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On the cyclic decomposition of circulant graphs into almost-bipartite graphs

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Abstract

It is known that if an almost bipartite graph G with n edges possesses a γ-labeling, then the complete graph K_{2nx+1} admits a cyclic G-decomposition. We introduce a variation of γ-labeling and show that whenever an almost bipartite graph G admits such a labeling, then there exists a cyclic G-decomposition of a family of circulant graphs. We also determine which odd length cycles admit the variant labeling.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_t$ the group of integers modulo t. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. The *order* and the *size* of a graph G are $|V(G)|$ and $|E(G)|$, respectively.

Let $V(K_t) = \{0, 1, \ldots, t-1\}$. The *length* of an edge $\{i, j\}$ in K_t is $\min\{|i-j|, t-|i-j|\}$. Note that if t is odd, then K_t consists of t edges of length i for $i = 1, 2, \ldots, \frac{t-1}{2}$. If t is even, then K_t consists of t edges of length i for $i = 1, 2, \ldots, \frac{t}{2}-1$, and $\frac{t}{2}$ edges of length $\frac{t}{2}$; moreover, in this case, the edges of length $\frac{t}{2}$ constitute a 1-factor in K_t.

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Let $V(K_t) = \mathbb{Z}_t$ and let G be a subgraph of K_t. By *clicking* G, we mean applying the permutation $i \to i+1$ to $V(G)$. Let H and G be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set $\Delta = \{G_1, G_2, \ldots, G_r\}$ of pairwise edge-disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{r} E(G_i)$. A G-decomposition of K_t is also known as a *(K_t, G)-design.* A (K_t, G)-design Δ is *cyclic* if clicking is an automorphism of Δ. For recent surveys on G-designs, see [1] and [5]. If we let $L \subseteq \{1, 2, \ldots, \lfloor t/2 \rfloor\}$, then the subgraph of K_t induced by all the edges with lengths in L is called a *circulant graph* and is denoted by $\langle L \rangle_t$. Of course, circulant graphs are *Cayley graphs* on cyclic groups. As noted earlier, $\langle \{t/2\} \rangle_t$ is a 1-factor in K_t when t is even. Otherwise, for $1 \le i < t/2$, it is easy to see that $\langle \{i\} \rangle_t$ consists of δ vertex disjoint $C_{t/\delta}$'s, where $\delta = \gcd(i, t)$.

Let k and n be positive integers and let G be a graph of size n. It would be of interest to know whether there exists a G-decomposition of the circulant $\langle [k, n+k-1] \rangle_{2n+2k-1}$. When $k = 1$, the circulant $\langle [k, n+k-1] \rangle_{2n+2k-1}$ is the complete graph K_{2n+1}. A popular conjecture of Ringel [15] states that there exists a (K_{2n+1}, G)-design for every tree G of size n. It is very likely that every tree of size n will decompose the circulant $\langle [k, n+k-1] \rangle_{2n+2k-1}$ for every positive integer k. In fact, it would be of interest to know what graphs of size n do not decompose $\langle [k, n+k-1] \rangle_{2n+2k-1}$ for some positive k.

A popular approach to dealing with Ringel's Conjecture is the use of graph labelings. In fact, numerous conjectures in graph labelings subsume Ringel's Conjecture (see [12]). For example, Kotzig (see [16]) conjectures that every tree admits what is called a ρ-labeling. This would imply that there is a cyclic (K_{2n+1}, G)-design for every tree G of size n. It can be conjectured similarly that there is a cyclic G-decomposition of $\langle [k, n+k-1] \rangle_{2n+2k-1}$ for every tree G of size n.

1.1 Extensions of Rosa-type Labelings

For any graph G, a one-to-one function $f : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [16], Rosa introduced a hierarchy of labelings. We generalize Rosa's labelings and add a few items to this hierarchy. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{E}(G) = \{\bar{f}(e) : e \in E(G)\}$. Let k be a positive integer and consider the following conditions:

ℓ1: $f(V(G)) \subseteq [0, 2(n+k-1)]$,

ℓ2: $f(V(G)) \subseteq [0, n+k-1]$,

ℓ3: $\bar{E}(G) = \{x_k, x_{k+1}, \ldots, x_{n+k-1}\}$, where for each $i \in [k, n+k-1]$ either $x_i = i$ or $x_i = 2(n+k-1) + 1 - i = 2(n+k) - 1 - i$,

ℓ4: $\bar{E}(G) = [k, n+k-1]$.

If in addition G is bipartite, with bipartition $\{A, B\}$ of $V(G)$ (with every edge in G having one end vertex in A and the other in B), consider also

$\ell 5$: for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,

$\ell 6$: there exists an integer λ (called the *boundary value* of f) such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

$\ell 1, \ell 3$: is called a *ρ_k-labeling*;

$\ell 1, \ell 4$: is called a *σ_k-labeling*;

$\ell 2, \ell 4$: is called a *β_k-labeling*.

A β_k-labeling is necessarily a σ_k-labeling which in turn is a ρ_k-labeling. When $k = 1$, these labelings correspond, respectively, to the β, σ, and ρ-labelings that were introduced by Rosa [16].

If G is bipartite and a ρ_k, σ_k or β_k-labeling of G also satisfies ($\ell 5$), then the labeling is *ordered* and is denoted by ρ_k^+, σ_k^+ or β_k^+, respectively. If in addition ($\ell 6$) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ_k^{++}, σ_k^{++} or β_k^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an *α-labeling* as introduced in [16]. Moreover, what we are calling a β_k-labeling was previously independently introduced as a *k-graceful labeling* by Slater [17] and by Maheo and Thuillier [14]. For $k > 1$, we shall refer to all the labelings introduced above simply as *k-labelings*. Labelings that are used in graph decompositions are called *Rosa-type* because of Rosa's original article [16] on the topic. For a survey of Rosa-type labelings and their graph decomposition applications, see [12]. A comprehensive dynamic survey on general graph labelings is maintained by Gallian [13].

