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# On $\rho$-labeling 2-regular graphs consisting of 5-cycles

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**Abstract**

Let $G$ be a graph of size $n$ with vertex set $V(G)$ and edge set $E(G)$. A *$\rho$-labeling* of $G$ is a one-to-one function $h : V(G) \to \{0, 1, \ldots, 2n\}$ such that $\{\min\{|h(u)-h(v)|, 2n+1-|h(u)-h(v)|\} : \{u, v\} \in E(G)\} = \{1, 2, \ldots, n\}$. Such a labeling of $G$ yields a cyclic $G$-decomposition of $K_{2n+1}$. It is conjectured by El-Zanati and Vanden Eynden that every 2-regular graph $G$ admits a $\rho$-labeling. We show that the vertex-disjoint union of any number of 5-cycles admits a $\rho$-labeling.

## 1 Introduction

If $a$ and $b$ are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo $n$. For a graph $G$, let $V(G)$ and $E(G)$ denote the vertex set of $G$ and the edge set of $G$, respectively. Let $rG$ denote the vertex-disjoint union of $r$ copies of $G$.

Let $V(K_k) = \mathbb{Z}_k$ and let $G$ be a subgraph of $K_k$. By *clicking* $G$, we mean applying the isomorphism $i \to i + 1$ to $V(G)$. Let $H$ and $G$ be graphs

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such that $G$ is a subgraph of $H$. A $G$-*decomposition* of $H$ is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of edge-disjoint subgraphs of $H$ each of which is isomorphic to $G$ and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. A $G$-decomposition of $K_k$ is known as a $G$-*design of order* $k$. A $G$-decomposition $\Gamma$ of $K_k$ is *cyclic* if clicking is a permutation of $\Gamma$.

The investigation of $G$-designs is a popular area of research in combinatorial design theory. For example, if $G$ is $K_k$, then a $G$-design of order $v$ is a $(v, k, 1)$-BIBD. If $G$ has $n$ edges, then $G$-designs of order $2n+1$ are of particular interest. In 1963, Ringel [8] conjectured that there is a $G$-design of order $2n+1$ for every tree $G$ with $n$ edges. In [9], Rosa introduced graph labelings as means of attacking Ringel's conjecture.

For any graph $G$, a one-to-one function $h : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of $G$. Let $G$ be a graph with $n$ edges and no isolated vertices and let $h$ be a labeling of $G$. Let $h(V(G)) = \{h(u) : u \in V(G)\}$. Define a function $\bar{h} : E(G) \to \mathbb{Z}^+$ by $\bar{h}(e) = |h(u) - h(v)|$, where $e = \{u, v\} \in E(G)$ and let $(\bar{h}(e))^* = \min\{\bar{h}(e), 2n+1-\bar{h}(e)\}$. We will refer to $\bar{h}(e)$ and $(\bar{h}(e))^*$ as the *label* and the *length* of $e$, respectively. If $F \subseteq E(G)$, then $\bar{h}(F) = \{\bar{h}(e) : e \in F\}$ and $(\bar{h}(F))^* = \{(\bar{h}(e))^* : e \in F\}$. We say $h$ is a $\rho$-*labeling* of $G$ if $h(V(G)) \subseteq [0, 2n]$ and $(\bar{h}(E(G))^* = [1, n]$. If $h(V(G)) \subseteq [0, n]$ and $\bar{h}(E(G) = [1, n]$, then $h$ is $\beta$-*labeling* or a *graceful* labeling of $G$.

Labelings are critical to the study of cyclic graph decompositions as seen in the following result from [9].

**Theorem 1.** *Let $G$ be a graph with $n$ edges. There exists a cyclic $G$-decomposition of $K_{2n+1}$ if and only if $G$ has a $\rho$-labeling.*

While a $\rho$-labeling is the most basic of Rosa's labelings, $\beta$-labelings (i.e., graceful) are by far the most popular. Graphs that admit a graceful labeling are called *graceful*. A conjecture that every tree is graceful is one of the best known conjectures in design theory. Unfortunately, graceful labelings are too restrictive for many classes of graphs. For example, $K_4$ is the largest complete graph that is graceful and $C_n$ is graceful if and only if $n \equiv 0$ or 3 (mod 4). For a comprehensive survey of graph labelings that lead to cyclic $G$-designs, we direct the reader to [5]. A dynamic survey on general graph labelings is maintained by Gallian [6].

