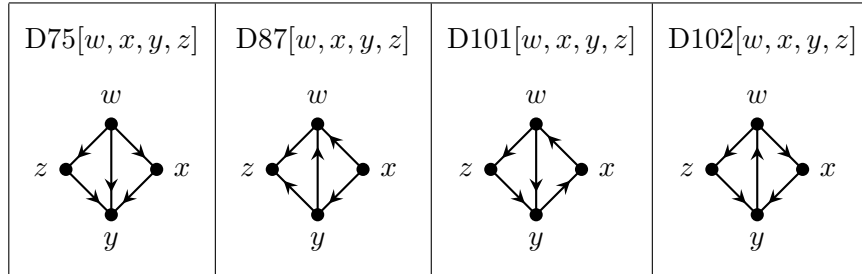


Figure 1. The four orientations of $K_4 - e$ that are self-complementary in $(K_4 - e)^*$.

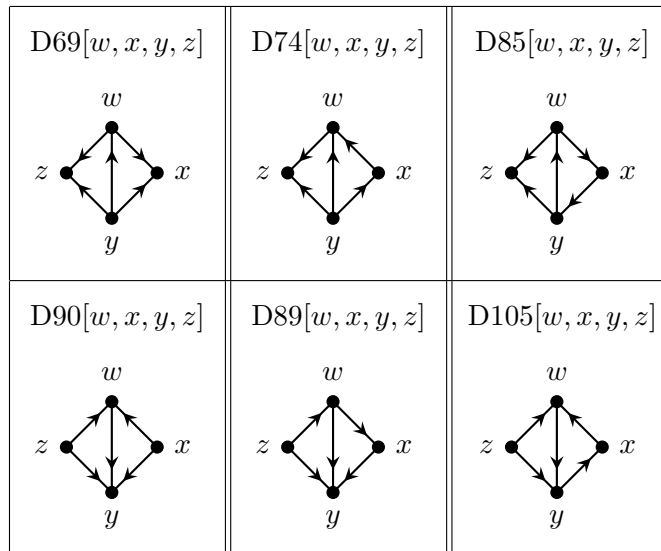


Figure 2. The six orientations of $K_4 - e$ that are not self-complementary in $(K_4 - e)^*$, shown paired with their reverse orientations.

Note that 6 of the digraphs of interest in this paper occur in pairs with respect to their reverse orientations (see Figure 2). Namely, $D69 \cong \text{Rev}(D90)$, $D74 \cong \text{Rev}(D89)$, and $D85 \cong \text{Rev}(D105)$. The remaining 4 digraphs of interest (see Figure 1) are isomorphic to their reverse orientations, e.g., $D75 \cong \text{Rev}(D75)$, which is shown in the proceeding section (see Lemma 4) to imply that these 4 digraphs are self-complementary in $(K_4 - e)^*$.

2. SOME BASIC RESULTS

The necessary conditions for a digraph D to decompose K_n^* include

- (A) $|V(D)| \leq n$,

- (B) $|E(D)|$ divides $n(n-1)$, and
 (C) both $\gcd\{\text{outdegree}(v) : v \in V(D)\}$ and $\gcd\{\text{indegree}(v) : v \in V(D)\}$ divide $n-1$.

Applying these necessary conditions to the 10 digraphs under consideration, we obtain the following necessary condition: For $D \in \{D69, D74, D75, D85, D87, D89, D90, D101, D102, D105\}$, a (K_n^*, D) -design exists only if $n \equiv 0$ or $1 \pmod{5}$.

The following observation was stated in [6].

Observation 1. *Let D and H be digraphs. A D -decomposition of H exists if and only if a $\text{Rev}(D)$ -decomposition of $\text{Rev}(H)$ exists.*

The fact that $K_n^* \cong \text{Rev}(K_n^*)$ leads to our next observation, also stated in [6].

Observation 2. *Let D be a digraph. A (K_n^*, D) -design exists if and only if a $(K_n^*, \text{Rev}(D))$ -design exists.*

2.1. Results for self-complementary digraphs

We note that the existence of $(K_4 - e)$ -decompositions of complete multigraphs (i.e., the spectrum of index λ) is known [12]. However, we present here the following theorem reduced to what is useful for characterizing the spectra of our 4 self-complementary digraphs.