Rosa-type labelings are critical to the study of cyclic graph decompositions as seen in the following two results from Rosa [16] and El-Zanati, Vanden Eynden and Punnim [11], respectively.

Theorem 1 *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

Theorem 2 *Let G be a graph with n edges that has a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

From a graph decompositions perspective, Theorem 2 offers a great advantage over Theorem 1. However, only bipartite graphs can admit an ordered labeling. By using k-labelings, we get extensions of the above theorems to cyclic G-decompositions of the corresponding circulant graphs.

Theorem 3 *Let G be a graph with n edges and let k be a positive integer. There exists a cyclic G-decomposition of $\langle[k, n+k-1]\rangle_{2n+2k-1}$ if and only if G has a ρ_k-labeling.*

Theorem 4 (See [10]) *Let G be a graph with n edges that has a ρ_k^+-labeling. Then there exists a cyclic G-decomposition of $\langle[k, nx+k-1]\rangle_{2nx+2k-1}$ for all positive integers x.*

The proof of Theorem 3 is straightforward. A ρ_k-labeling of G is an embedding of G in $K_{2n+2k-1}$ (with $V(K_{2n+2k-1}) = \mathbb{Z}_{2n+2k-1}$) so that there is one edge in $E(G)$ of each length ℓ for $k \le \ell \le n+k-1$. Moreover, $\langle[k, n+k-1]\rangle_{2n+2k-1} = K_{2n+2k-1} - \langle[1, k-1]\rangle_{2n+2k-1}$. It is easy to see how the result holds. A proof of Theorem 4 and other results related to cyclic decompositions of circulant graphs into bipartite graphs can be found in [10].

In [4], Blinco, El-Zanati, and Vanden Eynden introduced a variation of a ρ-labeling of an almost-bipartite graph G of size n that yields cyclic G-decompositions of K_{2nx+1} for every positive integer x. They called this labeling a γ-labeling.

A non-bipartite graph G is said to be *almost-bipartite* if $G-e$ is bipartite for some $e \in E(G)$. Note that if G is almost-bipartite with $e = \{\hat{b}, c\}$, then G is necessarily tripartite and $V(G)$ can be partitioned into three sets A, B and $C = \{c\}$ such that $\hat{b} \in B$ and e is the only edge joining an element of B to c.

Let G be an almost-bipartite graph with n edges with vertex tripartition A, B, C as above. A labeling h of the vertices of G is called a γ-*labeling* of G if the following conditions hold.

(g1) The function h is a ρ-labeling of G.

(g2) If $\{a, v\}$ is an edge of G with $a \in A$, then $h(a) < h(v)$.

(g3) We have $h(c) - h(\hat{b}) = n$.

It was shown in [4], that if a graph G with n edges admits a γ-labeling, then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.

Theorem 5 *Let G be a graph with n edges having a γ-labeling. Then G divides K_{2nx+1} cyclically for all positive integers x.*

Blinco et al. [4] showed that odd cycles other than C_3 admit γ-labelings. We extend the definition of a γ-labeling to what we call a γ_k^*-labeling.

Let G be an almost-bipartite graph with n edges and vertex-tripartition $A, B, C = \{c\}$ as in the definition of almost-bipartite. Let $k \le n$ be a positive integer and let h be a ρ_k-labeling of G. We call h a γ_k^*-*labeling* of G if the following conditions hold.

(g*1) We have $h(A) < h(B \cup C)$.

(g*2) For all $u, v \in B \cup C, h(u) - h(v) \ne 2n$.

(g*3) We have $h(c) - h(\hat{b}) = n$.

In this manuscript, we shall first show that a γ_k^*-labeling of an almost-bipartite graph G of size n yields cyclic G-decompositions of $\langle [k, nx+k-1] \rangle_{2nx+2k-1}$ for all positive integers x. We also show that all odd cycles of length n admit γ_k^*-labelings for all $k \leq n$, except for $(n,k) \in \{(3,1),(3,3),(5,3)\}$.

2 Main Result

Next we show that γ_k^*-labelings also decompose infinitely many graphs.

Theorem 6 *Let G be an almost-bipartite graph with n edges having a γ_k^*-labeling. Then there exists a cyclic G-decomposition of $\langle [k, nx+k-1] \rangle_{2nx+2k-1}$ for all positive integers x.*

Proof. Let G have n edges and let h be a γ_k^*-labeling for G, with A, B, C, c, and $\hat{b}$ as in the above definition. We will assume $k > 1$, since otherwise Theorem 5 applies. Let $B_1, B_2, \dots, B_x$ be x vertex-disjoint copies of B, and let $c_1, c_2, \dots, c_x$ be x new vertices. The vertex in B_i corresponding to $b \in B$ will be called b_i. Let $B^* = \bigcup_1^x B_i$ and $C^* = \{c_1, c_2, \dots, c_x\}$. We define a new graph G^* with vertex set $A \bigcup B^* \bigcup C^*$ and edges $\{a, v_i\}$, $1 \leq i \leq x$, whenever $a \in A$ and $\{a, v\}$ is an edge of G, and the edges $\{\hat{b}_i, c_i\}$, $1 \leq i \leq x$. Clearly G^* has nx edges and there is G-decomposition of G^*.

