On cyclic G-designs where G is the one-point union of two cycles

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Abstract

Let G be the one-point union of two cycles and suppose G has n edges. We show via various graph labelings that there exists a cyclic G-decomposition of K_{2nt+1} for every positive integer t.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_t$ the group of integers modulo t. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. The *order* and the *size* of a graph G are $|V(G)|$ and $|E(G)|$, respectively. If G_1 and G_2 are vertex-transitive graphs, $G_1 \mathbin{\dot\cup} G_2$ denotes the one-point union of G_1 and G_2.

Let $V(K_t) = \{0, 1, \ldots, t-1\}$. The *length* of an edge $\{i, j\}$ in K_t is $\min\{|i-j|, t-|i-j|\}$. Note that if t is odd, then K_t consists of t edges of length i for $i = 1, 2, \ldots, \frac{t-1}{2}$.

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Let $V(K_t) = \mathbb{Z}_t$ and let G be a subgraph of K_t. By *clicking* G, we mean applying the permutation $i \to i+1$ to $V(G)$. Note that clicking an edge does not change its length. Let H and G be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set $\Delta = \{G_1, G_2, \ldots, G_r\}$ of pairwise edge-disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{r} E(G_i)$. A G-decomposition of K_t is also known as a *(K_t, G)-design.* A (K_t, G)-design Δ is *cyclic* if clicking is an automorphism of Δ. The study of graph decompositions is generally known as the study of graph designs, or G-designs. For recent surveys on G-designs, see [1] and [6].

Let G be a graph of size n. A primary question in the study of graph designs is: *for what values of k does there exist a (K_k, G)-design*? For most studied graphs G, it is often the case that if $k \equiv 1 \pmod{2n}$, then there exists a (K_k, G)-design. A common approach to finding these designs is through the use of graph labelings.

In this paper, we will show via various graph labelings that if G of size n is the one-point union of two cycles, then there exists a cyclic G-decomposition of K_{2nt+1} for every positive integer t.

1.1 Graph Labelings

For any graph G, a one-to-one function $f : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [12], Rosa introduced a hierarchy of labelings. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{f}(E(G)) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

(ℓ1) $f(V(G)) \subseteq [0, 2n]$,

(ℓ2) $f(V(G)) \subseteq [0, n]$,

(ℓ3) $\bar{f}(E(G)) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,

(ℓ4) $\bar{f}(E(G)) = [1, n]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ consider also

(ℓ5) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,

(ℓ6) there exists an integer λ such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

(ℓ1), (ℓ3) is called a *ρ-labeling*;

(ℓ1), (ℓ4) is called a *σ-labeling*;

(ℓ2), (ℓ4) is called a *β-labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. Suppose G is bipartite. If a ρ, σ, or β-labeling of G satisfies condition (ℓ5), then the labeling is *ordered* and is denoted by ρ^+, σ^+ or β^+, respectively. If in addition (ℓ6) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++} or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an *α-labeling* as introduced in [12]. Labelings of the types above are called *Rosa-type* because of Rosa's original article [12] on the topic. (See [9] for a recent comprehensive survey of Rosa-type labelings). A dynamic survey on general graph labelings is maintained by Gallian [11].

Call a connected graph G *Eulerian* if every vertex of G has even degree. If a graph G with Eulerian components admits a σ-labeling, then we have the following well-known restriction on $|E(G)|$.

Theorem 1 (Parity Condition in [12]) *If a graph G with Eulerian components and n edges has a σ-labeling, then $n \equiv 0$ or $3 \pmod 4$. If in addition G is bipartite, then $n \equiv 0 \pmod 4$.*

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [12].

Theorem 2 *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G admits a ρ-labeling.*

Theorem 3 *Let G be a bipartite graph with n edges that admits an α-labeling. Then there exists a cyclic G-decomposition of K_{2nt+1} for all positive integers t.*

It is easy to see how Theorem 3 works. Let G have bipartition $\{A, B\}$ and let h be an α-labeling for G with $h(A) < h(B)$. Let $B_1, B_2, \ldots, B_t$ be t vertex-disjoint copies of B. The vertex in B_i corresponding to $b \in B$ will be called b_i. Let $B^* = \bigcup_{i=1}^t B_i$. We define a new graph G^* with vertex set $A \bigcup B^*$ and edges $\{a, b_i\}$, $1 \le i \le t$, whenever $a \in A$ and $\{a, b\}$ is an edge of G. Clearly G^* has nt edges and G divides G^*. Define a labeling h^* on G^* by

$$h^*(v) = \begin{cases} h(v) & v \in A, \\ h(b) + (i-1)n & v = b_i \in B_i. \end{cases}$$

The labeling h^* is an α-labeling of G^* (which is also a ρ-labeling) and thus the result follows by Theorem 2.

From a graph decompositions perspective, Theorem 3 offers a great advantage over Theorem 2. However, there are many classes of bipartite graphs (see [9]) that do not admit α-labelings. Theorem 3 was extended to cover graphs that admit ρ^+-labelings in [8].

Theorem 4 *Let G be a bipartite graph with n edges that admits an ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nt+1} for all positive integers t.*

Theorem 4 is set up in exactly the same way as Theorem 3 except that now we let $h^*(v) = h(b) + (i-1)2n$ for $v = b_i \in B_i$. It is conjectured by El-Zanati and Vanden Eynden (see [9]) that every bipartite graph admits a ρ^+-labeling.

Labelings that lead to results similar those of Theorem 4 have now been introduced for almost-bipartite graphs [4] and for tripartite graphs in general [7].

A non-bipartite graph G is said to be *almost-bipartite* if $G-e$ is bipartite for some $e \in E(G)$. Note that if G is almost-bipartite with $e = \{\hat{b}, c\}$, then G is necessarily tripartite and $V(G)$ can be partitioned into three sets A, B and $C = \{c\}$ such that $\hat{b} \in B$ and e is the only edge joining an element of B to c.

Let G be an almost-bipartite graph with n edges with vertex tripartition A, B, C as above. A labeling h of the vertices of G is called a γ-*labeling* of G if the following conditions hold:

(g1) The function h is a ρ-labeling of G.