In this manuscript, we will focus on $\rho$-labelings of $rC_5$, the the vertex-disjoint union of $r$ copies of $C_5$. Kotzig [7] has shown that $rC_5$ is never graceful. In the same paper, Kotzig showed that $rC_3$ is graceful only if $r = 1$. A subsequent result of Dinitz and Rodney [3] is equivalent to showing that $rC_3$ admits a $\rho$-labeling for all positive integers $r$. From results in [1], it can be concluded that every 2-regular bipartite graph admits a $\rho$-labeling. More recently, it was shown in [2] that $rC_{4x+1}$ has a $\rho$-labeling for $r \leq 10$ and $x \geq 1$. Here, we shall show that $rC_5$ has a $\rho$-labeling for all integers $r \geq 1$. This provides further evidence in support of a conjecture of El-Zanati and Vanden Eynden that every 2-regular graph admits a $\rho$-labeling.

# 2 Main Result

Let $C_5$ be the graph with vertex set $\{v_i : 1 \leq i \leq 5\}$ and edge set $\{\{v_i, v_{i+1}\} : 1 \leq i \leq 4\} \bigcup \{v_5, v_1\}$. For a positive integer $r$, let $G = rC_5$, the vertex-disjoint union of $r$ copies of $C_5$. For $1 \leq j \leq r$, let the $j^{\text{th}}$ component of $G$ have vertex set $\{v_{i,j} : 1 \leq i \leq 5\}$ and edge set $\{\{v_{i,j}, v_{i+1,j}\} : 1 \leq i \leq 4\} \bigcup \{v_{5,j}, v_{1,j}\}$. For $1 \leq i \leq 5$, let $V_i = \{v_{i,j} : 1 \leq j \leq r\}$. For $1 \leq i \leq 4$, and $1 \leq j \leq r$, let $e_{i,j}$ denote the edge $\{v_{i,j}, v_{i+1,j}\}$ and let $e_{5,j}$ denote the edge $\{v_{5,j}, v_{1,j}\}$. Finally, for $1 \leq i \leq 5$, let $E_i = \{e_{i,j} : 1 \leq j \leq r\}$.

**Theorem 2.** *Let $G = rC_5$. Then $G$ admits a $\rho$-labeling.*

*Proof.* We will consider two cases depending on whether $r$ is even or odd.
**Case 1:** $r$ is even.
Let $r = 2t$. Thus $|V(G)| = |E(G)| = 10t$. Let $h : V(G) \to [0, 20t]$ be defined as follows:
For $1 \leq j \leq 2t$, let

$$\begin{aligned}
h(v_{1,j}) &= j - 1, \\
h(v_{3,j}) &= 2t + j - 1, \\
h(v_{4,j}) &= 10t + 3j - 1, \\
h(v_{5,j}) &= 6t + 2j - 1,
\end{aligned}$$

and let

$$h(v_{2,j}) = \begin{cases} 20t - j & \text{if } 1 \leq j \leq t, \\ 6t - j & \text{if } t + 1 \leq j \leq 2t. \end{cases}$$

Figure 2 shows the case $r = 6$. Note that when restricted to each $V_i$, the function $h$ is either strictly increasing or strictly decreasing. Thus, $h(v_{i,j})$

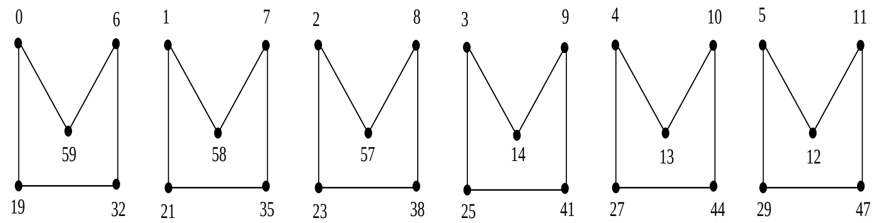


Figure 1: A $\rho$-labeling of $6C_5$.

and $h(v_{i,k})$ are equal if and only if $j = k$. Moreover,

$$\begin{aligned}
h(V_1) &= [0, 2t-1],\\
h(V_2) &= [19t, 20t-1] \cup [4t, 5t-1],\\
h(V_3) &= [2t, 4t-1],\\
h(V_4) &\subseteq [10t+2, 16t-1],\\
h(V_5) &\subseteq [6t+1, 10t-1].
\end{aligned}$$

Thus $h(V_i)$ and $h(V_j)$ are disjoint for $i \neq j$ and $h(V(G)) \subseteq [0, 20t]$. It remains to show that $(\bar{h}(E(G))^* = [1, 10t]$.