Theorem 3 [4]. *There exists a $(K_4 - e)$ -decomposition of K_n if and only if $n \equiv 0$ or $1 \pmod{5}$ and $n \geq 6$.*

Since there does not exist a $(K_4 - e)$ -decomposition of K_5 , we must address decompositions of K_5^* (see Section 3). To make use of the known spectrum of $K_4 - e$, we present the following.

Lemma 4. *Let D be an orientation of a graph G . Then D is isomorphic to $\text{Rev}(D)$ if and only if D is self-complementary in G^* .*

Proof. Let D' be the digraph with vertex set $V(G^*)$ and arc set $E(G^*) \setminus E(D)$. Note that $E(D') = \{(v, u) : (u, v) \in E(D)\}$, which implies that D' is both the reverse orientation of D and the complement of D in G^* . The result then follows. ■

Since there exists a (K, G) -design if and only if a (K^*, G^*) -design exists, we arrive at the following corollary of the above lemma.

Corollary 5. *Let D be an orientation of a simple graph G such that D is self-complementary in G^* . If there exists a (K, G) -design, then there exists a (K^*, D) -design.*

In light of Corollary 5, we can combine Theorem 3 and Example 11 (see Section 3) to characterize the spectra of the digraphs that are self-complementary in $(K_4 - e)^*$, namely D75, D87, D101, and D102 (as seen in Figure 1).

Theorem 6. *Let $D \in \{D75, D87, D101, D102\}$. There exists a (K_n^*, D) -design if and only if $n \equiv 0$ or $1 \pmod{5}$ and $n \geq 5$.*

2.2. Results for non-self-complementary digraphs

Our general constructions also use some basic results concerning decompositions of both complete graphs and complete multipartite graphs into complete graphs of orders 3 and 5. These are sometimes stated in the language of group divisible designs and/or pairwise balanced designs. Note that these background results concern graphs, as opposed to digraphs. Theorems 7, 8, and 9 can be found in the *Handbook of Combinatorial Designs* [8] (see [1] and [9]).

Theorem 7. *If n is odd, then a $\{K_3, K_5\}$ -decomposition of K_n exists.*

Theorem 8. *The necessary and sufficient conditions for the existence of a K_3 -decomposition of $K_{u \times m}$ are (i) $u \geq 3$, (ii) $(u - 1)m \equiv 0 \pmod{2}$, and (iii) $u(u - 1)m^2 \equiv 0 \pmod{6}$.*

Theorem 9. *If $u \geq 3$ and $u \equiv 0 \pmod{3}$, then there exists a K_3 -decomposition of $K_{u \times 2, 4}$.*

Our general constructions further rely on the following direct result of blowing up the vertices in the graphs of a decomposition. This well-known building block is a special case of Wilson’s Fundamental Construction.

Lemma 10. *Let $m, r, s, t, u_1, u_2, \dots, u_m$ all be positive integers. If there exists a $\{K_r, K_s\}$ -decomposition of K_{u_1, u_2, \dots, u_m} , then there exists a $\{K_{r \times t}, K_{s \times t}\}$ -decomposition of $K_{tu_1, tu_2, \dots, tu_m}$. In particular, if there exists a $(K_{u_1, u_2, \dots, u_m}, K_r)$ -design, then there exists a $(K_{tu_1, tu_2, \dots, tu_m}, K_{r \times t})$ -design.*

3. EXAMPLES OF SMALL DESIGNS

We now turn our attention to the designs of small order which will be used for the general constructions.

Given a digraph represented by the notation $D[a, b, c, d]$ and some $i \in \mathbb{Z}_n$, we define $D[a, b, c, d] + i = D[a + i, b + i, c + i, d + i]$ where all addition is performed in \mathbb{Z}_n . By convention, define $\infty + 1 = \infty$.

Example 11. There exists a (K_5^*, D) -design for $D \in \{D74, D75, D85, D87, D89, D101, D102, D105\}$.

Let $V(K_5^*) = \mathbb{Z}_4 \cup \{\infty\}$.

A $(K_5^*, D74)$ -design is given by $\{D74[0, \infty, 2, 1] + i : i \in \mathbb{Z}_4\}$.

A $(K_5^*, D75)$ -design is given by $\{D75[0, \infty, 3, 1] + i : i \in \mathbb{Z}_4\}$.

A $(K_5^*, D85)$ -design is given by $\{D85[0, \infty, 2, 1] + i : i \in \mathbb{Z}_4\}$.