The plan of the proof is to show that G^* has a ρ_k-labeling. Theorem 3 then applies. We define a labeling h^* on G^* by

$$h^*(v) = \begin{cases} h(v) & v \in A, \\ h(b) + 2n(i-1) & v = b_i \in B_i, \\ h(c) + 2n(x-i) & v = c_i. \end{cases}$$

To see that h^* is a ρ_k-labeling, first note that if v is a vertex of G^*, then $0 \leq h^*(v) \leq 2(n+k-1) + (x-1)2n = 2(nx+k-1)$. To see that h^* is one-to-one on $V(G^*)$, note that $h^*(A) = h(A) < h(B \cup C) \leq h^*(B \cup C)$. Thus, $h^*(A) \cap h^*(B \cup C) = \emptyset$. Moreover, if $i \neq j$ and $h^*(B_i) \cap h^*(B_j) \neq \emptyset$, then $h^*(b_i) = h^*(b'_j)$ for some $b, b' \in B$. Thus, $h(b) + 2n(i-1) = h(b') + 2n(j-1)$ and therefore $h(b) - h(b') = 2n(j-i)$. Since $h(b) - h(b') \leq 2(n+k-1) \leq 4n-2$, either $i - j = 0$ and $h(b) = h(b')$ or $i - j = 1$ and $h(b) - h(b') = 2n$. In either case, we contradict the definition of a γ_k^*-labeling. A similar argument shows that we cannot have $h^*(c_j) \in h^*(B_i)$.

Now let $\ell \in [k, nx+k-1]$. We will show that some edge of the new graph has label ℓ or $2nx + 2(k-1) + 1 - \ell$. First, we show that there exist integers q and r where either $\ell = 2nq + r$ or $2(nx+k-1) + 1 - \ell = 2nq + r$, with $0 \leq q < x$ and $k \leq r \leq n+k-1$. Let q' and r' be integers such that $\ell = 2nq' + r'$, where $q' \geq 0$

and $0 \leq r' < 2n$. If $k \leq r' \leq n+k-1$, then let $q = q'$ and $r = r'$. Otherwise, if $r' > n+k-1$, then

$$\begin{aligned}
2(nx+k-1)+1-\ell &= 2(n(x-1)+n+k-1)+1-(2nq'+r') \\
&= 2n(x-1-q')+2(n+k)-(r'+1) \\
&= 2nq+r, \text{where } q = x-1-q' \text{ and } r = 2(n+k)-(r'+1).
\end{aligned}$$

It is easy to verify that $0 \leq q < x$ and $k \leq r \leq k+n-1$.

Similarly, if $r' < k$, then

$$\begin{aligned}
2(nx+k-1)+1-\ell &= 2(nx+k-1)+1-(2nq'+r') \\
&= 2n(x-q')+2k-(r'+1) \\
&= 2nq+r, \text{where } q = x-q' \text{ and } r = 2k-(r'+1).
\end{aligned}$$

It is easy to verify that $0 \leq q < x$ and $k \leq r \leq k+n-1$.

Therefore, either $\ell = 2nq+r$ or $2(nx+k-1)+1-\ell = 2nq+r$, where $q \geq 0$ and $k \leq r \leq n+k-1$. Since h is a γ_k^*-labeling of G, there exists an edge e (in G) with label either r or $2(n+k)-1-r$.

Case 1. The label of e is r.

First, suppose $e = \{a, b\}$, where $a \in A$ and $b \in B$. Since $h(b) - h(a) = r$, we have

$$\begin{aligned}
h^*(b_{q+1}) - h^*(a) &= h(b) + 2n(q) - h(a) \\
&= r + 2nq.
\end{aligned}$$

Thus $h^*(b_{q+1}) - h^*(a) = \ell$ if $\ell = 2nq+r$ and $h^*(b_{q+1}) - h^*(a) = 2(nx+k-1)+1-\ell$ if $2(nx+k-1)+1-\ell = 2nq+r$.

Next, suppose $e = \{a, c\}$, where $a \in A$. Since $h(c) - h(a) = r$, we have

$$\begin{aligned}
h^*(c_{x-q}) - h^*(a) &= h(c) + 2n[x-(x-q)] - h(a) \\
&= 2nq + r.
\end{aligned}$$

Thus $h^*(c_{x-q}) - h^*(a) = \ell$ if $\ell = 2nq+r$ and $h^*(c_{x-q}) - h^*(a) = 2(nx+k-1)+1-\ell$ if $2(nx+k-1)+1-\ell = 2nq+r$.

Finally suppose $e = \{\hat{b}, c\}$. Thus in this case, $r = n$. We first note that for $t \in [1, x]$, $h^*(c_t) - h^*(\hat{b}_t) = h(c) + 2n(x-t) - [h(\hat{b}) + 2n(t-1)] = n + 2n(x+1-2t)$. Thus the label of the edge $\{\hat{b}_t, c_t\}$ is $|h^*(c_t) - h^*(\hat{b}_t)|$ which equals $n+2n(x+1-2t)$ when $t \leq \frac{1}{2}(x+1)$, and equals $2n[2t-(x+2)]+n$ when $t > \frac{1}{2}(x+1)$. As t runs over $[1, x]$ when x is even, the preceding values run through $n+2n(x-1), n+2n(x-3), \ldots, n+2n, n, n+2n(2), \ldots, n+2n(x-2)$. The set of outcomes is the same when n is odd.

Case 2. The label of e is $2(n+k-1)+1-r$.

