Investigating Spanning Bipartite Subgraphs of $^2K_5$

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Investigation Objective

During this lesson the student will work with the terminology used within Graph Theory as a field of Discrete Mathematics while investigating spanning bipartite subgraphs $^2K_5$. The students will construct all the spanning bipartite subgraphs of $^2K_5$ and then determine for each of the spanning bipartite subgraphs if there exists (a) an ordered; (b) a graceful; and/or (c) a graceful ordered labeling for the ^2K_n graph that the given spanning bipartite subgraph decomposes.

Teacher Investigation Notes

Step 1: Vocabulary Review

Objective: The students will be able to recall the definitions of the vocabulary that are useful for this module.

Lesson: The students will first match the vocabulary words with the respective definitions. The words isomorphic, edge length, bipartite, subgraph, spanning subgraph, complete graphs, incident, degree, and regular graph will be used. These words should have been learned during the introduction, but they are particularly useful to this module, so it will be necessary that the students remember them. The students will then investigate K_5, reviewing properties necessary to the investigation while telling you everything they know about K_5. If they do not mention anything about order or it being regular, prompt the students by asking how many edges are incident to the vertices or what the degrees of the vertices are. Once they realize that it is a regular degree 4 graph, prompt them to remember it is complete by asking what it means to have degree 4 at each vertex on a graph with five vertices. This should lead to them realizing that each vertex is incident to every other vertex making the graph complete.

Materials:
Student Vocabulary Worksheet found on page 7.
Student Vocabulary Worksheet ANSWER KEY found on page 8.

Step 2: Investigation of K_5

Objective: Students will investigate K_5 to further understand its properties and find all the subgraphs of K_5. Once they find the subgraphs, the students will be able to determine which of those subgraphs are spanning, bipartite, and/or complete.

Lesson: With K_5 as a reference, students should draw as many subgraphs of K_5 as possible and hopefully they can come up with them all. Discussion should happen first that in order to be a subgraph of K_5 a graph can have at most 5 vertices meaning students will be able to play with any combination of graphs with 1, 2, 3, 4, or 5 vertices. Seeing as five vertices are possible, though not all have to be used, mathematicans say that they are looking for all graphs of order 5. If a graph is of order 5 then we know it is a subgraph of K_5. NOTE: If a graph is of order 5 that does not imply that every vertex need be incident with an edge.

In an effort to help students discussion should center around the fact that you can start with a 'blank template' of five vertices and then try to find how many different ways can you use the vertices to gradually go from an empty graph to a complete graph. The students will be provided with a worksheet with 34 boxes and 5 vertices in each box. There are 34 graphs of order 5, 33 of which are true subgraphs of K_5; the 34^{th} graph is K_5. This worksheet has been differentiated as explained in the materials section below.

Once the students receive the appropriate worksheet, they will start drawing all the subgraphs that they can find. The students should first work to see what subgraphs they are able to find on their own. It is suggested that at some point in the activity the students then work in teams to find the subgraphs and as a team keep a master list of the ones found. This will help the students keep organized and help each other when checking to see if a graph found by a student is isomorphic to one already found. An answer key has been provided which outlines how many graphs of various sizes exist and what all the graphs are. This worksheet has been provided to help the teacher in assisting students should they get stuck.

After drawing all of the subgraphs of K_5, making sure they are all nonisomorphic, discussion should now center on the properties each graph of order five holds. Students should indicate for each graph if it is bipartite, spanning, and/or complete. While the students have a check list below each graph they will have an additional worksheet to record information and answer some additional questions. It is important to note that for the purposes of this full investigation the bipartite and spanning properties are critical. In an effort to help students determine bipartite they can try to color each graph with only two colors ensuring each edge has a different color at each end. Once the students have determined which graphs hold the bipartite, spanning, and connected graphs, the students will determine which graphs are both spanning and bipartite. They will then check their spanning bipartite graphs against the six shown to see which, if any match. NOTE: Their six spanning bipartite graphs should be isomorphic to the six on the worksheet and if they did everything correctly they will notice all six match.

Materials:

Find All Graphs of Order 5 Worksheet

Blank Version found on page 9.

This worksheet has 34 blank graphs with five vertices and the students need to find all the graphs of order five using any combination of 0-5 vertices with 0 - 10 edges. The students also have blank spaces for the students to classify the graphs as bipartite, spanning and/or connected. This worksheet is suggested for students who will persevere through the puzzle of finding all 34.

Size Hint Version found on page 10.

This worksheet has 34 blank graphs with five vertices and the students need to find all the graphs of order five. For each graph students are given a hint as to how many edges are needed and it is up to them to determine how to use that many edges with different vertex combinations. The students also have blank spaces for the students to classify the graphs as bipartite, spanning and/or connected.

Classification Version found on page 11.

This worksheet has all graphs of order 5 listed with blank spaces for the students to classify the graphs as bipartite, spanning, and/or connected.

Answer Key found on page 12.

Working with Classifications Worksheets

Working to Classify the Graphs of Order 5 found on page 13.

First give students the *Working to Classify the Graphs of Order 5* worksheet. If you have the students classify the graphs from the *Classification Version* worksheet found on page 11 then every student should have identical answers. If they classify the graph using their own drawings from the *Blank Version* or *Size Hint Version* on pages 9 and 10 respectively then they will have different answers for each classification and the same amount of graphs for each classification.

Matching Your 6 Spanning Bipartite Graphs found on page 14

Once you have checked with students that they have classified all graphs correctly and they have drawn all the graphs necessary for question 7, give them the *Matching Your 6 Spanning Bipartite Graphs* worksheet.

Answer Key found on page 15.

This answer key is compatible with the graph numbering found on the classification version worksheet and answer key worksheet from the previous set of worksheets. Should the students have found all the graphs on their own with the blank worksheet or the size hint worksheet the numbers will be different, but the number of graphs fitting each classification will be the same.

Step 3: Investigation of $^2K_5$

Objective: The students will be able to discuss a $^2K_5$ graph using mathematically correct language. They will be able to transform all of the bipartite spanning subgraphs that were created in part two into two fold graphs.

Lesson: The students will be asked to discuss everything they know about a $^2K_5$ graph. They should be able to mention that every edge in the K_5 graph is doubled, but the number of vertices remains the same. They should also talk about it being a regular degree eight graph. They should say that it has ten edges of length 1 and ten edges of length 2. If they are unable to come up with any of these facts on their own, then it is suggested to use prompts such as, How many edges are incident with each vertex? and What are the edge lengths of this graph?

Once the students have come up with these facts, it will be time to ask them to pick a few of their subgraphs from part two and make them two fold. Teachers should then highlight students and share what two fold graphs were created. It would be good to also find within the student work two similar graph that were made two fold by different means. Once students have an understanding of making graphs two fold, appreciating that there is more than one way to make graphs two fold, focus will return to the six spanning bipartite graphs of K_5 found at the end of part 2.

Teachers need to work with students having them realize that the 6 graphs that span K_5 in a bipartite fashion are also the only six graphs that span $^2K_5$ in a bipartite fashion. They then should realize that these six graphs form the families of graphs that span $^2K_5$ in a bipartite fashion as it is possible for one to all of the edges in each graph to have multiplicity two and still be a spanning bipartite subgraph. Teachers should then work with the students with one family of graphs, preferably class B, developing a means of systematically attacking the problems while avoiding isomorphism, thus ensuring success. Once the class has worked through finding all the two fold possibilities for Class B . When this is complete, students will use the same or similar strategies to finish finding all sixty spanning subgraphs of $^2K_5$.