A $(K_5^*, D87)$ -design is given by

$$\{D87[0, \infty, 1, 2], D87[0, 3, 2, \infty], D87[3, \infty, 2, 1], D87[3, 0, 1, \infty]\}.$$

A $(K_5^*, D101)$ -design is given by $\{D101[0, \infty, 3, 1] + i : i \in \mathbb{Z}_4\}$.

A $(K_5^*, D102)$ -design is given by

$$\{D102[0, \infty, 1, 2], D102[1, 3, 0, \infty], D102[2, \infty, 3, 0], D102[3, 1, 2, \infty]\}.$$

Applying Observation 2, we obtain the remaining designs.

Example 12. There exists a (K_6^*, D) -design for $D \in \{D69, D74, D85, D89, D90, D105\}$.

Let $V(K_6^*) = \mathbb{Z}_6$.

A $(K_6^*, D69)$ -design is given by $\{D69[0, 2, 1, 4] + i : i \in \mathbb{Z}_6\}$.

A $(K_6^*, D74)$ -design is given by $\{D74[0, 5, 1, 3] + i : i \in \mathbb{Z}_6\}$.

A $(K_6^*, D85)$ -design is given by $\{D85[0, 3, 5, 4] + i : i \in \mathbb{Z}_6\}$.

Applying Observation 2, we obtain the remaining designs.

Example 13. There exists a (K_{10}^*, D) -design for $D \in \{D74, D85, D89, D105\}$.

Let $V(K_{10}^*) = \mathbb{Z}_9 \cup \{\infty\}$.

A $(K_{10}^*, D74)$ -design is given by

$$\{D74[0, \infty, 4, 6] + i : i \in \mathbb{Z}_9\} \cup \{D74[0, 8, 1, 4] + i : i \in \mathbb{Z}_9\}.$$

A $(K_{10}^*, D85)$ -design is given by

$$\{D85[0, \infty, 4, 7] + i : i \in \mathbb{Z}_9\} \cup \{D85[0, 6, 1, 2] + i : i \in \mathbb{Z}_9\}.$$

Applying Observation 2, we obtain the remaining designs.

Example 14. There exists a (K_{11}^*, D) -design for $D \in \{D69, D74, D85, D89, D90, D105\}$.

Let $V(K_{11}^*) = \mathbb{Z}_{11}$.

A $(K_{11}^*, D69)$ -design is given by

$$\{D69[0, 2, 7, 5] + i : i \in \mathbb{Z}_{11}\} \cup \{D69[0, 1, 4, 3] + i : i \in \mathbb{Z}_{11}\}.$$

A $(K_{11}^*, D74)$ -design is given by

$$\{D74[0, 7, 1, 3] + i : i \in \mathbb{Z}_{11}\} \cup \{D74[0, 4, 10, 8] + i : i \in \mathbb{Z}_{11}\}.$$

A $(K_{11}^*, D85)$ -design is given by

$$\{D85[0, 6, 10, 7] + i : i \in \mathbb{Z}_{11}\} \cup \{D85[0, 9, 8, 2] + i : i \in \mathbb{Z}_{11}\}.$$

Applying Observation 2, we obtain the remaining designs.

Example 15. There exists a (K_{20}^*, D) -design for $D \in \{D74, D85, D89, D105\}$.

Let $V(K_{20}^*) = \mathbb{Z}_{19} \cup \{\infty\}$.

A $(K_{20}^*, D74)$ -design is given by

$$\begin{aligned} & \{D74[0, \infty, 13, 2] + i : i \in \mathbb{Z}_{19}\} \cup \{D74[0, 12, 1, 10] + i : i \in \mathbb{Z}_{19}\} \\ & \cup \{D74[0, 14, 16, 12] + i : i \in \mathbb{Z}_{19}\} \cup \{D74[0, 15, 18, 13] + i : i \in \mathbb{Z}_{19}\}. \end{aligned}$$

A $(K_{20}^*, D85)$ -design is given by

$$\begin{aligned} & \{D85[0, \infty, 2, 16] + i : i \in \mathbb{Z}_{19}\} \cup \{D85[0, 12, 1, 6] + i : i \in \mathbb{Z}_{19}\} \\ & \cup \{D85[0, 13, 17, 7] + i : i \in \mathbb{Z}_{19}\} \cup \{D85[0, 3, 18, 10] + i : i \in \mathbb{Z}_{19}\}. \end{aligned}$$

Applying Observation 2, we obtain the remaining designs.

Example 16. There exists a (K_{21}^*, D) -design for $D \in \{D69, D74, D85, D89, D90, D105\}$.