First, suppose $e = \{a, b\}$, where $a \in A$ and $b \in B$. Since $h(b) - h(a) = 2(n + k - 1) + 1 - r$, we have

$$\begin{aligned} h^*(b_{x-q}) - h^*(a) &= h(b) + 2n(x-q-1) - h(a) \\ &= h(b) - h(a) + 2nx - 2nq - 2n \\ &= 2(k+n-1) + 1 - r + 2nx - 2nq - 2n \\ &= 2(nx+k-1) + 1 - (2nq + r). \end{aligned}$$

Thus $h^*(b_{x-q}) - h^*(a) = 2(nx + k - 1) + 1 - \ell$ if $\ell = 2nq + r$ and $h^*(b_{x-q}) - h^*(a) = 2(nx + k - 1) + 1 - [2(nx + k - 1) + 1 - \ell] = \ell$ if $2(nx + k - 1) + 1 - \ell = 2nq + r$.

Finally, suppose $e = \{a, c\}$, where $a \in A$. Since $h(c) - h(a) = 2(n + k - 1) + 1 - r$, we have

$$\begin{aligned} h^*(c_{q+1}) - h^*(a) &= h(c) - h(a) + 2n(x-q-1) \\ &= 2(k+n-1) + 1 - r + 2n(x-q-1) \\ &= 2k + 2n - 1 - r + 2nx - 2nq - 2n \\ &= 2nx + 2k - 1 - (2nq + r) \\ &= 2(nx+k-1) + 1 - (2nq + r). \end{aligned}$$

Thus $h^*(c_{q+1}) - h^*(a) = 2(nx + k - 1) + 1 - \ell$ if $\ell = 2nq + r$ and $h^*(c_{q+1}) - h^*(a) = 2(nx + k - 1) + 1 - [2(nx + k - 1) + 1 - \ell] = \ell$ if $2(nx + k - 1) + 1 - \ell = 2nq + r$.

Since G^* has size nx and each of the nx edge lengths $k, k+1, \ldots, nx+k-1$, is the length of an edge, h^* is a ρ_k-labeling of G^*. Thus there is a cyclic G^*-decomposition of $\langle [k, nx + k - 1] \rangle_{2nx+2k-1}$. ∎

Below we show an example of a γ_4^*-labeling of C_5 and the three starters for a cyclic C_5-decomposition of $\langle [4, 18] \rangle_{37}$.

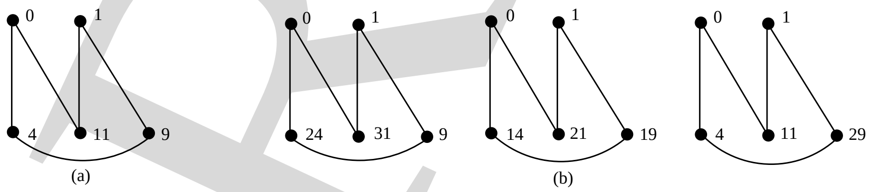

Figure 1: (a) A γ_4^*-labeling of C_5. (b) The three starters for a cyclic C_5-decomposition of $\langle [4, 18] \rangle_{37}$.

We note here that an almost-bipartite graph G of size n can fail to admit a γ_k^*-labeling for some $k < n$, even if G admits a γ-labeling. For example, C_5 has a γ-labeling, but it does not admit a γ_3^*-labeling.

3 On γ_k^*-labelings of Odd Cycles

In [4], it is shown that C_{2m+1} admits a γ-labeling if and only if $m \geq 2$. In this section, we show that every odd cycle of length n admits a γ_k^*-labeling for every positive $k \le n$ unless $n = 5$ and $k = 3$ or $n = 3$ and $k \in \{1, 3\}$. To simplify our consideration of the labelings, we will henceforth consider graphs whose vertices are named by distinct nonnegative integers, which are also their labels. Recall that by the label of the edge $\{x, y\}$ in such a graph we mean $|x - y|$. If G is a graph with n edges and if m is the label of an edge e, let $m^* = \min\{m, 2n + 1 - m\}$ (thus m^* is the length of e). If S is a set of edge labels, let $S^* = \{m^* : m \in S\}$.

We denote the path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \le i \le k-1$, by $(x_0, x_1, \ldots, x_k)$. In using this notation, we are thinking of traversing the path from x_0 to x_k so that x_0 is the first vertex, x_1 is the second vertex, and so on. Let $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$. If G_1 and G_2 are vertex-disjoint except for $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$. If the only vertices they have in common are $x_0 = y_k$ and $x_j = y_0$, then by $G_1 + G_2$ we mean the cycle $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_{k-1}, x_0)$.

Let $P(k)$ be the path with k edges and $k+1$ vertices $0, 1, \ldots, k$ given by $(0, k, 1, k-1, 2, k-2, \ldots, \lceil k/2 \rceil)$. Note that the set of vertices of this graph is $A \cup B$, where $A = [0, \lfloor k/2 \rfloor]$, $B = [\lfloor k/2 \rfloor + 1, k]$, and every edge joins a vertex from A to one from B. Furthermore the set of labels of the edges of $P(k)$ is $[1, k]$.

Now let a and b be nonnegative integers with $a \le b$ and let us add a to all the vertices of A and b to all the vertices of B. We will denote the resulting graph by $P(a, b, k)$. Note that this graph has the following properties.

P1: $P(a, b, k)$ is a path with first vertex a and second vertex $b + k$. If k is even, its last vertex is $a + k/2$.

P2: Each edge of $P(a, b, k)$ joins a vertex from $A' = [a, \lfloor k/2 \rfloor + a]$ to a vertex with a larger label from $B' = [\lfloor k/2 \rfloor + 1 + b, k + b]$.

P3: The set of edge labels of $P(a, b, k)$ is $[b - a + 1, b - a + k]$.