(g2) If $\{a, v\}$ is an edge of G with $a \in A$, then $h(a) < h(v)$.

(g3) We have $h(c) - h(\hat{b}) = n$.

Several classes of almost-bipartite graphs have been shown to have γ-labelings (see [9]). It was shown in [4] that γ-labelings yield results similar to ρ^+-labelings.

Theorem 5 *Let G be an almost-bipartite graph with n edges that admits a γ-labeling. Then there exists a cyclic G-deomposition of K_{2nt+1} for all positive integers t.*

We illustrate how Theorem 5 works. Let G have n edges and let h be a γ-labeling for G, with A, B, C, c, and $\hat{b}$ as in the above definition. Let $B_1, B_2, \ldots, B_t$ be t vertex-disjoint copies of B, and let $c_1, c_2, \ldots, c_t$ be t new vertices. The vertex in B_i corresponding to $b \in B$ will be called b_i. Let $B^* = \bigcup_{i=1}^{t} B_i$ and $C^* = \{c_1, c_2, \ldots, c_t\}$. We define a new graph G^* with vertex set $A \bigcup B^* \bigcup C^*$ and edges $\{a, v_i\}$, $1 \leq i \leq t$, whenever $a \in A$

and $\{a, v\}$ is an edge of G, and $\{\hat{b_i}, c_i\}$, $1 \leq i \leq t$. Clearly G^* has nt edges and G divides G^*. Define a labeling h^* on G^* by

$$h^*(v) = \begin{cases} h(v) & v \in A, \\ h(b) + (i-1)2n & v = b_i \in B_i, \\ h(c) + (t-i)2n & v = c_i. \end{cases}$$

The labeling h^* is a ρ-labeling of G^* and thus the result follows by Theorem 2.

The concept of a γ-labeling was generalized in [7] to cover tripartite graphs that are not necessarily almost-bipartite. Let G be a tripartite graph with n edges having the vertex tripartition $\{A, B, C\}$. A *σ-tripartite* labeling of G is a one-to-one function $h : V(G) \to [0, 2n]$ that satisfies

(s1) h is a σ-labeling of G.

(s2) If $\{a, v\} \in E(G)$ with $a \in A$, then $h(a) < h(v)$.

(s3) If $e = \{b, c\} \in E(G)$ with $b \in B$ and $c \in C$, then there exists an edge $e' = \{b', c'\} \in E(G)$ with $b' \in B$ and $c' \in C$ such that

$$|h(c') - h(b')| + |h(c) - h(b)| = n.$$

(s4) If $a \in A$ and $v \in B \cup C$, then $h(a) - h(v) \neq n$.

(s5) If $b \in B$ and $c \in C$, then $|h(b) - h(c)| \notin \{n, 2n\}$.

A σ-tripartite labeling of G is necessarily a σ-labeling of G. Thus the parity condition must be satisfied in order for G to admit a σ-tripartite labeling.

Also, a *ρ-tripartite* labeling of G is a one-to-one function $h : V(G) \to [0, 2n]$ that satisfies

(r1) h is a ρ-labeling of G.

(r2) If $\{a, v\} \in E(G)$ with $a \in A$, then $h(a) < h(v)$.

(r3) If $e = \{b, c\} \in E(G)$ with $b \in B$ and $c \in C$, then there exists an edge $e' = \{b', c'\} \in E(G)$ with $b' \in B$ and $c' \in C$ such that

$$|h(c') - h(b')| + |h(c) - h(b)| = 2n.$$

(r4) If $b \in B$ and $c \in C$, then $|h(b) - h(c)| \neq 2n$.

We note that a γ-labeling of a graph G is necessarily a ρ-tripartite labeling of G. For the purposes of this manuscript, we will consider almost-bipartite graphs separately from general tripartite graphs.

The following theorem shows that the above tripartite labelings yield results similar to γ-labelings:

Theorem 6 *If a tripartite graph G with n edges has a σ-tripartite or a ρ-tripartite labeling, then there exists a cyclic G-decompositition of K_{2nt+1} for all postitive integers t.*

Theorem 6 works in a similar way to Theorem 5. Let G have n edges and let h be a σ-tripartite or a ρ-tripartite labeling for G, with A, B, and C as in the above definitions. Let $B_1, B_2, \ldots, B_t$ be t vertex-disjoint copies of B, and let $C_1, C_2, \ldots, C_t$ be t vertex-disjoint copies of C. The vertex in B_i corresponding to $b \in B$ will be called b_i. Similarly, the vertex in C_i corresponding to $c \in C$ will be called c_i. Let $B^* = \bigcup_{i=1}^t B_i$ and $C^* = \bigcup_{i=1}^t C_i$. We define a new graph G^* with vertex set $A \bigcup B^* \bigcup C^*$ and edges $\{a, v_i\}$, $1 \le i \le t$, whenever $a \in A$ and $\{a, v\}$ is an edge of G, and $\{b_i, c_i\}$, $1 \le i \le t$, whenever $\{b, c\}$ is an edge of G with $b \in B$ and $c \in C$. Clearly G^* has nt edges and G divides G^*. Define a labeling h^* on G^* by

$$h^*(v) = \begin{cases} h(v) & v \in A, \\ h(b) + (i-1)\delta n & v = b_i \in B_i, \\ h(c) + (t-i)\delta n & v = c_i \in C_i. \end{cases}$$

We use $\delta = 1$ if h is a σ-tripartite labeling of G and $\delta = 2$ if h is a ρ-tripartite labeling. Either way, the labeling h^* is a either a σ-labeling or a ρ-labeling of G^* and the result follows by Theorem 2.

Let $r \ge 3$ and $s \ge 4$ be integers and let $G = C_r \uplus C_s$. We will show that G admits: an α-labeling if $r \equiv s \equiv 0 \pmod 4$, a ρ^+-labeling if $r \equiv s \equiv 2 \pmod 4$, a γ-labeling if $r + s \equiv 1$ or $3 \pmod 4$ and $G \ne C_3 \uplus C_4$, a σ-tripartite labeling if r and s are both odd and $r + s \equiv 0 \pmod 4$, and a ρ-tripartite labeling if r and s are both odd and $r + s \equiv 2 \pmod 4$ or $G = C_3 \uplus C_4$.