Materials:

Let's Investigate $^2K_5$ Worksheet found on page 16
Let's Investigate $^2K_5$ Worksheet ANSWER KEY found on page 17
Creating $^2K_5$ Spanning Bipartite Subgraphs worksheet found on page 18
$^2K_5$ Spanning Bipartite Subgraph Listing found on page 20

Step 4: Decomposition

Objective: The students will take all sixty spanning bipartite subgraphs of $^2K_5$ and cyclically decompose the appropriate ^2K_n with ideally an ordered graceful labeling or at the very least two labelings where one graceful and one ordered labeling.

Lesson: Now that the students have all 60 subgraphs of $^2K_5$ that are spanning and bipartite it is time to find cyclic-decompositions labelings. The students should count the number of edges present in each graph and then determine the appropriate ^2K_n that the graph decomposes. Some useful hints for students include:

1. Pair up the edges: 2 of length 1, 2 of length 2, 2 of length 3, etc. The students will stop with either 2 of the same length or one of the left over length. Students should refer back to the work with edge lengths. They should realize that:
   (a) if there are less edges of the longest length, the smaller lengths (in this case one of the longest length) then the graph will decompose an even graph, namely $^2K_{2n}$ where n is the longest length.
   (b) if there are the same number of edges of the longest length as the smaller lengths (in this case two of each length) then the graph will decompose an odd graph, namely $^2K_{2n+1}$ where n is the longest length.
2. When students get the hang of the first hint and are still seeking another, faster, method, have the students look for a connection between the number of edges present in the graph and the ^2K_n being decomposed. Hopefully they will notice that for n edges, the graph is decomposing $^2K_{n+1}$.

Once they know how many edges of each length are needed and the ^2K_n environment they are working in, they can determine labelings using the numbers 0 through $n-1$. For an ordered labeling, they will want all of the numbers on the top to be less than the numbers on the bottom that they are connected to by an edge. For graceful labelings, there must be no wrap-arounds. A graceful ordered labeling is the goal but some graphs do not emit this type of labeling, they emit graceful and ordered labelings separately, just not at the same time. It is suggested that teachers have the class start with the same three or four graphs and then share all the possible labelings. Some teachable moments that can occur:

1. Multiple students find different labelings. Have the students share the labelings then discuss:

(a) Are they all correct? Do they all have the same lengths?

(b) Which are ordered?

(c) Which are ordered if you rotate 180 degrees?

(d) Which are graceful?

(e) Which are ordered and graceful?

(f) Can you get from one labeling to another? Here you can get at clicking by first saying "Add 5 to every labeling (remembering for the labelings that go to n in ^2K_n, that turns into 0)." This should generate discussion.

2. Have students notice that some graphs have an ordered graceful labeling while other graphs do not. Have them also notice that you can have an orderd labeling and a graceful labeling independent of eachother.

3. Have students notice that is more than one answer and just because you find something, and find many cases that are identical, that doesn't imply it isn't possible. (i.e. You get ordered and graceful seperately) does not mean that you can't get ordered and graceful at the same time.)

IMPORTANT: When the students work on the labelings it is important to know that graph B-4,1(b) does not emit an ordered or a graceful cyclic decomposition labeling for $^2K_5$. While it is a subgraph of $^2K_5$ it does not cyclically decompose this graph. For more on why see the paper written on the research on which this module is based.

Student Investigation Notes

Step 1: Vocabulary Review

Objective: You will be able to recall the definitions of the vocabulary that are useful for this investigation.

Lesson: Today we are going to be working on a groundbreaking investigation into ${}^2K_5$ but first we must make sure that we have strong foundation in Graph Theory. As the budding mathematician you are you appreciate the beauty of mathematics and how as mathematicians we use precise language to communicate ideas across our global community. Since you will be embarking on this adventure into new uncharted mathematical territory a firm grasp of mathematical vocabulary is key!

To help ensure you have the necessary vocabulary skills to be successful in this adventure you will first practice what has been taught so far. On the vocabulary worksheet you will receive from your teacher there are two tasks for you. First you will match vocabulary words to their correct definition. Then you will work on drawing and discussing what you know about K_5.

Good luck young mathematician! A wonderful journey awaits.

Materials: Student Vocabulary Worksheet.

Step 2: Investigation of K_5

Objective: Students will investigate K_5 to further understand its properties and find all the subgraphs of K_5. Once they find the subgraphs, the students will be able to determine which of those subgraphs are spanning, bipartite, and/or complete.

Lesson: Today we are going to eventually be working in ${}^2K_5$ and looking at every spanning subgraph which is bipartite and has at least one 2-fold edge. But before we can run away with this investigation and explore this yet unexplored area of mathematics, we first need to walk into discovering what we need in our tool box.

We've just discussed what we know about K_5 and now it is time to look at graphs of order 5. Can you think of what the connection may be between K_5 and graphs of order 5? This is something you should think about as you work through the task at hand. Today you will be working through a series of worksheets as outlined below.

1. Using the first worksheet provided by your teacher you will:
   - A. First: Find all thirty-four graphs of order 5. I will give you a hint, you have already drawn one graph of order 5 and now you only need to find 33 more! When you feel like you're nearing a road block, don't give up, ask your teacher if you can work with a classmate. The trick will be putting your graph together and staying away from graphs that are ***isomorphic*** to each other!
   - B. Second: Classify which graphs are: **Bipartite** by placing a mark next to the B.
2. Using the next worksheet provided by your teacher you will analyze your classification groups.
3. Using the final worksheet provided by your teacher you will finally start working with the special classification of graphs that we are interested in today.

This step of the investigation is the most puzzling, but you will soon find it to also be the most rewarding. Good luck young mathematician!

Materials: Finding All Graphs of Order 5 Worksheet, Working to Classify Graphs of Order 5 Worksheet, Matching your Spanning Bipartite Graphs Worksheet.

Step 3: Investigation of $^2K_5$

Objective: The students will be able to discuss a $^2K_5$ graph using mathematically correct language. They will be able to transform all of the bipartite spanning subgraphs that were created in part two into two fold graphs.

Lesson: Today we are going to eventually be working in $^2K_5$ and looking at every spanning subgraph which is bipartite and has at least one 2-fold edge. But before we can run away with this investigation and explore this yet unexplored area of mathematics, we first need to walk into discovering what we need in our tool box. To start you will be answering everything you know about $^2K_5$. Once you have demonstrated your outstanding knowledge on this topic, you will then work with the six spanning bipartite subgraphs of K_5 which you found in step 2 and determine every possible way to make at least one edge 2-fold. Note, you will start by making one edge two fold and work up to having every edge present two-fold.

Materials: Investigating $^2K_5$ Worksheet, Creating $^2K_5$ Spanning Bipartite Subgraphs Worksheet.

Step 4: Decomposition

You've made it! With all the hard work leading up to this, you've made it to the final step, the original research step!

It is time to take all 60 spanning bipartite subgraphs of $^2K_5$ that have at least one edge of multiplicity two (one two fold edge) determine what graph ^2K_n it decomposes and find a labeling for that decomposition. Your ultimate goal is to find labelings that are both ordered and graceful. Remember ordered means that a number on a 'top' vertex is smaller than the number(s) on the vertex/vertices it is connected to on 'bottom.' Also recall that graceful for two fold graphs means that there are no wrap-around edges. Wrap-around edges come when the shortest distance from one vertex to another is found by going counter-clockwise, rather than simply finding the difference of the two vertex labels. The easiest way to make sure that you have graceful labelings is that the difference of each pair of labels used for each edge lengths is the actual length.

Good luck young mathematician! With hard work, perseverance, and the assistance of those around you, you will help advance mathematics and the study of graph theory into an entirely new world.

Exploring Subgraphs of ${}^2K_5$ Vocabulary Review

Today we will be exploring the idea of subraphs! But first, let us review some of the vocabulary that is necessary to the ideas of this investigation so we can communicate with mathematical precision and accuracy.

Match the vocab word with it's definition.