The paths $P(6)$ and $P(4, 7, 6)$ are shown in Figure 2 below.

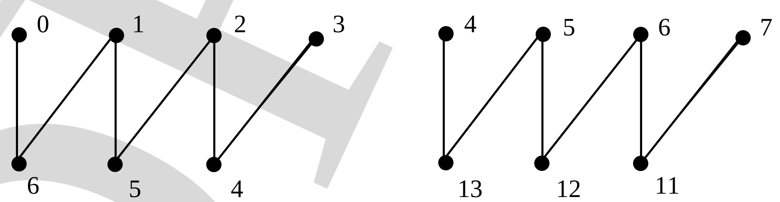

Figure 2: The paths $P(6)$ and $P(4, 7, 6)$.

Theorem 7 *If G is an odd cycle with n edges and $k \in [1, n]$, then G has a γ_k^*-labeling, unless $(n, k) \in \{(3, 1), (3, 3), (5, 3)\}$.*

Proof. It is easy to verify by inspection that C_3 does not admit a γ_k^* -labeling for $k \in \{1, 3\}$ and that $(0, 2, 5, 0)$ is a γ_2^*-labeling of C_3. Similarly, it can be shown that

C_5 does not admit a γ_3^*-labeling. We divide the remaining problem into 12 cases, based on restrictions on n and k. We give a detailed proof for Case 1 and leave out some of the easy to verify details in the remaining cases. We also provide an example with each case.

Case 1. G is a C_{4m+1} where k is even and $k \leq 2m$.
Thus $n = 4m+1$ and G is embedded in $K_{8m+2k+1}$. We take G to be $G_1 + G_2 + G_3 + (2m-1, 6m+k-1, 2m+k-2, 0)$, where

$$G_1 = P(0, 4m+1, k-2),$$
$$G_2 = P(k/2-1, 2m+3k/2-3, 2m-k+2),$$
$$G_3 = P(m, m+k-1, 2m-2).$$

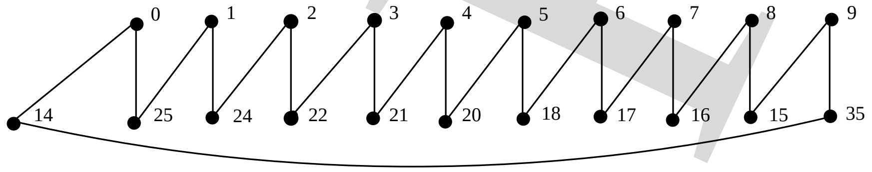

Figure 3: A γ_6^*-labeling of C_{21}.

First, we show that $G_1 + G_2 + G_3 + (2m-1, 6m+k-1, 2m+k-2, 0)$ is a cycle of length $4m+1$. Note that by P1, the first vertex of G_1 is 0 and the last is $k/2-1$, the first vertex of G_2 is $k/2-1$ and the last is m, the first vertex of G_3 is m and the last is $2m-1$. For $1 \leq i \leq 3$, let A_i and B_i denote the sets labeled A' or B' in P2, corresponding to the path G_i. Then using P2, we compute

$$A_1 = [0, k/2-1], \qquad B_1 = [4m+k/2+1, 4m+k-1],$$
$$A_2 = [k/2-1, m], \qquad B_2 = [3m+k-1, 4m+k/2-1],$$
$$A_3 = [m, 2m-1], \qquad B_3 = [2m+k-1, 3m+k-3].$$

Thus, $A_1 \leq A_2 \leq A_3 < B_3 < B_2 < B_1$. Also note that $V(G_1) \cap V(G_2) = \{k/2-1\}$, $V(G_2) \cap V(G_3) = \{m\}$ and that, otherwise, G_i and G_j are vertex-disjoint. Therefore, $G_1 + G_2 + G_3$ is a path P of length $4m-2$ with first vertex 0 and last vertex $2m-1$. Since $V(P) \cap \{2m-1, 6m+k-1, 2m+k-2, 0\} = \{2m-1, 0\}$, the graph $G_1+G_2+G_3+(2m-1, 6m+k-1, 2m+k-2, 0)$ is a cycle of length $4m+1$. Moreover, if we let $A = A_1 \cup A_2 \cup A_3$, $\hat{b} = 2m+k-2$, $B = \{\hat{b}\} \cup B_1 \cup B_2 \cup B_3$, $c = 6m+k-1$, and $C = \{c\}$, then $\max(B \cup C) - \min(B \cup C) = (6m+k-1) - (2m+k-2) = c - \hat{b} = 4m+1 < 8m+2 = 2n$. Thus conditions (g^*2) and (g^*3) for a γ_k^*-labeling are satisfied.

Therefore it remains to show that the set of edge-lengths of G is $[k, 4m+k]$. Let E_i denote the set of edge labels in G_i for $1 \leq i \leq 3$. By P3, we have

$$E_1 = [4m+2, 4m+k-1],$$
$$E_2 = [2m+k-1, 4m],$$
$$E_3 = [k, 2m+k-3].$$

We note that when $k = 2$, the sets B_1 and E_1 are empty. Similarly, in the case when $m = 1$ and $k = 2$, the sets B_1, B_3, E_1, and E_3 will be empty. However, these cases do not change the proof in any way.

Additionally, the path $(2m-1, 6m+k-1, 2m+k-2, 0)$ consists of edges with labels $4m+k$, $4m+1$, and $2m+k-2$. Thus, the edges of C_{4m+1} have labels $(\cup_{i=1}^{3} E_i) \cup \{4m+k, 4m+1, 2m+k-2\} = [k, 4m+k]$. Since no edge in G has a label larger than $4m+k$, the set of edge labels of G is also the set of edge lengths of G. Thus, we have a γ_k^*-labeling of C_{4m+1}.