1.2 Some Known Results

Several authors have investigated labelings of the one-point union of various graphs. We direct the interested reader to Gallian's graph labelings survey [11] for a detailed list of results. We will only cite the most relevant results for our problem. Most of the previous investigations have focused on graceful and α-labelings of one-point union graphs.

Bodendiek, Schumacher, and Wegner [5] proved that the one-point union of any two cycles is graceful when the number of edges is congruent to 0 or 3 modulo 4. Figueroa-Centeno, Ichishima, and Muntaner-Batle [10] have shown that if $m \equiv 0 \pmod 4$ then the one-point union of 2, 3, or 4 copies of C_m admits an α-labeling, and if $m \equiv 2 \pmod 4$ then the one-point union of 2 or 4 copies of C_m admits an α-labeling. They conjecture that the one-point union of n copies of C_m admits an α-labeling if and only if $mn \equiv 0 \pmod 4$. Let $C_n^{(t)}$ denote the one-point union of t cycles of length n. Bermond, Brouwer, and Germa [2] and Bermond, Kotzig, and Turgeon [3] proved that $C_3^{(t)}$ is graceful if and only if $t \equiv 0$ or $1 \pmod 4$.

1.3 Additional Notation and Definitions

For ease of notation, we will henceforth consider graphs whose vertices are (distinct) nonnegative integers. Each vertex will be its own label, so the label of the edge $\{x, y\}$ in such a graph will be simply $|x - y|$.

We denote the directed path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \le i \le k-1$, by $(x_0, x_1, \ldots, x_k)$. The *first vertex* of this path is x_0, the *second vertex* is x_1, and the *last vertex* is x_k. If $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$ are directed paths with $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$.

Let $P(k)$ be the path with k edges and $k+1$ vertices $0, 1, \ldots, k$ given by $(0, k, 1, k-1, 2, k-2, \ldots, \lceil k/2 \rceil)$. Note that the set of vertices of this graph is $A \cup B$, where $A = [0, \lfloor k/2 \rfloor]$, $B = [\lfloor k/2 \rfloor + 1, k]$, and every edge joins a vertex of A to one of B. Furthermore the set of labels of the edges of $P(k)$ is $[1, k]$.

Now let a and b be nonnegative integers with $a \le b$ and let us add a to all the vertices of A and b to all the vertices of B. We will denote the resulting graph by $P(a, b, k)$. Note that this graph has the following properties.

P1 $P(a, b, k)$ is a path with first vertex a and second vertex $b + k$. If k is even, its last vertex is $a + k/2$.

P2 Each edge of $P(a, b, k)$ joins a vertex of $A' = [a, \lfloor k/2 \rfloor + a]$ to a larger vertex of $B' = [\lfloor k/2 \rfloor + 1 + b, k + b]$.

P3 The set of edge labels of $P(a, b, k)$ is $[b - a + 1, b - a + k]$.

Figure 1 shows examples of this path notation.

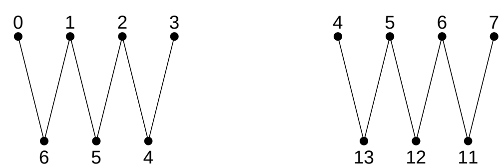

Figure 1: The paths $P(6)$ and $P(4, 7, 6)$.

2 Main Results

Lemma 7 *If a graph G is the one-point union of even cycles C_r and C_s where $r \equiv s \pmod 4$, then G has an α-labeling.*

Proof. We will consider two cases.

Case 1 $r \equiv s \equiv 0 \pmod 4$.
Let $G = C_{4x} \dot{\cup} C_{4y}$ where $x, y \geq 1$. Let $C_{4x} = G_1 + G_2 + (2x-1, b_1, 0)$ and $C_{4y} = G_3 + G_4 + (2x+2y-2, b_2, 2x-1)$ where $b_1 = 2x+4y$, $b_2 = 2x+4y-1$, and

$$G_1 = P(0, 2x+4y, 2x),$$
$$G_2 = P(x, x+4y+1, 2x-2),$$
$$G_3 = P(2x-1, 2x+2y, 2y-2),$$
$$G_4 = P(2x+y-2, 2x+y-2, 2y).$$

(Note: In the case when $x = 1$, the path G_2 is empty, and when $y = 1$, the path G_3 is empty. However, this does not change the proof in any way.)

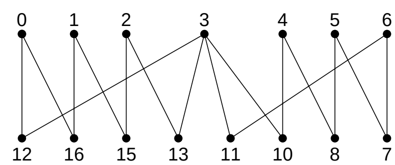

Figure 2: An α-labeling of $C_8 \dot{\cup} C_8$.

First, we show that $G_1 + G_2 + (2x-1, b_1, 0)$ is a cycle of length $4x$, and $G_3 + G_4 + (2x+2y-2, b_2, 2x-1)$ is a cycle of length $4y$. Note that by **P1**, the first vertex of G_1 is 0, and the last is x; the first vertex of G_2 is x, and the last is $2x-1$; the first vertex of G_3 is $2x-1$, and the last is $2x+y-2$; and the first vertex of G_4 is $2x+y-2$, and the last is $2x+2y-2$. For $1 \leq i \leq 4$, let A_i and B_i denote the sets labeled A' and B' in **P2** corresponding to the path G_i. Then using **P2**, we compute

$$A_1 = [0, x], \qquad B_1 = [3x+4y+1, 4x+4y],$$
$$A_2 = [x, 2x-1], \qquad B_2 = [2x+4y+1, 3x+4y-1],$$
$$A_3 = [2x-1, 2x+y-2], \qquad B_3 = [2x+3y, 2x+4y-2],$$
$$A_4 = [2x+y-2, 2x+2y-2], \qquad B_4 = [2x+2y-1, 2x+3y-2].$$

Thus,

$$A_1 \leq A_2 \leq A_3 \leq A_4 < B_4 < B_3 < b_2 < b_1 < B_2 < B_1. \tag{1}$$

Note that $V(G_1) \cap V(G_2) = \{x\}$, $V(G_2) \cap V(G_3) = \{2x-1\}$, and $V(G_3) \cap V(G_4) = \{2x+y-2\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1+G_2+(2x-1, b_1, 0)$ is a cycle of length $4x$, and $G_3+G_4+(2x+2y-2, b_2, 2x-1)$ is a cycle of length $4y$. Furthermore, $V(C_{4x}) \cap V(C_{4y}) = \{2x-1\}$; therefore, G is a graph composed of two cycles that share a single vertex.