Vocabulary Word	Answer	Answer Choices
1. Complete Graph	_____	A. When two graphs can look the same after moving the vertices around.
2. Degree	_____	B. When a graph can be properly colored with exactly two colors.
3. Incident	_____	C. A graph whose vertex and edge sets are subsets of another graph's.
4. Isomorphic	_____	D. A graph that shares its vertex set with another graph while using a subset of the edge set of the other graph.
5. Bipartite	_____	E. A graph such that every vertex is adjacent to every other vertex.
6. Subgraph	_____	F. The relationship between a vertex and an edge.
7. Spanning Subgraph	_____	G. The number of edges incident with the vertex.
8. Adjacent	_____	H. Two edges that share a vertex or two vertices that share an edge.

Now, let's investigate K_5! Draw K_5 below.

Describe any attributes of K_5 you notice in relation to the following words.

Vertices

Edges

Degree

Adjacent/Incident

Are there any other attributes of K_5 that you didn't list above?

Exploring Subgraphs of ${}^2K_5$ Vocabulary Review

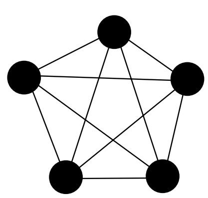
ANSWER KEY

Today we will be exploring the idea of subraphs! But first, let us review some of the vocabulary that is necessary to the ideas of this investigation so we can communicate with mathematical precision and accuracy.

Match the vocab word with it's definition.

Vocabulary Word	Answer	Answer Choices
1. Complete Graph	**E**	A. When two graphs can look the same after moving the vertices around.
2. Degree	**G**	B. When a graph can be properly colored with exactly two colors.
3. Incident	**F**	C. A graph whose vertex and edge sets are subsets of another graph's.
4. Isomorphic	**A**	D. A graph that shares its vertex set with another graph while using a subset of the edge set of the other graph.
5. Bipartite	**B**	E. A graph such that every vertex is adjacent to every other vertex.
6. Subgraph	**C**	F. The relationship between a vertex and an edge.
7. Spanning Subgraph	**D**	G. The number of edges incident with the vertex.
8. Adjacent	**H**	H. Two edges that share a vertex or two vertices that share an edge.

Now, let's investigate K_5! Draw K_5 below.



Describe any attributes of K_5 you notice in relation to the following words.

Vertices

There are 5 vertices. That is why $n = 5$.

Edges

There are 10 edges. The number of edges in a complete graph is $(n(n-1))/2$ where n is the number of vertices, which is why we know there are 10 edges in K_5.

Degree

The degree of each vertex is 4. This is a 4 regular graph, so the degree of each vertex is 4.

Adjacent/Incident

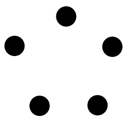
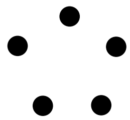
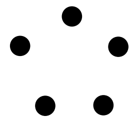
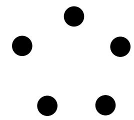
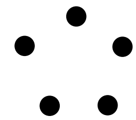
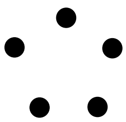
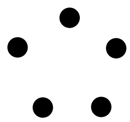
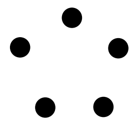
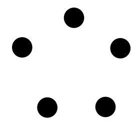
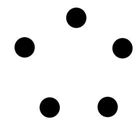
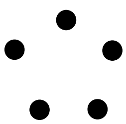
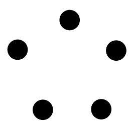
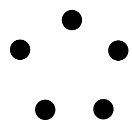
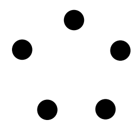
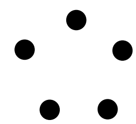
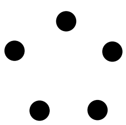
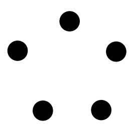
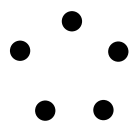
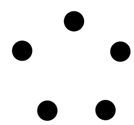
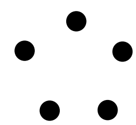
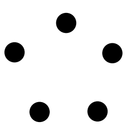
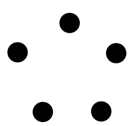
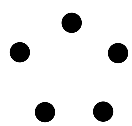
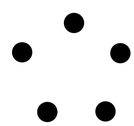
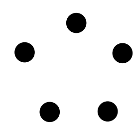
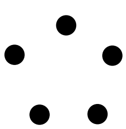
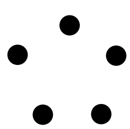
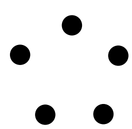
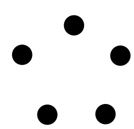
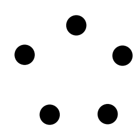
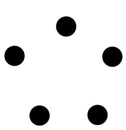
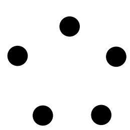
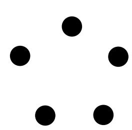
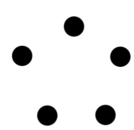
Every vertex is adjacent to every other vertex. Since this is a complete graph, every vertex is adjacent to every other vertex. Each vertex is incident to 4 edges.

Are there any other attributes of K_5 that you didn't list above?

For K5, there are five edges of length 1 and five edges of length 2. In order to number the vertices, $0-4$ are used.

Finding All Graphs of Order 5

Below you will find thirty-four graphs of order 5 and size 0. It is your task to find all possible graphs of order 5 with any possible combination of edges. Good luck!

1. B: ___ 	**2.** B: ___ 	**3.** B: ___ 	**4.** B: ___ 	**5.** B: ___ 
6. B: ___ 	**7.** B: ___ 	**8.** B: ___ 	**9.** B: ___ 	**10.** B: ___ 
11. B: ___ 	**12.** B: ___ 	**13.** B: ___ 	**14.** B: ___ 	**15.** B: ___ 
16. B: ___ 	**17.** B: ___ 	**18.** B: ___ 	**19.** B: ___ 	**20.** B: ___ 
21. B: ___ 	**22.** B: ___ 	**23.** B: ___ 	**24.** B: ___ 	**25.** B: ___ 
26. B: ___ 	**27.** B: ___ 	**28.** B: ___ 	**29.** B: ___ 	**30.** B: ___ 
31. B: ___ 	**32.** B: ___ 	**33.** B: ___ 	**34.** B: ___ 	

Once you have found every graph and have checked that you do have all 34 (Sorry! No accidental isomorphic repeats allowed) determine if the graph is ***bipartite (B)*** by placing checkmarks where necessary. You might want to check with your graphs with your teacher before you start!

Finding All Graphs of Order 5

Below you will find thirty-four graphs of order 5 and size 0. It is your task to find all possible graphs of order 5 with any possible combination of edges. Good luck!