Let $V(K_{21}^*) = \mathbb{Z}_{21}$.

A $(K_{21}^*, D69)$ -design is given by

$$\begin{aligned} & \{D69[0, 4, 1, 12] + i : i \in \mathbb{Z}_{21}\} \cup \{D69[0, 7, 2, 16] + i : i \in \mathbb{Z}_{21}\} \\ & \cup \{D69[0, 6, 19, 13] + i : i \in \mathbb{Z}_{21}\} \cup \{D69[0, 17, 20, 9] + i : i \in \mathbb{Z}_{21}\}. \end{aligned}$$

A $(K_{21}^*, D74)$ -design is given by

$$\begin{aligned} & \{D74[0, 13, 4, 14] + i : i \in \mathbb{Z}_{21}\} \cup \{D74[0, 2, 5, 20] + i : i \in \mathbb{Z}_{21}\} \\ & \cup \{D74[0, 18, 16, 1] + i : i \in \mathbb{Z}_{21}\} \cup \{D74[0, 8, 17, 7] + i : i \in \mathbb{Z}_{21}\}. \end{aligned}$$

A $(K_{21}^*, D85)$ -design is given by

$$\begin{aligned} & \{D85[0, 18, 1, 15] + i : i \in \mathbb{Z}_{21}\} \cup \{D85[0, 17, 2, 7] + i : i \in \mathbb{Z}_{21}\} \\ & \cup \{D85[0, 16, 19, 11] + i : i \in \mathbb{Z}_{21}\} \cup \{D85[0, 12, 20, 9] + i : i \in \mathbb{Z}_{21}\}. \end{aligned}$$

Applying Observation 2, we obtain the remaining designs.

Example 17. There exists a (K_{25}^*, D) -design for $D \in \{D69, D90\}$.

Let $V(K_{25}^*) = \mathbb{Z}_5 \times \mathbb{Z}_5$. A $(K_{25}^*, D69)$ -design is given by

$$\begin{aligned} & \{D69[(1, i), (0, 1 + i), (1, 2 + i), (0, i)] : i \in \mathbb{Z}_5\} \\ & \cup \{D69[(2, i), (0, 1 + i), (2, 2 + i), (0, i)] : i \in \mathbb{Z}_5\} \\ & \cup \{D69[(3, i), (0, 1 + i), (3, 2 + i), (0, i)] : i \in \mathbb{Z}_5\} \\ & \cup \{D69[(0, 1 + i), (1, i), (4, i), (0, i)] : i \in \mathbb{Z}_5\} \\ & \cup \{D69[(0, 2 + i), (1, i), (4, 3 + i), (0, i)] : i \in \mathbb{Z}_5\} \\ & \cup \{D69[(1, 3 + i), (1, i), (0, 3 + i), (0, i)] : i \in \mathbb{Z}_5\} \\ & \cup \{D69[(2, 1 + i), (1, 1 + i), (2, 4 + i), (1, i)] : i \in \mathbb{Z}_5\} \\ & \cup \{D69[(3, i), (1, 1 + i), (3, 3 + i), (1, i)] : i \in \mathbb{Z}_5\} \end{aligned}$$

$$\begin{aligned}
& \cup \{D69[(0, 4 + i), (2, 1 + i), (2, 2 + i), (1, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(1, 1 + i), (2, i), (4, 2 + i), (1, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(0, 2 + i), (2, 2 + i), (0, 1 + i), (2, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(1, i), (2, 1 + i), (3, 1 + i), (2, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(3, 3 + i), (2, 1 + i), (4, i), (2, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(3, 4 + i), (4, i), (1, 2 + i), (2, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(0, 3 + i), (4, i), (3, 1 + i), (3, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(0, 4 + i), (4, 2 + i), (4, 1 + i), (3, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(1, 1 + i), (4, i), (1, i), (3, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(1, 4 + i), (4, 1 + i), (4, 3 + i), (3, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(2, 4 + i), (4, 3 + i), (1, 2 + i), (3, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(2, 1 + i), (4, 1 + i), (2, i), (3, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(2, 3 + i), (4, 1 + i), (4, 4 + i), (3, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(2, 2 + i), (4, 4 + i), (4, i), (3, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(3, 4 + i), (4, 4 + i), (0, i), (3, i)] : i \in \mathbb{Z}_5\} \\
& \cup \{D69[(3, 3 + i), (4, 1 + i), (0, i), (4, i)] : i \in \mathbb{Z}_5\}.
\end{aligned}$$

Applying Observation 2, we obtain a $(K_{25}^*, D90)$ -design.