Case 2. G is a C_{4m+1} where k is even and $k > 2m$.
Thus $n = 4m+1$ and G is again embedded in $K_{8m+2k+1}$. We take G to be $G_1 + G_2 + G_3 + (2m-1, 4m+k+1, k, 0)$, where

$$\begin{aligned} G_1 &= P(0, 2m+k+2, 2m-2),\\ G_2 &= P(m-1, 5m, k-2m),\\ G_3 &= P(k/2-1, 3k/2-1, 4m-k). \end{aligned}$$

If we let $\hat{b} = k$ and $c = 4m+k+1$ and proceed as in Case 1, it is easy to verify that we have a γ_k^*-labeling of C_{4m+1}.

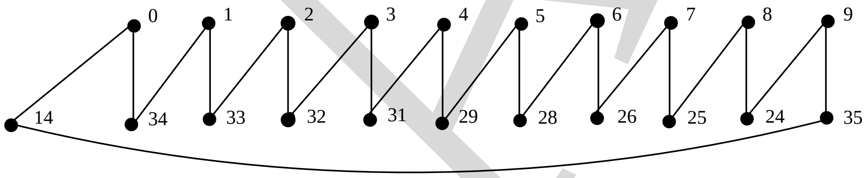

Figure 4: A γ_{14}^*-labeling of C_{21}.

Case 3. G is a C_{4m+1} where k is odd and $k \le 2m-1$.
We take G to be $G_1 + G_2 + G_3 + (2m-1, 2m+k-1, 6m+k, 0)$, where

$$\begin{aligned} G_1 &= P(0, 4m+1, k-1),\\ G_2 &= P((k-1)/2, 2m+(3k+1)/2, 2m-k-1),\\ G_3 &= P(m-1, k+m-1, 2m). \end{aligned}$$

If we let $\hat{b} = 2m+k-1$ and $c = 6m+k$, it is easy to verify that we have a γ_k^*-labeling of C_{4m+1}.

Case 4. G is a C_{4m+1} where k is odd, $m \ne 1$, and $k = 2m+1$.
Note that in this case, G is embedded in K_{12m+3}. Recall that C_5 does not admit a γ_3^*-labeling. If $m = 2$ and $k = 5$, a γ_5^*-labeling of C_9 is given by $(0, 13, 1, 12, 2, 8, 3, 10, 19, 0)$. Otherwise, we take G to be $G_1 + G_2 + (2m-3, 12m-4, 2m-2, 4m, 2m-1, 4m+2, 8m+3, 0)$, where

$$\begin{aligned} G_1 &= P(0, 4m+1, 2m),\\ G_2 &= P(m, 3m+5, 2m-6). \end{aligned}$$

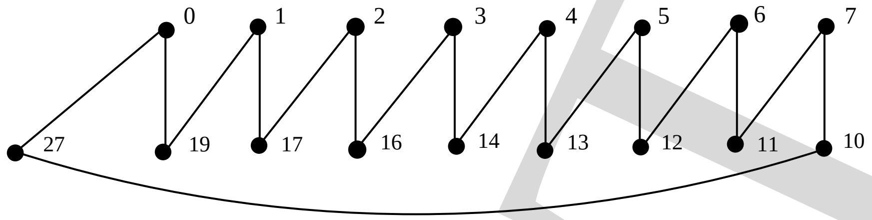

Figure 5: A γ_3^*-labeling of C_{17}.

If we let $\hat{b} = 4m+2$ and $c = 8m+3$, it is easy to verify that we have a a γ_k^*-labeling of C_{4m+1}.

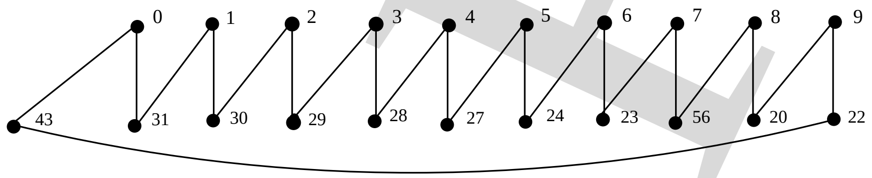

Figure 6: A γ_{11}^*-labeling of C_{21}.

Case 5. G is a C_{4m+1} where k is odd, $k > 2m+1$ and $k \neq 4m+1$.
We take G to be $G_1+G_2+G_3+(2m-1, 8m+k-1, 4m+k-2, 0, 4m+k, 1, 4m+k+6, 2)$, where

$$\begin{aligned} G_1 &= P(2, 2m+k+3, 2m-6), \\ G_2 &= P(m-1, 5m, k-2m-1), \\ G_3 &= P((k-3)/2, (3k-5)/2, 4m-k+1). \end{aligned}$$

If we let $\hat{b} = 4m+k-2$ and $c = 8m+k-1$, it is easy to verify that we have a a γ_k^*-labeling of C_{4m+1}.

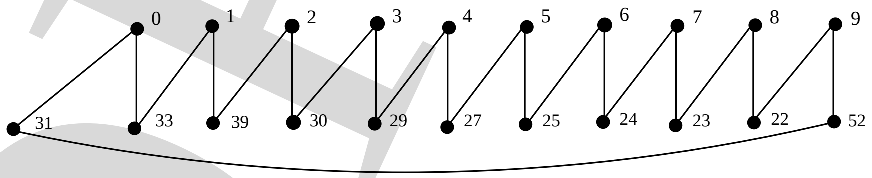

Figure 7: A γ_{13}^*-labeling of C_{21}.