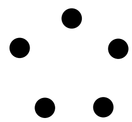
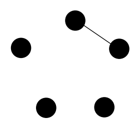
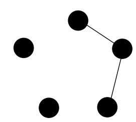
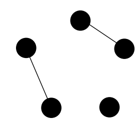
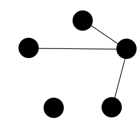
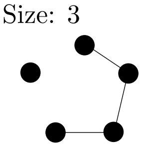
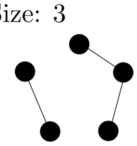
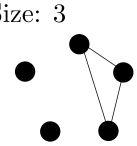
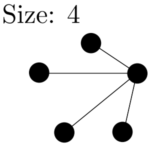
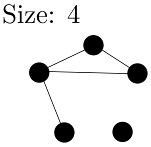
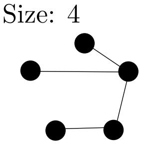
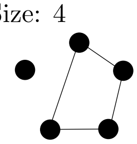
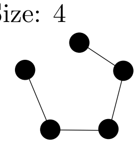
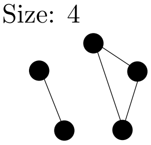
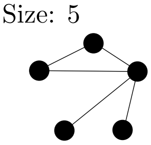
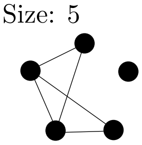
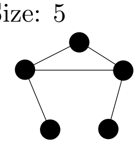
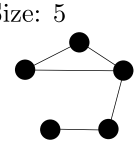
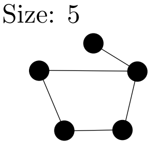
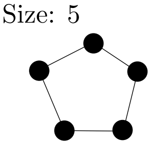
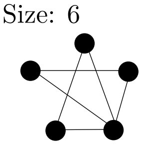
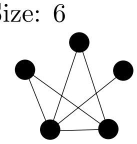
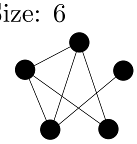
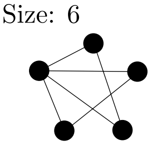
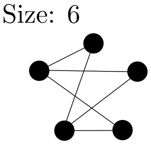
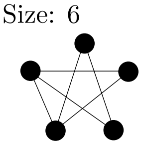
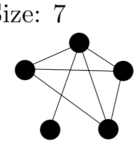
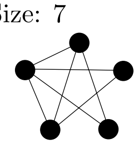
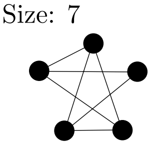
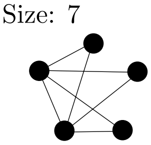
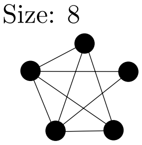
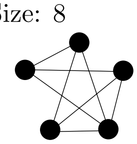
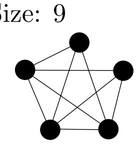
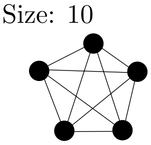
1. B: ___ Size: 0	**2.** B: ___ Size: 1	**3.** B: ___ Size: 2	**4.** B: ___ Size: 2	**5.** B: ___ Size: 3
6. B: ___ Size: 3	**7.** B: ___ Size: 3	**8.** B: ___ Size: 3	**9.** B: ___ Size: 4	**10.** B: ___ Size: 4
11. B: ___ Size: 4	**12.** B: ___ Size: 4	**13.** B: ___ Size: 4	**14.** B: ___ Size: 4	**15.** B: ___ Size: 5
16. B: ___ Size: 5	**17.** B: ___ Size: 5	**18.** B: ___ Size: 5	**19.** B: ___ Size: 5	**20.** B: ___ Size: 5
21. B: ___ Size: 6	**22.** B: ___ Size: 6	**23.** B: ___ Size: 6	**24.** B: ___ Size: 6	**25.** B: ___ Size: 6
26. B: ___ Size: 6	**27.** B: ___ Size: 7	**28.** B: ___ Size: 7	**29.** B: ___ Size: 7	**30.** B: ___ Size: 7
31. B: ___ Size: 8	**32.** B: ___ Size: 8	**33.** B: ___ Size: 9	**34.** B: ___ Size: 10	

Once you have found every graph and have checked that you do have all 34 (Sorry! No accidental isomorphic repeats allowed) determine if the graph is ***bipartite (B)*** by placing checkmarks where necessary. You might want to check with your graphs with your teacher before you start!

Finding All Graphs of Order 5

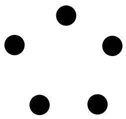
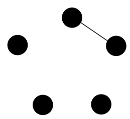
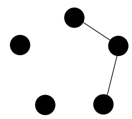
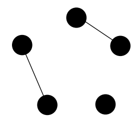
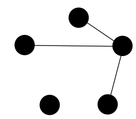
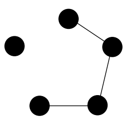
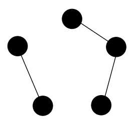
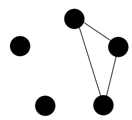
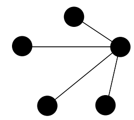
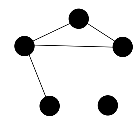
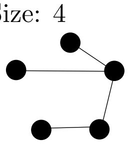
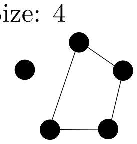
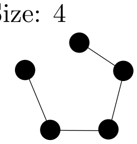
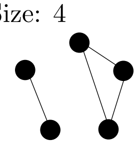
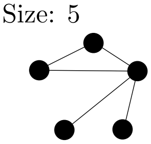
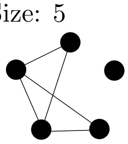
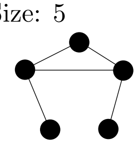
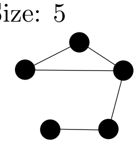
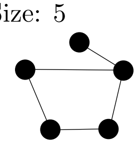
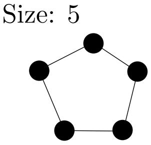
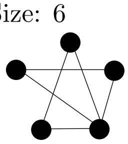
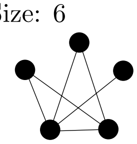
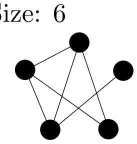
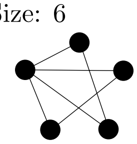
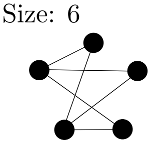
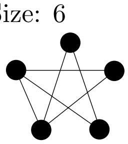
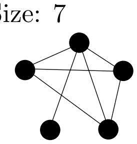
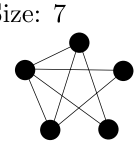
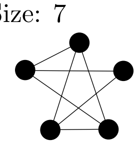
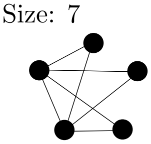
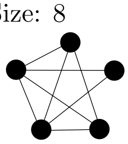
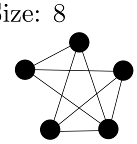
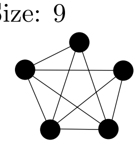
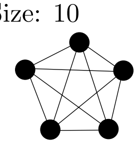
Classification Sheet

Below you will find thirty-four graphs of order 5 with sizes between 0 and 10. It is your task to classify these graphs as **bipartite**. Good luck!

1. B: ___ Size: 0 	**2.** B: ___ Size: 1 	**3.** B: ___ Size: 2 	**4.** B: ___ Size: 2 	**5.** B: ___ Size: 3 
6. B: ___ Size: 3 	**7.** B: ___ Size: 3 	**8.** B: ___ Size: 3 	**9.** B: ___ Size: 4 	**10.** B: ___ Size: 4 
11. B: ___ Size: 4 	**12.** B: ___ Size: 4 	**13.** B: ___ Size: 4 	**14.** B: ___ Size: 4 	**15.** B: ___ Size: 5 
16. B: ___ Size: 5 	**17.** B: ___ Size: 5 	**18.** B: ___ Size: 5 	**19.** B: ___ Size: 5 	**20.** B: ___ Size: 5 
21. B: ___ Size: 6 	**22.** B: ___ Size: 6 	**23.** B: ___ Size: 6 	**24.** B: ___ Size: 6 	**25.** B: ___ Size: 6 
26. B: ___ Size: 6 	**27.** B: ___ Size: 7 	**28.** B: ___ Size: 7 	**29.** B: ___ Size: 7 	**30.** B: ___ Size: 7 
31. B: ___ Size: 8 	**32.** B: ___ Size: 8 	**33.** B: ___ Size: 9 	**34.** B: ___ Size: 10 	