Example 18. There exists a (K_{30}^*, D) -design for $D \in \{D69, D90\}$.

Let $V(K_{30}^*) = \mathbb{Z}_5 \times \mathbb{Z}_6$.

A $(K_{30}^*, D69)$ -design is given by

$$\begin{aligned}
& \{D69[(1, i), (0, 1 + i), (1, 2 + i), (0, i)] : i \in \mathbb{Z}_6\} \\
& \cup \{D69[(1, 4 + i), (0, 1 + i), (2, i), (0, i)] : i \in \mathbb{Z}_6\} \\
& \cup \{D69[(2, 2 + i), (0, 1 + i), (2, 4 + i), (0, i)] : i \in \mathbb{Z}_6\} \\
& \cup \{D69[(3, i), (0, 1 + i), (3, 2 + i), (0, i)] : i \in \mathbb{Z}_6\} \\
& \cup \{D69[(3, 4 + i), (0, 1 + i), (4, i), (0, i)] : i \in \mathbb{Z}_6\} \\
& \cup \{D69[(0, 1 + i), (1, i), (4, 2 + i), (0, i)] : i \in \mathbb{Z}_6\} \\
& \cup \{D69[(0, i), (1, 1 + i), (0, 3 + i), (1, i)] : i \in \mathbb{Z}_6\} \\
& \cup \{D69[(0, 4 + i), (2, i), (4, 1 + i), (1, i)] : i \in \mathbb{Z}_6\} \\
& \cup \{D69[(1, 1 + i), (2, i), (2, 1 + i), (1, i)] : i \in \mathbb{Z}_6\} \\
& \cup \{D69[(1, 4 + i), (2, i), (2, 3 + i), (1, i)] : i \in \mathbb{Z}_6\} \\
& \cup \{D69[(1, 3 + i), (2, i), (3, i), (1, i)] : i \in \mathbb{Z}_6\} \\
& \cup \{D69[(1, 5 + i), (2, i), (3, 4 + i), (1, i)] : i \in \mathbb{Z}_6\}
\end{aligned}$$

$$\begin{aligned}
 &\cup \{D69[(2, 4 + i), (2, i), (3, 1 + i), (1, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(3, 2 + i), (2, i), (4, i), (1, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(4, 3 + i), (2, i), (4, 4 + i), (1, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(0, i), (2, 1 + i), (0, 2 + i), (2, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(3, 5 + i), (3, i), (2, 5 + i), (2, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(0, i), (3, 1 + i), (0, 4 + i), (3, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(0, 2 + i), (4, 1 + i), (4, i), (3, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(1, 4 + i), (4, i), (4, 3 + i), (3, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(2, 4 + i), (4, 1 + i), (0, 1 + i), (3, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(2, 1 + i), (4, 3 + i), (1, 3 + i), (3, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(2, 2 + i), (4, 3 + i), (4, 1 + i), (3, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(2, 3 + i), (4, 3 + i), (4, 5 + i), (3, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(3, 1 + i), (4, 2 + i), (1, 1 + i), (3, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(3, 3 + i), (4, 5 + i), (1, 2 + i), (3, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(0, 5 + i), (4, 2 + i), (0, 4 + i), (4, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(2, 2 + i), (4, 1 + i), (1, 2 + i), (4, i)] : i \in \mathbb{Z}_6\} \\
 &\cup \{D69[(3, 3 + i), (4, 1 + i), (3, 1 + i), (4, i)] : i \in \mathbb{Z}_6\}.
 \end{aligned}$$

Applying Observation 2, we obtain a $(K_{30}^*, D90)$ -design.

Example 19. There exists a $(K_{3 \times 5}^*, D)$ -design for $D \in \{D69, D74, D85, D89, D90, D105\}$.

Let $V(K_{3 \times 5}^*) = \mathbb{Z}_{15}$ with vertex partition $\{V_i : i \in \mathbb{Z}_3\}$, where $V_i = \{j \in \mathbb{Z}_{15} : j \equiv i \pmod{3}\}$.