We now compute the resulting edge labels. For $1 \leq j \leq t$, we have

$$\begin{aligned}
\bar{h}(e_{1,j}) &= 20t - 2j + 1,\\
\bar{h}(e_{2,j}) &= 18t - 2j + 1,\\
\bar{h}(e_{3,j}) &= 8t + 2j,\\
\bar{h}(e_{4,j}) &= 4t + j,\\
\bar{h}(e_{5,j}) &= 6t + j.
\end{aligned}$$

Since the edge labels $\bar{h}(e_{1,j})$ and $\bar{h}(e_{2,j})$ exceed $10t$, their corresponding edge lengths are $(\bar{h}(e_{1,j}))^* = 2j$ and $(\bar{h}(e_{2,j}))^* = 2t + 2j$. Similarly, for $t+1 \leq j \leq 2t$, we compute

$$\begin{aligned}
\bar{h}(e_{1,j}) &= 6t - 2j + 1,\\
\bar{h}(e_{2,j}) &= 4t - 2j + 1,\\
\bar{h}(e_{3,j}) &= 8t + 2j,\\
\bar{h}(e_{4,j}) &= 4t + j,\\
\bar{h}(e_{5,j}) &= 6t + j.
\end{aligned}$$

In this case, $\bar{h}(e_{3,j})$ is the only label that exceeds $10t$. The corresponding edge length is $(\bar{h}(e_{3,j}))^* = 12t - 2j + 1$. Thus,

$$\begin{aligned}
(\bar{h}(E_1))^* &= \{2j : 1 \le j \le t\} \cup \{6t - 2j + 1 : t + 1 \le j \le 2t\},\\
(\bar{h}(E_2))^* &= \{2t - 2j + 1 : 1 \le j \le t\} \cup \{4t - 2j + 1 : t + 1 \le j \le 2t\},\\
(\bar{h}(E_3))^* &= \{8t + 2j : 1 \le j \le t\} \cup \{12t - 2j + 1 : t + 1 \le j \le 2t\},\\
(\bar{h}(E_4))^* &= \{4t + j : 1 \le j \le 2t\},\\
(\bar{h}(E_5))^* &= \{6t + j : 1 \le j \le 2t\}.
\end{aligned}$$

The above sets can be rewritten as:

$$\begin{aligned}
(\bar{h}(E_1))^* &= \{2m : 1 \le m \le t\} \cup \{2m - 1 : t + 1 \le m \le 2t\},\\
(\bar{h}(E_2))^* &= \{2m - 1 : 1 \le m \le t\} \cup \{2m : t + 1 \le m \le 2t\},\\
(\bar{h}(E_3))^* &= \{m : 8t + 1 \le m \le 10t\},\\
(\bar{h}(E_4))^* &= \{m : 4t + 1 \le m \le 6t\},\\
(\bar{h}(E_5))^* &= \{m : 6t + 1 \le m \le 8t\}.
\end{aligned}$$

Thus, $(\bar{h}(E(G))^* = [1, 10t]$ and $h$ is a $\rho$-labeling of $G$.
**Case 2:** $r$ is odd.
Let $r = 2t + 1$. Thus $|V(G)| = |E(G)| = 10t$. Let $h : V(G) \to [0, 20t + 10]$ be defined as follows:
For $1 \le j \le 2t + 1$, let

$$\begin{aligned}
h(v_{1,j}) &= j - 1,\\
h(v_{3,j}) &= 2t + j,\\
h(v_{4,j}) &= 10t + 3j + 4,\\
h(v_{5,j}) &= 6t + 2j + 2,
\end{aligned}$$

and let

$$h(v_{2,j}) = \begin{cases} 20t - j + 10 & \text{if } 1 \le j \le t,\\ 6t - j + 3 & \text{if } t + 1 \le j \le 2t + 1.\end{cases}$$

Figure 2 shows the case $r = 5$. Note that when restricted to each $V_i$, the function $h$ is either strictly increasing or strictly decreasing. Thus, $h(v_{i,j})$ and $h(v_{i,k})$ are equal if and only if $j = k$. Moreover,

$$\begin{aligned}
h(V_1) &= [0, 2t],\\
h(V_2) &= [19t + 10, 20t + 9] \cup [4t + 2, 5t + 2],\\
h(V_3) &= [2t + 1, 4t + 1],\\
h(V_4) &\subseteq [10t + 7, 16t + 7],\\
h(V_5) &\subseteq [6t + 4, 10t + 4].
\end{aligned}$$