Case 6. G is a C_{4m+1} and $k = 4m+1$.
Thus in this case G is embedded in K_{16m+3}. We take G to be $G_1 + G_2 + (2m-1, 6m+1, 10m+2, 0)$, where

$$\begin{aligned} G_1 &= P(0, 6m+1, 2m), \\ G_2 &= P(m, 5m+2, 2m-2). \end{aligned}$$

If we let $\hat{b} = 6m + 1$ and $c = 10m + 2$, it is easy to verify that we have a a γ_k^*-labeling of C_{4m+1}.

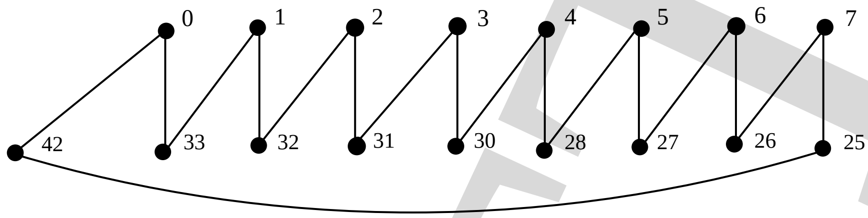

Figure 8: A γ_{17}^*-labeling of C_{17}.

Case 7. G is a C_{4m+3} where k is even and $k \leq 2m$.
In this case, $n = 4m+3$ and G is embedded in the complete graph $K_{8m+2k+5}$. We take G to be $G_1+(k/2-1, 4m+k/2+3, k/2+1)+G_2+G_3+(2m+1, 2m+k+1, 6m+k+4, 0)$, where

$$G_1 = P(0, 4m + 4, k - 2),$$
$$G_2 = P(k/2 + 1, 2m + 3k/2 + 2, 2m - k),$$
$$G_3 = P(m + 1, m + k + 1, 2m).$$

If we let $\hat{b} = 2m + k + 1$ and $c = 6m + k + 4$, it is easy to verify that we have a a γ_k^*-labeling of C_{4m+3}.

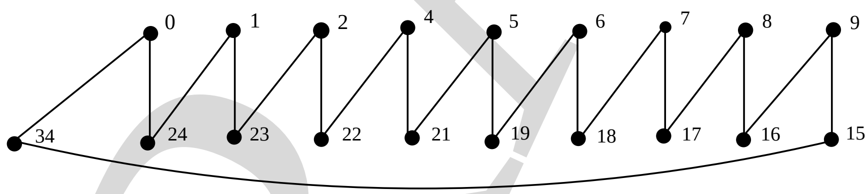

Figure 9: A γ_6^*-labeling of C_{19}.

Case 8. G is a C_{4m+3} where k is even and $k > 2m$.
We take G to be $G_1 + G_2 + G_3 + (2m, 2m + k, 6m + k + 3, 0)$, where

$$G_1 = P(0, 2m + k + 2, 2m),$$
$$G_2 = P(m, 5m + 3, k - 2m - 2),$$
$$G_3 = P(k/2 - 1, 3k/2 - 1, 4m - k + 2).$$

If we let $\hat{b} = 2m + k$ and $c = 6m + k + 3$, it is easy to verify that we have a a γ_k^*-labeling of C_{4m+3}.

Case 9. G is a C_{4m+3} where k is odd and $k \leq 2m + 1$.
We start with the case $k = 1$. If $m = 1$, we take the cycle $(0, 9, 4, 8, 5, 7, 14, 0)$. For, $m \geq 2$, we take G to be $(0, 4m + 5, 4) + G_1 + (m + 4, 3m + 4, m + 6) + G_2 + (2m + 4, 2m+5, 6m+8, 0)$, where $G_1 = P(4, 2m+4, 2m)$, and $G_2 = P(m+6, m+7, 2m-4)$.

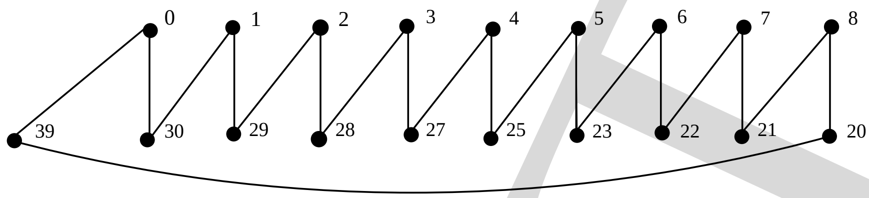

Figure 10: A γ_{12}^*-labeling of C_{19}.

It is easy to verify that G is a C_{4m+3} that admits a γ_1^*-labeling (with $\hat{b} = 2m+5$ and $c = 6m+8$).

For $k \geq 3$, we take G to be $(0, 4m+k+3, 2) + G_1 + G_2 + G_3 + (2m+1, 2m+k+1, 6m+k+4, 0)$, where

$$\begin{aligned} G_1 &= P(2, 4m+5, k-3), \\ G_2 &= P((k+1)/2, (4m+3k+3)/2, 2m-k+1), \\ G_3 &= P(m+1, m+k+1, 2m). \end{aligned}$$

If we let $\hat{b} = 2m+k+1$ and $c = 6m+k+4$, it is easy to verify that we have a a γ_k^*-labeling of C_{4m+3}.