A $(K_{3 \times 5}^*, D69)$ -design is given by

$$\{D69[0, 8, 10, 11] + i : i \in \mathbb{Z}_{15}\} \cup \{D69[0, 4, 5, 7] + i : i \in \mathbb{Z}_{15}\}.$$

A $(K_{3 \times 5}^*, D74)$ -design is given by

$$\{D74[0, 2, 10, 11] + i : i \in \mathbb{Z}_{15}\} \cup \{D74[0, 13, 5, 4] + i : i \in \mathbb{Z}_{15}\}.$$

A $(K_{3 \times 5}^*, D85)$ -design is given by

$$\{D85[0, 7, 5, 1] + i : i \in \mathbb{Z}_{15}\} \cup \{D85[0, 8, 10, 14] + i : i \in \mathbb{Z}_{15}\}.$$

Applying Observation 2, we obtain the remaining designs.

Example 20. There exists a $(K_{5 \times 5}^*, D)$ -design for $D \in \{D69, D74, D85, D89, D90, D105\}$.

First, let $V(K_{5 \times 5}^*) = \mathbb{Z}_{25}$ with vertex partition $\{V_i : i \in \mathbb{Z}_5\}$, where $V_i = \{j \in \mathbb{Z}_{25} : j \equiv i \pmod{5}\}$.

A $(K_{5 \times 5}^*, D69)$ -design is given by

$$\begin{aligned} & \{D69[1, 14, 0, 24] + i : i \in \mathbb{Z}_{25}\} \cup \{D69[2, 8, 0, 18] + i : i \in \mathbb{Z}_{25}\} \\ & \cup \{D69[3, 12, 0, 22] + i : i \in \mathbb{Z}_{25}\} \cup \{D69[4, 11, 0, 21] + i : i \in \mathbb{Z}_{25}\}. \end{aligned}$$

A $(K_{5 \times 5}^*, D90)$ -design follows from Observation 2.

Next, let $D \in \{D74, D89, D105, D85\}$. A $(K_{5 \times 5}, K_5)$ -design can be obtained by removing one parallel class from an affine plane of order 5. Thus, there exists a $(K_{5 \times 5}^*, K_5^*)$ -design. Since a (K_5^*, D) -design exists by Example 11, the desired $(K_{5 \times 5}^*, D)$ -design exists.

4. MAIN RESULTS

We first show some nonexistence results for $(D69, K_n^*)$ - and $(D90, K_n^*)$ -designs. Interestingly for $n \equiv 0 \pmod{5}$, these designs do not exist for $n \in \{5, 10, 15, 20\}$ (see Theorem 21) but do exist for $n \in \{25, 30\}$ (see Examples 17 and 18). By Wilson’s Theorem [14], there exists an integer n_0 such that for all $n \geq n_0$ that satisfy the necessary conditions there exists both a $(D69, K_n^*)$ -design and a $(D90, K_n^*)$ -design. We conjecture that $n_0 = 25$ for this pair of digraphs.

Theorem 21. *There does not exist a D69- or D90-decomposition of K_{5k}^* for $1 \leq k \leq 4$.*

Proof. We prove by contradiction that a D69-decomposition of K_{5k}^* cannot exist. Note that by Observation 2, a D90-decomposition must also not exist.

Let Δ be a D69-decomposition of K_{5k}^* . Given a vertex $v \in V(K_{5k}^*)$, let $n_w(v)$ denote the number of D69-blocks in Δ where vertex w in $D69[w, x, y, z]$ is identified with vertex v . Define $n_x(v)$, $n_y(v)$, and $n_z(v)$ similarly. Thus, the following must hold:

$$\begin{aligned} 1n_w(v) + 2n_x(v) + 0n_y(v) + 2n_z(v) &= 5k - 1, \\ 2n_w(v) + 0n_x(v) + 3n_y(v) + 0n_z(v) &= 5k - 1. \end{aligned}$$

Substituting $\bar{n}(v) = n_x(v) + n_z(v)$, the above equations can be parameterized as

$$\begin{aligned} (1) \quad & n_w(v) = 5k - 1 - 2\bar{n}(v), \\ (2) \quad & n_y(v) = -\frac{1}{3}(5k - 1 - 4\bar{n}(v)). \end{aligned}$$

Since $\bar{n}(v)$, $n_w(v)$, and $n_y(v)$ must all be nonnegative integers, we have that

$$\left. \begin{aligned} 0 &\leq 5k - 1 - 2\bar{n}(v) \\ 0 &\leq -\frac{1}{3}(5k - 1 - 4\bar{n}(v)) \end{aligned} \right\} \implies \frac{1}{4}(5k - 1) \leq \bar{n}(v) \leq \frac{1}{2}(5k - 1).$$

Furthermore, equation (2) implies that $5k - 1 - 4\bar{n}(v)$ must be a multiple of 3; hence, $k + \bar{n}(v) + 1 \equiv 0 \pmod{3}$.