Finding All Graphs of Order 5

FULL ANSWER KEY

1. Size: 0  B: yes	**2.** Size: 1  B: yes	**3.** Size: 2  B: yes	**4.** Size: 2  B: yes	**5.** Size: 3  B: yes
6. Size: 3  B: yes	**7.** Size: 3  B: yes	**8.** Size: 3  B: no	**9.** Size: 4  B: yes	**10.** Size: 4  B: no
11. Size: 4  B: yes	**12.** Size: 4  B: yes	**13.** Size: 4  B: yes	**14.** Size: 4  B: no	**15.** Size: 5  B: no
16. Size: 5  B: no	**17.** Size: 5  B: no	**18.** Size: 5  B: no	**19.** Size: 5  B: no	**20.** Size: 5  B: yes
21. Size: 6  B: no	**22.** Size: 6  B: no	**23.** Size: 6  B: no	**24.** Size: 6  B: no	**25.** Size: 6  B: yes
26. Size: 6  B: no	**27.** Size: 7  B: no	**28.** Size: 7  B: no	**29.** Size: 7  B: no	**30.** Size: 7  B: no
31. Size: 8  B: no	**32.** Size: 8  B: no	**33.** Size: 9  B: no	**34.** Size: 10  B: no	

Working to Classify Graphs of Order 5

As a mathematician you will soon learn that cool things happen when we look at a smaller group of a larger set of items. To help us find some small groups to investigate it is now your turn to work on classifying the 34 graphs of order 5. To do these classifications is is important to ***ignore any unused vertices...pretend they're no longer there!***

1. Which (if any) of the 34 graphs of order five are subgraphs of K_5?

2. Which (if any) of the 34 graphs of order five are complete?

3. Which (if any) of the 34 graphs of order five are bipartite?

4. Which (if any) of the 34 graphs of order five are spanning?

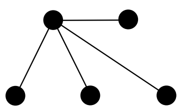
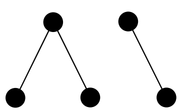
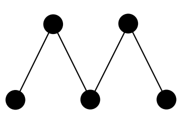
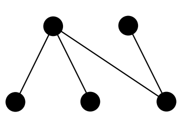
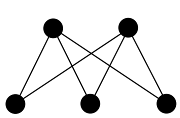
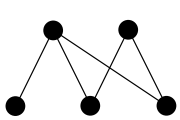
5. Which (if any) of the 34 graphs of order five are connected?

6. Which (if any) of the 34 graphs of order five are spanning bipartite?

7. Draw the graphs listed in question 6 in a bipartite fashion. (Vertices of one color on top connected to the vertices of the other color on bottom)

Matching Your 6 Spanning Bipartite Graphs

It is time to match your spanning bipartite graphs listed in question 6 and drawn in question 7 on the *Working to Classify Graphs of Order 5* worksheet to the six spanning bipartite graphs shown below. You will soon see that through the steps of this activity so far you have stumbled upon the only six spanning bipartite subgraphs of K_5 and have laid the very important foundation for us during the rest of the investigation!!!!!

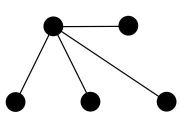
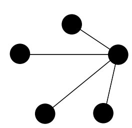
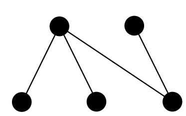
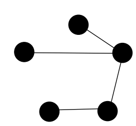
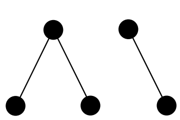
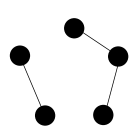
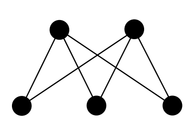
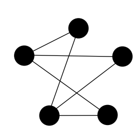
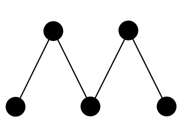
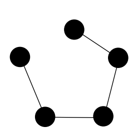
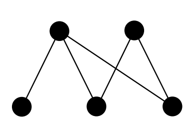
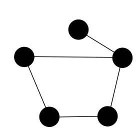
Our Graph	**Your Graph**
	
	
	
	
	
	

Working to Classify Graphs of Order 5
ANSWER KEY

As a mathematician you will soon learn that cool things happen when we look at a smaller group of a larger set of items. To help us find some small groups to investigate it is now your turn to work on classifying the 34 graphs of order 5. **NOTE:** This answer key is compatible with the graph numbering found on the classification version worksheet and answer key worksheet from the previous set of worksheets. Should the students have found all the graphs on their own with the blank worksheet or the size hint worksheet the numbers will be different, but the number of graphs fitting each classification will be the same.

1. Which (if any) of the 34 graphs of order five are subgraphs of K_5?
   All graphs except for graph #34
2. Which (if any) of the 34 graphs of order five are complete?
   Graph #34
3. Which (if any) of the 34 graphs of order five are bipartite?
   Graphs # 2, 3, 4, 5, 6, 7, 9, 11, 12, 13, 20 and 25
4. Which (if any) of the 34 graphs of order five are spanning?
   Graphs # 7, 9, 11, 13, 14, 15, and 18 - 34
5. Which (if any) of the 34 graphs of order five are connected?
   Graphs # 2, 3, 5, 6, 8, 9, 10, 11, 12, 13, and 15 - 34
6. Which (if any) of the 34 graphs of order five are spanning bipartite?
   Graphs # 2, 11, 14, 16, 21, and 26
7. Draw the graphs listed in question 6 in a bipartite fashion. (Vertices of one color on top connected to the vertices of the other color on bottom)
   Answers will vary

Matching Your 6 Spanning Bipartite Graphs
ANSWER KEY

Our Graph	**Your Graph**		**Our Graph**	**Your Graph**
Simple A 	Graph 2 		Simple D 	Simple 14 
Simple B 	Graph 11 		Simple E 	Simple 26 
Simple C 	Graph 16 		Simple F 	Graph 21 

Let's Investigate $^2K_5$!

To begin, draw $^2K_5$ below.

Describe any attributes of $^2K_5$ you notice in relation to the following words:

1. Vertices

2. Edges

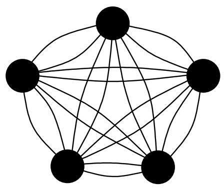
3. Degree

4. Incident/Adjacent

5. Are there any other attributes of $^2K_5$ that you didn't notice above?

Let's Investigate $^2K_5$!
ANSWER KEY

To begin, draw $^2K_5$ below.



Describe any attributes of $^2K_5$ you notice in relation to the following words.

1. Vertices

   There are 5 vertices. Even though the graph is now 2-fold, the number of vertices remains at 5 since it is $^2K_5$.

2. Edges

   There are 20 edges.The number of edges in a 2-fold graph is $(n(n-1))/2 * 2$ which is $n(n-1)$. Thus, there are 20 edges. For a λ-fold graph, the number of edges is $(n(n-1))/2 * \lambda$

3. Degree

   Each vertex has degree 8. The degree of each vertex doubles from K_5 to $^2K_5$. This is because it is now 2-fold. Since the degree of each vertex is 8, it is an 8-regular graph.

4. Incident/Adjacent

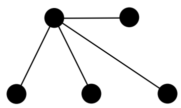
   Every vertex is adjacent to every other vertex. There are 8 edges incident to each vertex. When we make a complete graph into a 2-fold graph, it is still complete so ever vertex is adjacent to every other vertex.

5. Are there any other attributes of $^2K_5$ that you didn't notice above?

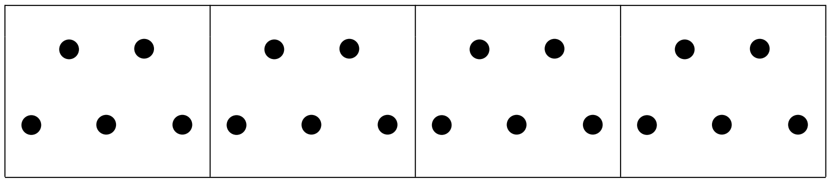
   There are ten edges of length 1 and ten edges of length 2.