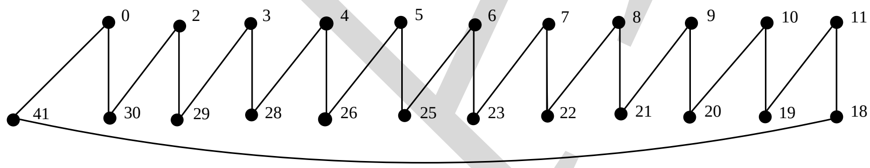

Figure 11: A γ_7^*-labeling of C_{23}.

Case 10. G is a C_{4m+3} where k is odd and $m = 2m+3$.
Note that in this case, G is embedded in K_{12m+11}. We take G to be $(0, 6m+9, 4) + G_1 + G_2 + G_3 + (2m+3, 4m+6, 8m+9, 0)$, where

$$\begin{aligned} G_1 &= P(4, 6m+6, 2), \\ G_2 &= P(5, 4m+8, 2m-2), \\ G_3 &= P(m+4, 3m+7, 2m-2). \end{aligned}$$

If we let $\hat{b} = 4m+6$ and $c = 8m+9$, it is easy to verify that we have a a γ_k^*-labeling of C_{4m+3}.

Case 11. G is a C_{4m+3} where k is odd and $k > 2m+3$, with $k \neq 4m+3$.
We take G to be $(0, 4m+k+3, 2) + G_1 + G_2 + ((k-1)/2, (8m+k+7)/2, (k+3)/2) + G_3 + (2m+2, 2m+k+2, 6m+k+5, 0)$, where

$$\begin{aligned} G_1 &= P(2, 2m+k+2, 2m), \\ G_2 &= P(m+2, 5m+6, k-2m-5), \\ G_3 &= P((k+3)/2, (3k+3)/2, 4m-k+1). \end{aligned}$$

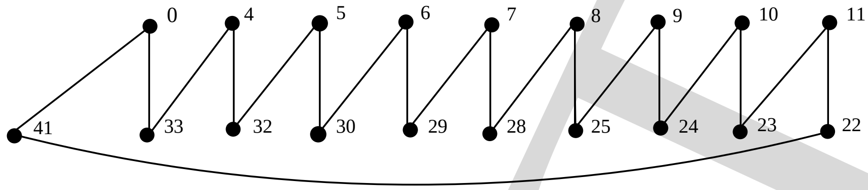

Figure 12: A γ_{11}^*-labeling of C_{19}.

If we let $\hat{b} = 2m + k + 2$ and $c = 6m + k + 5$, it is easy to verify that we have a a γ_k^*-labeling of C_{4m+3}.

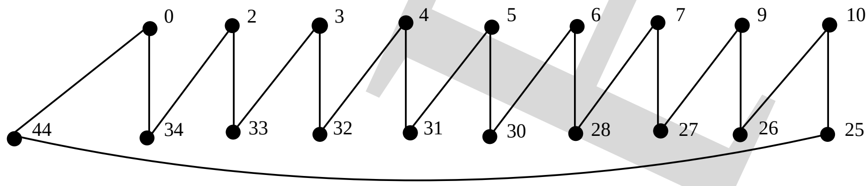

Figure 13: A γ_{15}^*-labeling of C_{19}.

Case 12. G is a C_{4m+3} where $k = 4m + 3$.
First we note that in this case G is embedded in the complete graph K_{16m+11}. We take G to be $(0, 8m + 6, 2) + G_1 + G_2 + (2m + 1, 6m + 5, 10m + 8, 0)$, where

$$G_1 = P(2, 6m + 5, 2m),$$
$$G_2 = P(m + 2, 5m + 6, 2m - 2).$$

If we let $\hat{b} = 6m + 5$ and $c = 10m + 8$, it is easy to verify that we have a a γ_k^*-labeling of C_{4m+3}.

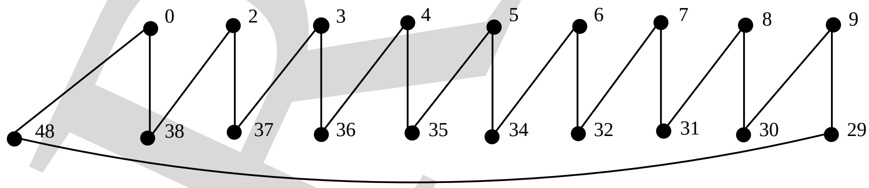

Figure 14: A γ_{19}^*-labeling of C_{19}.

This completes the proof. ∎

It is well known that there exists a cyclic C_3-decomposition of K_{2nx+1} for every positive integer x. Thus in light of Theorem 6, we have the following corollary to Theorem 7. Similar results for even cycles and other bipartite graphs can be found in [10].

Corollary 8 *Let $n \geq 3$ be odd and $k \in [1, n]$ with $(n, k) \notin \{(3, 3), (5, 3)\}$. Then there exists a cyclic C_n-decomposition of $\langle [k, nx + k - 1] \rangle_{2nx+2k-1}$ for every positive integer x.*

4 Other Decompositions of Circulant Graphs

Work on decompositions of circulant graphs has focused on decompositions into perfect matchings or into Hamilton cycles. The graph $\langle L\rangle_t$ has a 1-factorization if and only if L has an element of even order [18]. In [2], Alspach asks whether every $2k$-regular Cayley graph on a finite abelian group has a decomposition into k Hamilton cycles. Many results have been obtained on this problem (see [3] and the references therein), but the general problem is unsolved, even in the case of circulant graphs. Decompositions of low degree circulant graphs into cycles, paths, and circuits are investigated in [8], where a number of results are settled for decompositions of $\langle[1,2]\rangle_n$ and of $\langle[1,3]\rangle_n$. Decompositions of circulant graphs into combinations of Hamilton cycles and various other cycles can be found in [6, 7, 9].

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