Next, consider the case when $k = 1$. The above conditions require that for every $v \in V(K_5^*)$, we have $1 \leq \bar{n}(v) \leq 2$ and $\bar{n}(v) \equiv 1 \pmod{3}$. Thus, $\bar{n}(v)$ can only equal 1, and by equation (1), $n_w(v) = 2$ for every $v \in V(K_5^*)$. However, this would imply $|\Delta| = 10$, which is a contradiction (because $|\Delta| = 4$ when $k = 1$). Similarly if k is 2, 3, or 4, then $\bar{n}(v)$ can only equal 3, 5, or 7, respectively, which further yields only one value for $n_w(v)$: 3, 4, or 5, respectively, for every $v \in V(K_{5k}^*)$. However, this would imply $|\Delta|$ is a multiple of $5k$, which is a contradiction because

$$|\Delta| = \frac{|E(K_{5k}^*)|}{|E(D69)|} = \frac{5k(5k - 1)}{5} = k(5k - 1),$$

which is not divisible by $5k$. ■

Next we turn our attention to developing the general constructions needed to piece together the small designs presented in Section 3 and show sufficiency of the necessary conditions for the remaining four non-self-complementary digraphs.

Theorem 22. *Let $D \in \{D74, D85, D89, D105\}$. If $n \equiv 0 \pmod{5}$ with $n \geq 5$, then a (K_n^*, D) -design exists.*

Proof. Let $D \in \{D74, D85, D89, D105\}$ and let $n \equiv 0$ or $5 \pmod{10}$.

Case 1. $n \equiv 0 \pmod{10}$. Let $n = 10x = 5(2x)$ for some positive integer x . When x is 1 or 2 the result follows from Examples 13 and 15, respectively, so we now consider when $x \geq 3$. Let H_1, H_2, \dots, H_x be disjoint sets of 2 vertices each.

Subcase 1a. $x \equiv 0$ or $1 \pmod{3}$. Let $K_{x \times 2}$ have vertex partition $\{H_i : 1 \leq i \leq x\}$. By Theorem 8, a $(K_{x \times 2}, K_3)$ -design exists. Therefore, by Lemma 10 a $(K_{x \times 10}, K_{3 \times 5})$ -design exists. Let H'_i be the set obtained from H_i after blowing up each vertex in $K_{x \times 2}$ by 5. Now consider K_n^* to have vertex set $\bigcup_{i=1}^x H'_i$ where each H'_i induces a K_{10}^* . Thus, K_n^* decomposes into copies of K_{10}^* and $K_{3 \times 5}^*$. Since both a (K_{10}^*, D) -design and a $(K_{3 \times 5}^*, D)$ -design exist by Examples 13 and 19, respectively, we have our desired (K_n^*, D) -design.

Subcase 1b. $x \equiv 2 \pmod{3}$. Let $H_0 = H_{x-1} \cup H_x$ and let $K_{(x-2) \times 2, 4}$ have vertex partition $\{H_i : 0 \leq i \leq x - 2\}$. By Theorem 9, a $(K_{(x-2) \times 2, 4}, K_3)$ -design exists. Therefore, by Lemma 10 a $(K_{(x-2) \times 10, 20}, K_{3 \times 5})$ -design exists. Let H'_i be the set obtained from H_i after blowing up each vertex in $K_{(x-2) \times 2, 4}$ by 5. Now consider K_n^* to have vertex set $\bigcup_{i=0}^{x-2} H'_i$ where H'_0 induces a K_{20}^* and, for $1 \leq i \leq x - 2$, each H'_i induces a K_{10}^* . Thus, K_n^* decomposes into copies of K_{10}^* , K_{20}^* , and $K_{3 \times 5}^*$. Since a (K_{10}^*, D) -design, a (K_{20}^*, D) -design, and a $(K_{3 \times 5}^*, D)$ -design exist by Examples 13, 15, and 19, respectively, we have our desired (K_n^*, D) -design.