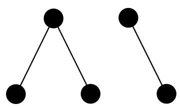
Creating ${}^2K_5$ Spanning Bipartite Subgraphs

All of the spanning subgraphs of K_5 are shown below. You will notice that they are classified by the underlying 1-fold structure that you have just found. Your job will be to find every ${}^2K_5$ spanning bipartite subgraphs. Because we are staying in K_5 with spanning bipartite subgraphs, we know the ${}^2K_5$ graphs are based upon these six spanning bipartite structures you found for K_5. To find all the ${}^2K_5$ spanning bipartite subgraphs we are going to give you two hints. First, the number of possible for each structure is equal to the number of blank dot pictures provided. Second, to find every possibility, gradually add in two-fold edges until you have every edge two fold. Just be cautious of creating two graphs that are isomorphic to each other. Best of luck young researcher! Once you have these graphs we will be ready to embark with new areas of mathematics.

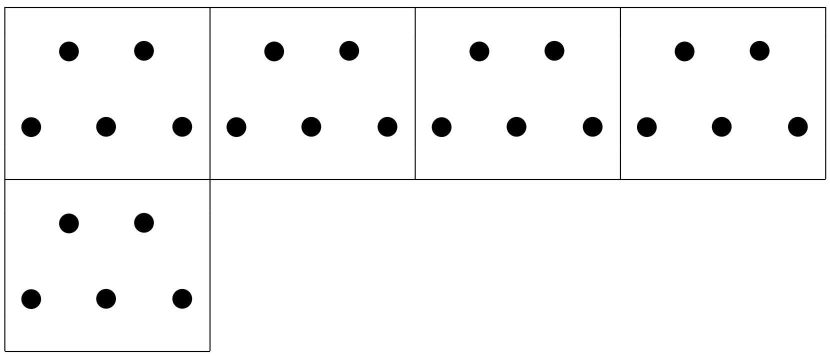


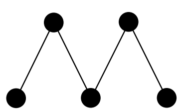
Simple Graph Type A



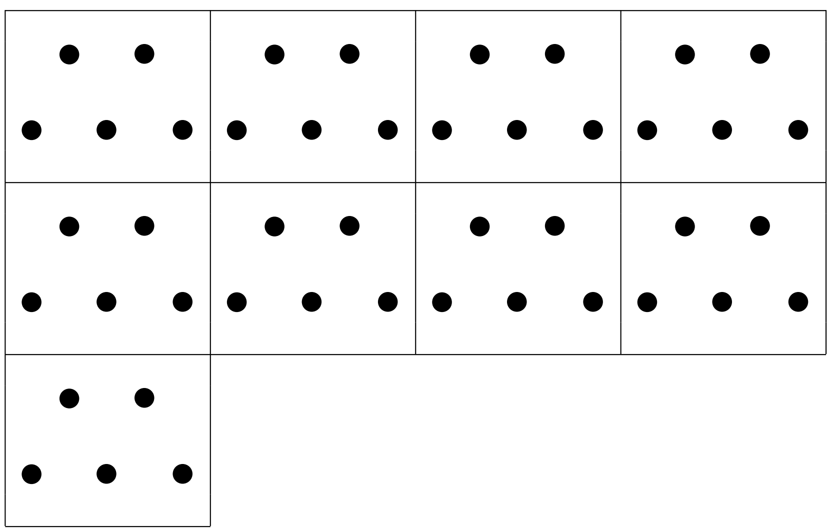


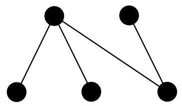
Simple Graph Type B



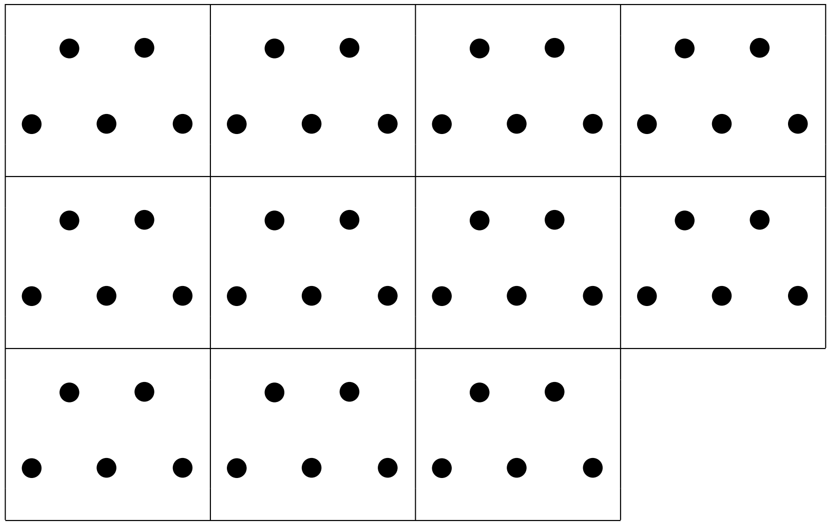


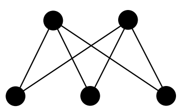
Simple Graph Type C



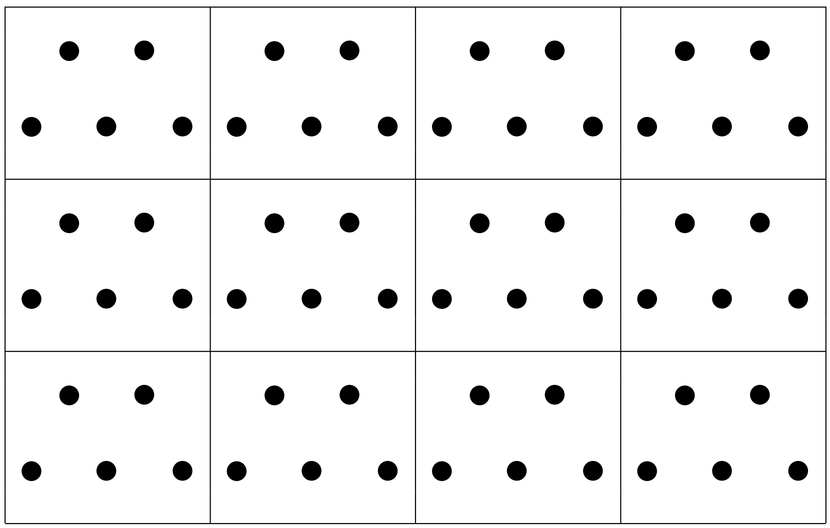


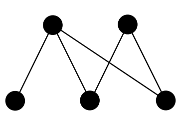
Simple Graph Type D



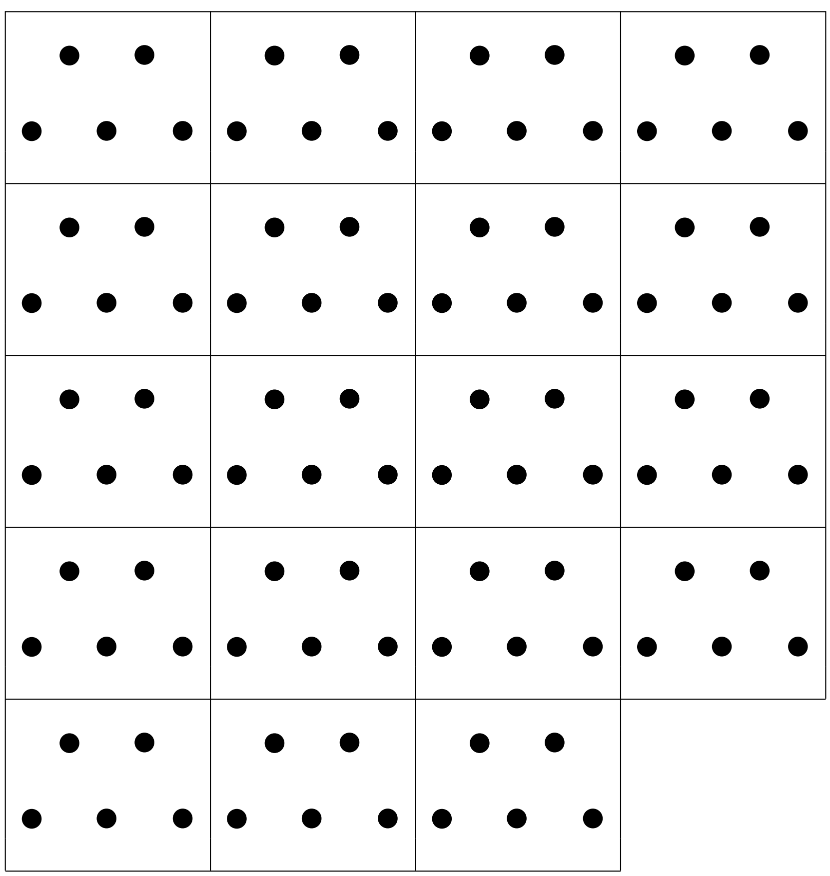


Simple Graph Type E





Simple Graph Type F

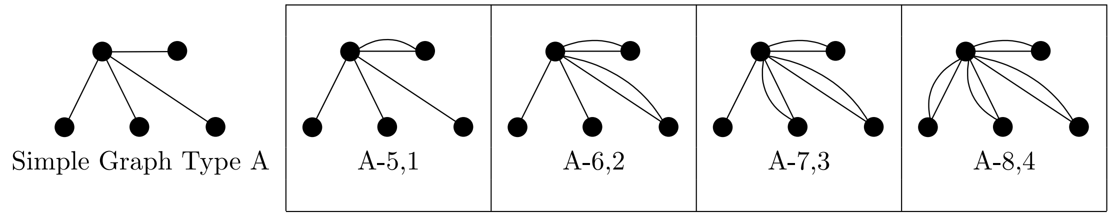


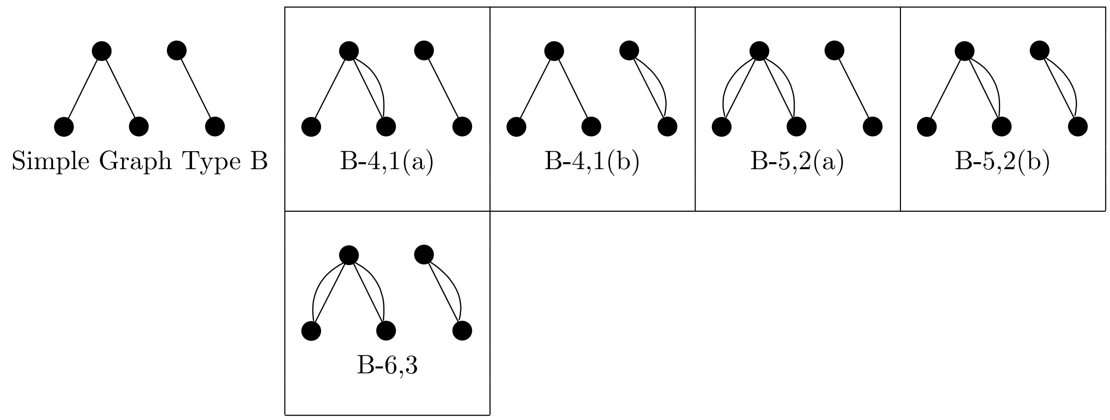
$^2K_5$ Spanning Bipartite Subgraph Listing

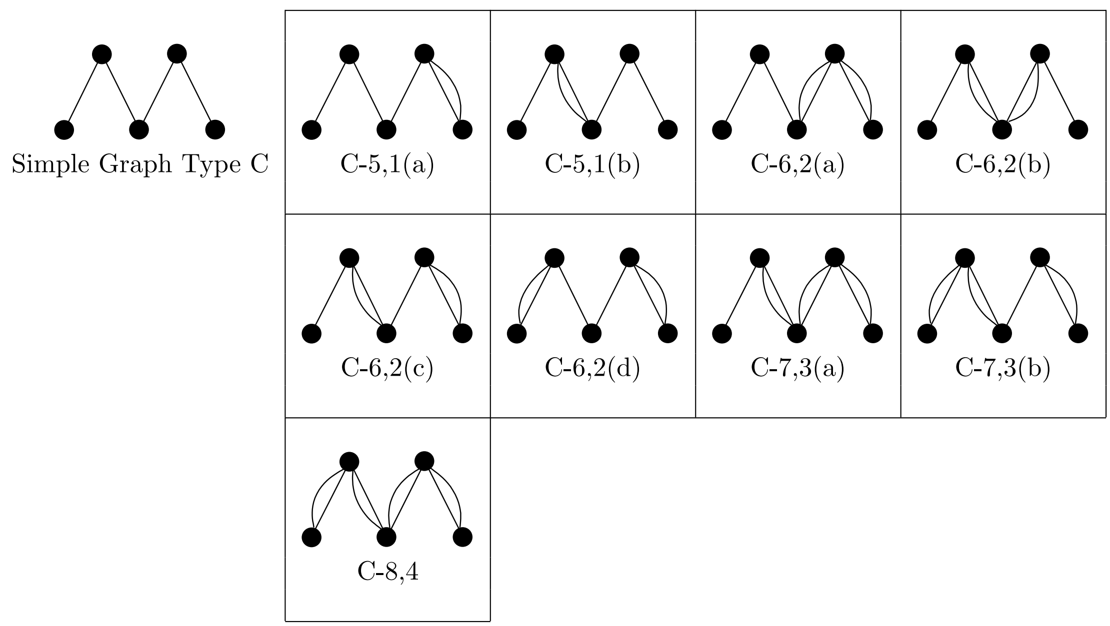
All of the spanning subgraphs of $^2K_5$ are shown below. You will notice that they are classified by the underlying 1-fold structure. There are six such structures. Within each structure classification the naming process is:

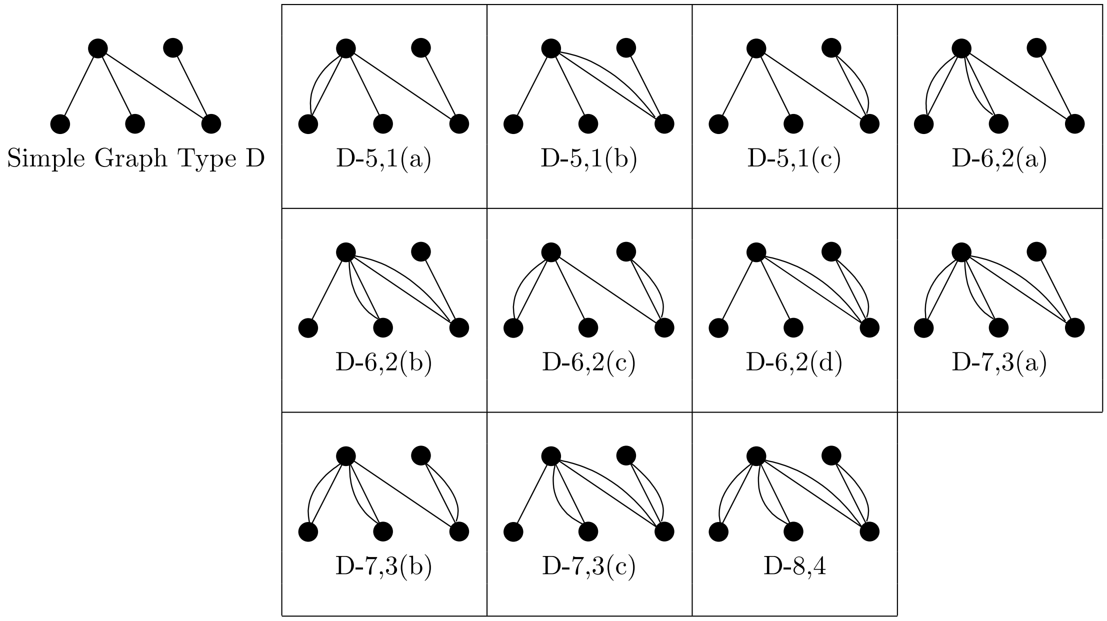
Classification - Number of Edges, Number of 2-Fold Pairs