Next, let E_i denote the set of edge labels in G_i for $1 \leq i \leq 4$. By **P3**, we have edge labels

$$\begin{aligned}
E_1 &= [2x+4y+1, 4x+4y], \\
E_2 &= [4y+2, 2x+4y-1], \\
E_3 &= [2y+2, 4y-1], \\
E_4 &= [1, 2y].
\end{aligned}$$

Moreover, the path $(2x-1, b_1, 0)$ consists of edges with labels $4y+1$ and $2x+4y$, and the path $(2x+2y-2, b_2, 2x-1)$ consists of edges with labels $2y+1$ and $4y$. Thus, the edge set of G has one edge of each label i where $1 \leq i \leq 4x+4y$, and condition $(\ell 4)$ for an α-labeling is satisfied.

Finally, let $A = \bigcup_{i=1}^{4} A_i$ and $B = \bigcup_{i=1}^{4} B_i \cup \{b_1, b_2\}$. Then, $\{A, B\}$ is a bipartition of $V(G)$. Conditions $(\ell 2)$ and $(\ell 6)$ for an α-labeling are clear from (1). Thus, we have an α-labeling of G.

Case 2 $r \equiv s \equiv 2 \pmod 4$.
Let $G = C_{4x+2} \dot{\cup} C_{4y+2}$ where $x, y \geq 1$. Let $C_{4x+2} = G_1 + G_2 + (2x, b_1, 0)$ and $C_{4y+2} = G_3 + G_4 + (2x+2y, b_2, 2x)$ where $b_1 = 2x+4y+2$, $b_2 = 2x+4y+3$, and

$$\begin{aligned}
G_1 &= P(0, 2x+4y+2, 2x+2), \\
G_2 &= P(x+1, x+4y+4, 2x-2), \\
G_3 &= P(2x, 2x+2y+3, 2y-2), \\
G_4 &= P(2x+y-1, 2x+y-1, 2y+2).
\end{aligned}$$

(Note: In the case when $x = 1$, the path G_2 is empty, and when $y = 1$, the path G_3 is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have an α-labeling of G. $\blacksquare$

It is necessary here to define a new operator on the edge label values and sets. Let G be a graph with n edges. If m is the label of an edge, let $m^* = \min\{m, 2n+1-m\}$ be the length of the edge, and let $S^* = \{m^* : m \in S\}$ be the corresponding set of edge lengths. Thus if the set of vertices of G is a subset of $[0, 2n]$ and the set E of edge labels of G satisfies $E^* = [1, n]$, then G has a ρ-labeling.

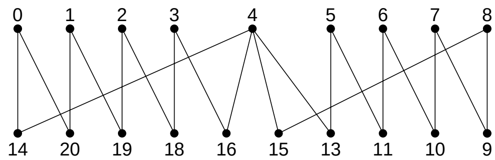

Figure 3: An α-labeling of $C_{10} \,\dot{\cup}\, C_{10}$.

Lemma 8 *If a graph G is the one-point union of even cycles C_r and C_s where $r \not\equiv s \pmod 4$, then G has a uniformly-ordered ρ-labeling.*

Proof. Let $G = C_{4x+2} \,\dot{\cup}\, C_{4y}$ where $x, y \geq 1$. Let $C_{4x+2} = G_1 + G_2 + (2x, b_1, 0)$ and $C_{4y} = G_3 + G_4 + (2x+2y-1, b_2, 2x)$ where $b_1 = 4x+4y+3$, $b_2 = 2x+2y$, and

$$\begin{aligned} G_1 &= P(0, 2x+4y+3, 2x-2), \\ G_2 &= P(x-1, x+4y-1, 2x+2), \\ G_3 &= P(2x, 2x+2y, 2y), \\ G_4 &= P(2x+y, 2x+y+1, 2y-2). \end{aligned}$$

(Note: In the case when $x = 1$, the path G_1 is empty, and when $y = 1$, the path G_4 is empty. However, this does not change the proof in any way.)

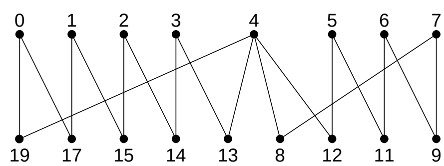

Figure 4: A uniformly-ordered ρ-labeling of $C_{10} \,\dot{\cup}\, C_8$.

First, we show that $G_1 + G_2 + (2x, b_1, 0)$ is a cycle of length $4x+2$, and $G_3 + G_4 + (2x+2y-1, b_2, 2x)$ is a cycle of length $4y$. Note that by **P1**, the first vertex of G_1 is 0, and the last is $x-1$; the first vertex of G_2 is $x-1$, and the last is $2x$; the first vertex of G_3 is $2x$, and the last is $2x+y$; and the first vertex of G_4 is $2x+y$, and the last is $2x+2y-1$. For $1 \leq i \leq 4$, let A_i and B_i denote the sets labeled A' and B' in **P2** corresponding to the path G_i. Then using **P2**, we compute

$$\begin{aligned} A_1 &= [0, x-1], & B_1 &= [3x+4y+3, 4x+4y+1], \\ A_2 &= [x-1, 2x], & B_2 &= [2x+4y+1, 3x+4y+1], \\ A_3 &= [2x, 2x+y], & B_3 &= [2x+3y+1, 2x+4y], \\ A_4 &= [2x+y, 2x+2y-1], & B_4 &= [2x+2y+1, 2x+3y-1]. \end{aligned}$$