Case 2. $n \equiv 5 \pmod{10}$. Let $n = 10x + 5 = 5(2x + 1)$ for some positive integer x . Let $H_1, H_2, \dots, H_{2x+1}$ be sets consisting of a single vertex each. By Theorem 7 a $\{K_3, K_5\}$ -decomposition of K_{2x+1} exists. Therefore, by Lemma 10 a $\{K_{3 \times 5}, K_{5 \times 5}\}$ -decomposition of $K_{(2x+1) \times 5}$ exists. Let H'_i be the set obtained from H_i after blowing up each vertex in K_{2x+1} by 5. Now consider K_n^* to have vertex set $\bigcup_{i=1}^{2x+1} H'_i$ where each H'_i induces a K_5^* . Thus, K_n^* decomposes into copies of K_5^* , $K_{3 \times 5}^*$, and $K_{5 \times 5}^*$. Since a (K_5^*, D) -design, a $(K_{3 \times 5}^*, D)$ -design, and a $(K_{5 \times 5}^*, D)$ -design all exist by Examples 11, 19, and 20, respectively, we have our desired (K_n^*, D) -design. ■

Theorem 23. *Let $D \in \{D69, D74, D85, D89, D90, D105\}$. If $n \equiv 1 \pmod{5}$ with $n \geq 6$, then a (K_n^*, D) -design exists.*

Proof. Let $D \in \{D69, D74, D85, D89, D90, D105\}$ and let $n \equiv 1$ or $6 \pmod{10}$.

Case 1. $n \equiv 1 \pmod{10}$. Let $n = 10x + 1 = 5(2x) + 1$ for some positive integer x . When x is 1 or 2 the result follows from Examples 14 and 16, respectively, so we now consider when $x \geq 3$.

Subcase 1a. $x \equiv 0$ or $1 \pmod{3}$. Here we can consider $V(K_n^*) = \left(\bigcup_{i=1}^x H'_i\right) \cup \{\infty\}$, where each H'_i is defined as in Subcase 1a of the proof of Theorem 22 with the modification that each $H'_i \cup \{\infty\}$ induces a K_{11}^* . Similarly to the proof of that Subcase 1a, the desired (K_n^*, D) -design can be constructed using (K_{11}^*, D) -designs—in place of (K_{10}^*, D) -designs—along with $(K_{3 \times 5}^*, D)$ -designs, which exist by Examples 14 and 19, respectively.

Subcase 1b. $x \equiv 2 \pmod{3}$. Here we can consider $V(K_n^*) = \left(\bigcup_{i=0}^{x-2} H'_i\right) \cup \{\infty\}$, where each H'_i is defined as in Subcase 1b of the proof of Theorem 22 with the modifications that $H'_0 \cup \{\infty\}$ induces a K_{21}^* and, for $1 \leq i \leq x - 2$, each $H'_i \cup \{\infty\}$ induces a K_{11}^* . Similarly to the proof of that Subcase 1b, the desired (K_n^*, D) -design can be constructed using (K_{11}^*, D) -designs and a (K_{21}^*, D) -design—in place of (K_{10}^*, D) - and (K_{20}^*, D) -designs—along with $(K_{3 \times 5}^*, D)$ -designs, which exist by Examples 14, 16, and 19, respectively.

Case 2. $n \equiv 6 \pmod{10}$. Here we can consider $V(K_n^*) = \left(\bigcup_{i=1}^{2x+1} H'_i\right) \cup \{\infty\}$, where each H'_i is defined as in Case 2 of the proof of Theorem 22 with the modification that $H'_i \cup \{\infty\}$ induces a K_6^* . Similarly to the proof of that Case 2, the desired (K_n^*, D) -design can be constructed by using (K_6^*, D) -designs—in place of (K_5^*, D) -designs—along with $(K_{3 \times 5}^*, D)$ -designs and $(K_{5 \times 5}^*, D)$ -designs, which exist by Examples 12, 19, and 20. ■

Our results from this section along with those in Theorem 6 are summarized in the following two main theorems.

Theorem 24. *Let $D \in \{D74, D75, D85, D87, D89, D101, D102, D105\}$. There exists a (K_n^*, D) -design if and only if $n \equiv 0$ or $1 \pmod{5}$ and $n \geq 5$.*

Theorem 25. *Let $D \in \{D69, D90\}$. There exists a (K_n^*, D) -design if $n \equiv 1 \pmod{5}$ and $n \geq 6$.*

Finally, we formally state our conjecture regarding the open results for the $\{D69, D90\}$ pair of digraphs.

Conjecture 26. *Let $D \in \{D69, D90\}$. There exists a (K_n^*, D) -design if $n \equiv 0 \pmod{5}$ and $n \geq 25$.*

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