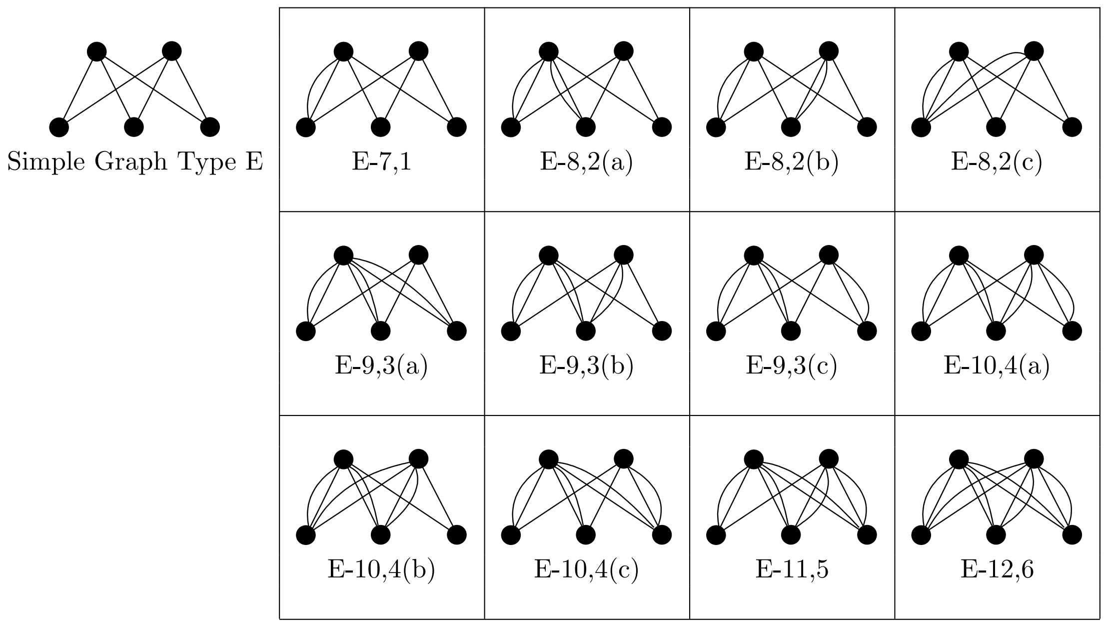
When there were multiple graphs within the same classification that had the same number of edges and the same number of 2-fold pairs, they are denoted by a, b, c, etc .

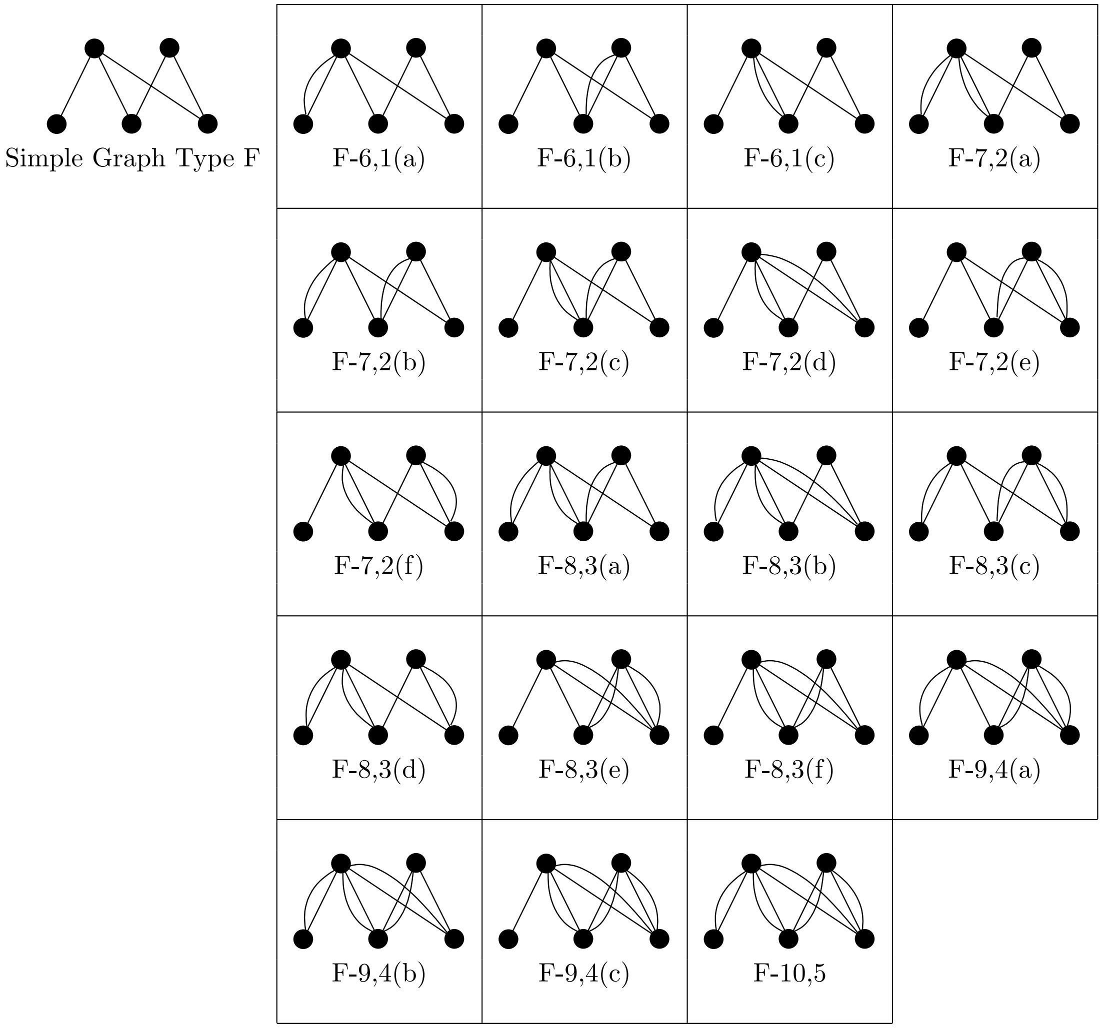







Simple Graph Type D
D-5,1(a)
D-5,1(b)
D-5,1(c)
D-6,2(a)
D-6,2(b)
D-6,2(c)
D-6,2(d)
D-7,3(a)
D-7,3(b)
D-7,3(c)
D-8,4


Simple Graph Type E
E-7,1
E-8,2(a)
E-8,2(b)
E-8,2(c)
E-9,3(a)
E-9,3(b)
E-9,3(c)
E-10,4(a)
E-10,4(b)
E-10,4(c)
E-11,5
E-12,6


Simple Graph Type F
F-6,1(a)
F-6,1(b)
F-6,1(c)
F-7,2(a)
F-7,2(b)
F-7,2(c)
F-7,2(d)
F-7,2(e)
F-7,2(f)
F-8,3(a)
F-8,3(b)
F-8,3(c)
F-8,3(d)
F-8,3(e)
F-8,3(f)
F-9,4(a)
F-9,4(b)
F-9,4(c)
F-10,5