Thus,

$$A_1 \le A_2 \le A_3 \le A_4 < b_2 < B_4 < B_3 < B_2 < B_1 < b_1. \qquad (2)$$

Note that $V(G_1) \cap V(G_2) = \{x-1\}$, $V(G_2) \cap V(G_3) = \{2x\}$, and $V(G_3) \cap V(G_4) = \{2x+y\}$; otherwise, G_i and G_j are vertex-disjoint for $i \ne j$. Therefore, $G_1+G_2+(2x, b_1, 0)$ is a cycle of length $4x+2$, and $G_3+G_4+(2x+2y-1, b_2, 2x)$ is a cycle of length $4y$. Furthermore, $V(C_{4x+2}) \cap V(C_{4y}) = \{2x\}$; therefore, G is a graph composed of two cycles that share a single vertex.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 4$. By **P3**, we have edge labels

$$\begin{aligned} E_1 &= [2x+4y+4, 4x+4y+1], \\ E_2 &= [4y+1, 2x+4y+2], \\ E_3 &= [2y+1, 4y], \\ E_4 &= [2, 2y-1] \end{aligned}$$

yielding edge lengths of the same values. Moreover, the path $(2x, b_1, 0)$ consists of edges with lengths $2x+4y+3$ and $(4x+4y+3)^* = 4x+4y+2$, and the path $(2x+2y-1, b_2, 2x)$ consists of edges with lengths 1 and $2y$. Thus, the edge set of G has one edge of each length i where $1 \le i \le 4x+4y+2$, and condition ($\ell 3$) for a ρ^{++}-labeling is satisfied.

Finally, let $A = \bigcup_{i=1}^{4} A_i$, $B = \bigcup_{i=1}^{4} B_i \cup \{b_1, b_2\}$. Conditions ($\ell 1$) and ($\ell 6$) of a ρ^{++}-labeling are clear from (2). Thus, we have a uniformly-ordered ρ-labeling of G. ∎

Lemma 9 *If a graph G is the one-point union of C_r and C_s where $r \not\equiv s \pmod 2$ and $\{r, s\} \ne \{3, 4\}$, then G has a γ-labeling.*

Proof. We will consider four cases.

Case 1 $r \equiv 0 \pmod 4$ and $s \equiv 1 \pmod 4$.
Let $G = C_{4x} \,\dot\cup\, C_{4y+1}$ where $x, y \ge 1$. We will consider two subcases.

Case 1.1 $x = 1$.
Let $C_4 = (a_1, b_1, a_2, b_2, a_1)$ and $C_{4y+1} = G_1 + G_2 + (2y-1, \hat{b}, c, a_1, b_3, 1)$ where $a_1 = 0$, $b_1 = 2y+4$, $a_2 = 2y$, $b_2 = 2y+5$, $\hat{b} = 4y+5$, $c = 8y+10$, $b_3 = 8y+9$, and

$$\begin{aligned} G_1 &= P(1, 6y+8, 2y-2), \\ G_2 &= P(y, 5y+6, 2y-2). \end{aligned}$$

(Note: In the case when $y = 1$, the paths G_1 and G_2 are empty. However, this does not change the proof in any way.)

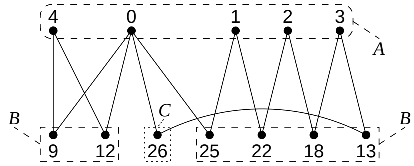

Figure 5: A γ-labeling of $C_4 \,\dot\cup\, C_9$.

First, we show that $G_1 + G_2 + (2y-1, \hat{b}, c, a_1, b_3, 1)$ is a cycle of length $4y+1$. Note that by **P1**, the first vertex of G_1 is 1, and the last is y; and the first vertex of G_2 is y, and the last is $2y-1$. For $1 \le i \le 2$, let A_i and B_i denote the sets labeled A' and B' in **P2** corresponding to the path G_i. Then using **P2**, we compute

$$A_1 = [1, y], \qquad B_1 = [7y+8, 8y+6],$$
$$A_2 = [y, 2y-1], \qquad B_2 = [6y+6, 7y+4].$$

Thus,

$$a_1 < A_1 \le A_2 < a_2 < b_1 < b_2 < \hat{b} < B_2 < B_1 < b_3 < c. \tag{3}$$

Note that $V(G_1) \cap V(G_2) = \{y\}$; otherwise, G_1 and G_2 are vertex-disjoint. Therefore, $G_1 + G_2 + (2y-1, \hat{b}, c, a_1, b_3, 1)$ is a cycle of length $4y+1$. Furthermore, $V(C_4) \cap V(C_{4y+1}) = \{a_1 = 0\}$; therefore, G is a graph composed of two cycles that share a single vertex.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 2$. By **P3**, we have edge labels

$$E_1 = [6y+8, 8y+5],$$
$$E_2 = [4y+7, 6y+4],$$

yielding edge lengths

$$E_1^* = \{8y+11-\ell : \ell \in E_1\} = [6, 2y+3],$$
$$E_2^* = \{8y+11-\ell : \ell \in E_2\} = [2y+7, 4y+4].$$

Moreover, the path $(a_1, b_1, a_2, b_2, a_1)$ consists of edges with lengths $2y+4$, 4, 5, and $2y+5$; and the path $(2y-1, \hat{b}, c, a_1, b_3, 1)$ consists of edges with lengths $2y+6$, $4y+5$, $(8y+10)^* = 1$, $(8y+9)^* = 2$, and $(8y+8)^* = 3$. Thus, the edge set of G has one edge of each length i where $1 \le i \le 4y+5$. It is clear from (3) that $V(G) \subseteq [0, 8y+10]$. Hence, the defined labeling is a ρ-labeling, and condition (g1) for a γ-labeling is satisfied.

Finally, let $A = A_1 \cup A_2 \cup \{a_1, a_2\}$, $B = B_1 \cup B_2 \cup \{\hat{b}, b_1, b_2, b_3\}$, and $C = \{c\}$. Then, $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (g2) of a

γ-labeling is clear from (3). Note that $c - \hat{b} = (8y+10) - (4y+5) = 4y+5$, the number of edges of G. Thus, condition (g3) is satisfied, and we have a γ-labeling of G.