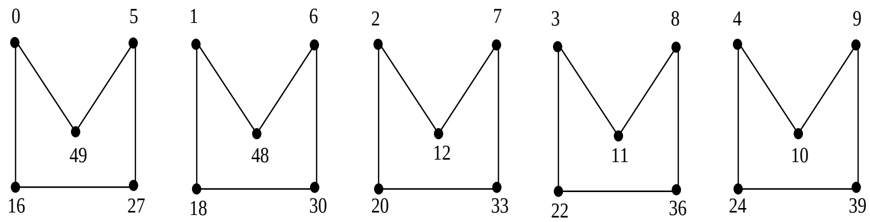


Figure 2: A $\rho$-labeling of $5C_5$.

Thus $h(V_i)$ and $h(V_j)$ are disjoint for $i \neq j$ and $h(V(G)) \subseteq [0, 20t+10]$. It remains to show that $(\bar{h}(E(G))^* = [1, 10t+5]$.

We now compute the resulting edge labels. For $1 \leq j \leq t$, we have

$$\begin{aligned}
\bar{h}(e_{1,j}) &= 20t - 2j + 11,\\
\bar{h}(e_{2,j}) &= 18t - 2j + 10,\\
\bar{h}(e_{3,j}) &= 8t + 2j + 4,\\
\bar{h}(e_{4,j}) &= 4t + j + 2,\\
\bar{h}(e_{5,j}) &= 6t + j + 3.
\end{aligned}$$

Since the edge labels $\bar{h}(e_{1,j})$ and $\bar{h}(e_{2,j})$ exceed $10t+5$, their corresponding edge lengths are $(\bar{h}(e_{1,j}))^* = 2j$ and $(\bar{h}(e_{2,j}))^* = 2t + 2j + 1$. Similarly, for $t + 1 \leq j \leq 2t + 1$, we compute

$$\begin{aligned}
\bar{h}(e_{1,j}) &= 6t - 2j + 4,\\
\bar{h}(e_{2,j}) &= 4t - 2j + 3,\\
\bar{h}(e_{3,j}) &= 8t + 2j + 4,\\
\bar{h}(e_{4,j}) &= 4t + j + 2,\\
\bar{h}(e_{5,j}) &= 6t + j + 3.
\end{aligned}$$

In this case, $\bar{h}(e_{3,j})$ is the only label that exceeds $10t+5$. The corresponding edge length is $(\bar{h}(e_{3,j}))^* = 12t - 2j + 7$. Thus,

$$\begin{aligned}
(\bar{h}(E_1))^* &= \{2j : 1 \leq j \leq t\} \cup \{6t - 2j + 4 : t + 1 \leq j \leq 2t + 1\},\\
(\bar{h}(E_2))^* &= \{2t + 2j + 1 : 1 \leq j \leq t\} \cup \{4t - 2j + 3 : t + 1 \leq j \leq 2t + 1\},\\
(\bar{h}(E_3))^* &= \{8t + 2j + 4 : 1 \leq j \leq t\} \cup \{12t - 2j + 7 : t + 1 \leq j \leq 2t + 1\},\\
(\bar{h}(E_4))^* &= \{4t + j + 2 : 1 \leq j \leq 2t + 1\},\\
(\bar{h}(E_5))^* &= \{6t + j + 3 : 1 \leq j \leq 2t + 1\}.
\end{aligned}$$

The above sets can be rewritten as:

$$\begin{aligned}
(\bar{h}(E_1))^* &= \{2m : 1 \le m \le 2t+1\},\\
(\bar{h}(E_2))^* &= \{2m-1 : 1 \le m \le 2t+1\},\\
(\bar{h}(E_3))^* &= \{m : 8t+5 \le m \le 10t+5\},\\
(\bar{h}(E_4))^* &= \{m : 4t+3 \le m \le 6t+3\},\\
(\bar{h}(E_5))^* &= \{m : 6t+4 \le m \le 8t+4\}.
\end{aligned}$$

Thus, $(\bar{h}(E(G))^* = [1, 10t+5]$ and $h$ is a $\rho$-labeling of $G$. □

In light of Theorem 1 and Theorem 2, we have the following corollary.

**Corollary 3.** *If $G = rC_5$, then there exists a cyclic $G$-decomposition of $K_{10r+1}$.*

In a forthcoming article [4], we extend the results from this paper to show that every 2-regular graph consisting of $m$-cycles admits a $\rho$-labeling.

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