Case 1.2 $x > 1$.
Let $C_{4x} = G_1 + G_2 + (2x+2y-2, b_1, a_1, b_2, 2y)$ and $C_{4y+1} = G_3 + G_4 + (2y-1, \hat{b}, c, a_1, b_3, 1)$ where $b_1 = 4x+2y$, $a_1 = 0$, $b_2 = 4x+2y+1$, $\hat{b} = 4x+4y+1$, $c = 8x+8y+2$, $b_3 = 4x+8y+4$, and

$$\begin{aligned} G_1 &= P(2y, 2x+2y+2, 2x-4), \\ G_2 &= P(x+2y-2, x+2y-1, 2x), \\ G_3 &= P(1, 4x+6y+4, 2y-2), \\ G_4 &= P(y, 4x+5y+2, 2y-2). \end{aligned}$$

(Note: In the case when $x = 2$, the path G_1 is empty, and when $y = 1$, the paths G_3 and G_4 are empty. However, this does not change the proof in any way.)

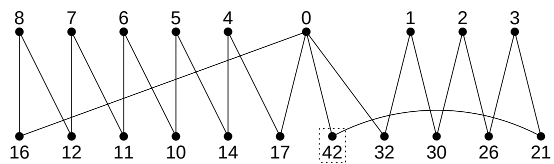

Figure 6: A γ-labeling of $C_{12} \uplus C_9$.

First, we show that $G_1 + G_2 + (2x+2y-2, b_1, a_1, b_2, 2y)$ is a cycle of length $4x$, and $G_3 + G_4 + (2y-1, \hat{b}, c, a_1, b_3, 1)$ is a cycle of length $4y+1$. Note that by **P1**, the first vertex of G_1 is $2y$, and the last is $x+2y-2$; the first vertex of G_2 is $x+2y-2$, and the last is $2x+2y-2$; the first vertex of G_3 is 1, and the last is y; and the first vertex of G_4 is y, and the last is $2y-1$. For $1 \le i \le 4$, let A_i and B_i denote the sets labeled A' and B' in **P2** corresponding to the path G_i. Then using **P2**, we compute

$$\begin{aligned} A_1 &= [2y, x+2y-2], & B_1 &= [3x+2y+1, 4x+2y-2], \\ A_2 &= [x+2y-2, 2x+2y-2], & B_2 &= [2x+2y, 3x+2y-1], \\ A_3 &= [1, y], & B_3 &= [4x+7y+4, 4x+8y+2], \\ A_4 &= [y, 2y-1], & B_4 &= [4x+6y+2, 4x+7y]. \end{aligned}$$

Thus,

$$a_1 < A_3 \le A_4 < A_1 \le A_2 < B_2 < B_1 < b_1 < b_2 < \hat{b} < B_4 < B_3 < b_3 < c. \quad (4)$$

Note that $V(G_1) \cap V(G_2) = \{x+2y-2\}$ and $V(G_3) \cap V(G_4) = \{y\}$; otherwise, G_i and G_j are vertex-disjoint for $i \ne j$. Therefore, $G_1 + G_2 + (2x+2y-$

$2, b_1, a_1, b_2, 2y)$ is a cycle of length $4x$, and $G_3 + G_4 + (2y-1, \hat{b}, c, a_1, b_3, 1)$ is a cycle of length $4y+1$. Furthermore, $V(C_{4x}) \cap V(C_{4y+1}) = \{a_1 = 0\}$; therefore, G is a graph composed of two cycles that share a single vertex.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 4$. By **P3**, we have edge labels

$$\begin{aligned} E_1 &= [2x+3, 4x-2], \\ E_2 &= [2, 2x+1], \\ E_3 &= [4x+6y+4, 4x+8y+1], \\ E_4 &= [4x+4y+3, 4x+6y] \end{aligned}$$

yielding edge lengths

$$\begin{aligned} E_1^* &= E_1 = [2x+3, 4x-2], \\ E_2^* &= E_2 = [2, 2x+1], \\ E_3^* &= \{8x+8y+3-\ell : \ell \in E_3\} = [4x+2, 4x+2y-1], \\ E_4^* &= \{8x+8y+3-\ell : \ell \in E_4\} = [4x+2y+3, 4x+4y]. \end{aligned}$$

Moreover, the path $(2x+2y-2, b_1, a_1, b_2, 2y)$ consists of edges with lengths $2x+2$, $4x+2y$, $4x+2y+1$, and $4x+1$; and the path $(2y-1, \hat{b}, c, a_1, b_3, 1)$ consists of edges with lengths $4x+2y+2$, $4x+4y+1$, $(8x+8y+2)^* = 1$, $(4x+8y+4)^* = 4x-1$, and $(4x+8y+3)^* = 4x$. Thus, the edge set of G has one edge of each length i where $1 \le i \le 4x+4y+1$. It is clear from (4) that $V(G) \subseteq [0, 8x+8y+2]$. Hence, the defined labeling is a ρ-labeling, and condition (g1) for a γ-labeling is satisfied.

Finally, let $A = \bigcup_{i=1}^4 A_i \cup \{a_1\}$, $B = \bigcup_{i=1}^4 B_i \cup \{\hat{b}, b_1, b_2, b_3\}$, and $C = \{c\}$. Then, $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (g2) of a γ-labeling is clear from (4). Note that $c - \hat{b} = (8x+8y+2) - (4x+4y+1) = 4x+4y+1$, the number of edges of G. Thus, condition (g3) is satisfied, and we have a γ-labeling of G.

Case 2 $r \equiv 0 \pmod 4$ and $s \equiv 3 \pmod 4$.
Let $G = C_{4x+3} \,\dot{\cup}\, C_{4y}$ where $x \ge 0$ and $y \ge 1$ but $(x, y) \ne (0, 1)$. We will consider two subcases.

Case 2.1 $x = 0$ and $y > 1$.
If $y = 2$, let $C_3 = (0, 3, 14, 0)$ and $C_8 = (1, 8, 4, 10, 5, 7, 6, 14, 1)$. It is easily checked that this is a γ-labeling with $A = \{0, 1, 4, 5, 6\}$, $B = \{\hat{b} = 3\} \cup \{7, 8, 10\}$, and $C = \{14\}$.

If $y \ge 3$, let $C_3 = (0, \hat{b}, c, 0)$ and $C_{4y} = G_1 + G_2 + (2y-2, b_1, a_1, b_2, 0)$ where $\hat{b} = 4y+1$, $c = 8y+4$, $b_1 = 2y$, $a_1 = 2y-1$, $b_2 = 4y+5$ and

$$\begin{aligned} G_1 &= P(0, 2y+6, 2y-6), \\ G_2 &= P(y-3, y, 2y+2). \end{aligned}$$

(Note: In the case when $y = 3$, the path G_1 is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a γ-labeling of G.

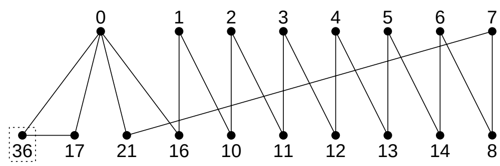

Figure 7: A γ-labeling of $C_3 \dot{\cup} C_{16}$.

Case 2.2 $x > 0$.
Let $C_{4x+3} = G_1 + G_2 + (2x+1, b_1, a_1, \hat{b}, c, 2)$ and $C_{4y} = G_3 + G_4 + (2x + 2y, b_2, 2x+1)$ where $b_1 = 2x + 4y + 3$, $a_1 = 0$, $\hat{b} = 1$, $c = 4x + 4y + 4$, $b_2 = 2x + 4y + 2$, and

$$\begin{aligned} G_1 &= P(2, 2x + 4y + 5, 2x - 2), \\ G_2 &= P(x + 1, x + 4y + 3, 2x), \\ G_3 &= P(2x + 1, 2x + 2y + 3, 2y - 2), \\ G_4 &= P(2x + y, 2x + y + 1, 2y). \end{aligned}$$

(Note: In the case when $x = 1$, the path G_1 is empty, and when $y = 1$, the path G_3 is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a γ-labeling of G.

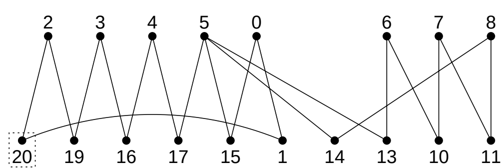

Figure 8: A γ-labeling of $C_{11} \dot{\cup} C_8$.

Case 3 $r \equiv 2 \pmod 4$ and $s \equiv 1 \pmod 4$.
Let $G = C_{4x+1} \dot{\cup} C_{4y+2}$ where $x, y \geq 1$. Let $C_{4x+1} = G_1 + G_2 + (2x, c, \hat{b}, a_1, b_1, 2)$ and $C_{4y+2} = G_3 + G_4 + (2x + 2y, b_2, 2x)$ where $c = 4x + 4y + 4$, $\hat{b} = 1$, $a_1 = 0$, $b_1 = 8x + 4y + 4$, $b_2 = 2x + 4y + 4$, and

$$G_1 = P(2, 6x + 4y + 5, 2x - 2),$$
$$G_2 = P(x + 1, 5x + 4y + 5, 2x - 2),$$
$$G_3 = P(2x, 2x + 2y + 4, 2y - 2),$$
$$G_4 = P(2x + y - 1, 2x + y, 2y + 2).$$

(Note: In the case when $x = 1$, the paths G_1 and G_2 are empty, and when $y = 1$ the path G_3 is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a γ-labeling of G.

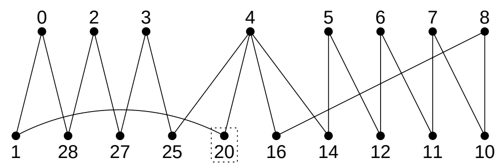

Figure 9: A γ-labeling of $C_9 \dot\cup C_{10}$.

Case 4 $r \equiv 2 \pmod 4$ and $s \equiv 3 \pmod 4$.
Let $G = C_{4x+2} \dot\cup C_{4y+3}$ where $x \geq 1$ and $y \geq 0$. We will consider three subcases.

Case 4.1 $y = 0$.
Let $C_{4x+2} = G_1 + G_2 + (2x, b_1, a_1, b_2, 1)$ and $C_3 = (2x, \hat{b}, c, 2x)$ where $b_1 = 6x + 7$, $a_1 = 0$, $b_2 = 8x + 10$, $\hat{b} = 2x + 3$, $c = 6x + 8$, and

$$G_1 = P(1, 6x + 8, 2x),$$
$$G_2 = P(x + 1, 5x + 9, 2x - 2).$$

(Note: In the case when $x = 1$, the path G_2 is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a γ-labeling of G.

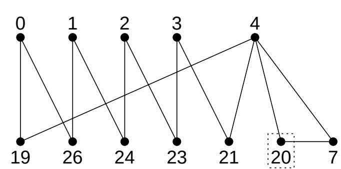

Figure 10: A γ-labeling of $C_{10} \dot\cup C_3$.

Case 4.2 $y > 0$ and $x \leq y + 2$.
Let $C_{4x+2} = G_1 + (2x, b_1, 0)$ and $C_{4y+3} = G_2 + G_3 + G_4 + (2x + 2y, \hat{b}, c, 2x)$ where $b_1 = 6x + 6y + 5$, $\hat{b} = 2x + 2y + 1$, $c = 6x + 6y + 6$, and

$$\begin{aligned} G_1 &= P(0, 4x + 8y + 9, 4x), \\ G_2 &= P(2x, 6x + 2y + 6, 2y - 2), \\ G_3 &= P(2x + y - 1, 4x + 3y + 5, 2x - 2), \\ G_4 &= P(3x + y - 2, 7x + y - 1, -2x + 2y + 4). \end{aligned}$$

(Note: In the case when $x = 1$, the path G_3 is empty; when $y = 1$, the path G_2 is empty; and when $x = y + 2$, the path G_4 is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a γ-labeling of G.

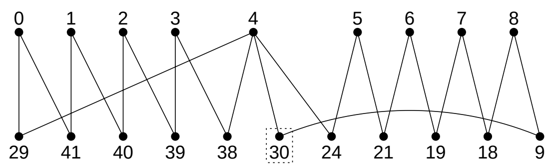

Figure 11: A γ-labeling of $C_{10} \uplus C_{11}$.

Case 4.3 $y > 0$ and $x > y + 2$.
Let $C_{4x+2} = G_1 + G_2 + (2x, b_1, 0)$ and $C_{4y+3} = G_3 + G_4 + (2x + 2y, \hat{b}, c, 2x)$ where $b_1 = 6x + 6y + 5$, $\hat{b} = 2x + 2y + 1$, $c = 6x + 6y + 6$, and

$$\begin{aligned} G_1 &= P(0, 6x + 6y + 5, 2x + 2y + 4), \\ G_2 &= P(x + y + 2, 5x + 9y + 10, 2x - 2y - 4), \\ G_3 &= P(2x, 6x + 2y + 6, 2y - 2), \\ G_4 &= P(2x + y - 1, 6x + y + 1, 2y + 2). \end{aligned}$$

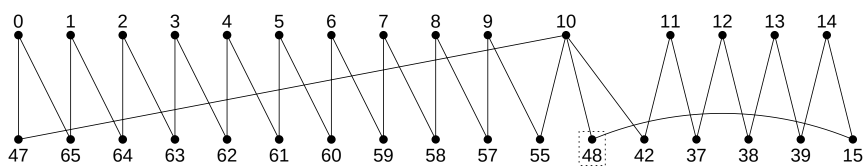

Figure 12: A γ-labeling of $C_{22} \uplus C_{11}$.

(Note: In the case when $y = 1$, the path G_3 is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a γ-labeling of G. ∎

Lemma 10 *If a graph G is the one-point union of odd cycles C_r and C_s where $r \not\equiv s \pmod 4$, then G has a σ-tripartite labeling.*

Proof. Let $G = C_{4x+1} \dot{\cup} C_{4y+3}$ where $x \geq 1$ and $y \geq 0$. We will consider four cases.

Case 1 $y = 0$.
Let $C_{4x+1} = G_1 + G_2 + (2x-1, b_1, c_1, 0)$ and $C_3 = (a_1, b_2, c_2, a_1)$ where $b_1 = 2x+1$, $c_1 = 4x+4$, $a_1 = 3x+3$, $b_2 = 3x+4$, $c_2 = 5x+5$, and

$$G_1 = P(0, 2x+3, 2x),$$
$$G_2 = P(x, x+2, 2x-2).$$

(Note: In the case when $x = 1$, the path G_2 is empty. However, this does not change the proof in any way.)

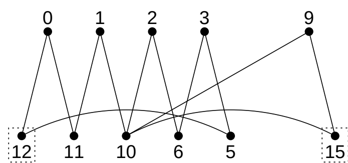

Figure 13: A σ-tripartite labeling of $C_9 \dot{\cup} C_3$.

First, we show that $G_1+G_2+(2x-1, b_1, c_1, 0)$ is a cycle of length $4x+1$. Note that by **P1**, the first vertex of G_1 is 0, and the last is x; and the first vertex of G_2 is x, and the last is $2x-1$. For $1 \leq i \leq 2$, let A_i and B_i denote the sets labeled A' and B' in **P2** corresponding to the path G_i. Then using **P2**, we compute

$$A_1 = [0, x], \qquad B_1 = [3x+4, 4x+3],$$
$$A_2 = [x, 2x-1], \qquad B_2 = [2x+2, 3x].$$

Thus,

$$A_1 \leq A_2 < b_1 < B_2 < a_1 < b_2 \leq B_1 < c_1 < c_2. \tag{5}$$

Note that $V(G_1) \cap V(G_2) = \{x\}$; otherwise, G_1 and G_2 are vertex-disjoint. Therefore, $G_1+G_2+(2x-1, b_1, c_1, 0)$ is a cycle of length $4x+1$. Furthermore, $V(C_{4x+1}) \cap V(C_3) = \{b_2 = 3x+4\}$; therefore, G is a graph composed of two cycles that share a single vertex.

Next, let E_i denote the set of edge labels in G_i for $1 \leq i \leq 2$. By **P3**, we have edge labels

$$E_1 = [2x+4, 4x+3],$$
$$E_2 = [3, 2x].$$

Moreover, the path $(2x-1, b_1, c_1, 0)$ consists of edges with labels 2, $2x+3$, and $4x+4$; and the path (a_1, b_2, c_2, a_1) consists of edges with labels 1, $2x+1$, and $2x+2$. Thus, the edge set of G has one edge of each label i where $1 \leq i \leq 4x+4$. It is clear from (5) that $V(G) \subseteq [0, 8x+8]$. Hence, the defined labeling is a σ-labeling, and condition (s1) for a σ-tripartite labeling is satisfied.

Now, let $A = A_1 \cup A_2 \cup \{a_1\}$, $B = B_1 \cup B_2 \cup \{b_1, b_2\}$, and $C = \{c_1, c_2\}$. Then, $\{A, B, C\}$ is a tripartition of $V(G)$. Since a_1 is only adjacent to b_2 and c_2, condition (s2) of a σ-tripartite labeling is clear from (5). Note that $|b_1 - c_1| + |b_2 - c_2| = (2x+3) + (2x+1) = 4x+4$, the number of edges of G. Thus, condition (s3) is satisfied. Also, $a = v + (4x+4)$, where $a \in A$ and $v \in B \cup C$, is impossible, since by (5) we have

$$v + (4x+4) \geq b_1 + 4x + 4 = 6x + 5 > 3x + 3 = \max A.$$

Thus, condition (s4) holds.

Finally, suppose $b \in B$ and $c \in C$. The equation $|b - c| = 8x + 8$ is impossible since all vertices are in $[0, 8x+8]$ and $\{0\} \in A$. Likewise, $|b - c_1| = 4x + 4$ is impossible since $c_1 = 4x + 4$ and $0 < B < 8x + 4$. The case remains that $c_2 - b = 4x + 4$, which gives $b = x + 1$. This contradicts (5), since $x + 1 < b_1$. Thus, condition (s5) holds, and we have a σ-tripartite labeling of G.

Case 2 $1 \leq y < x$.
Let $C_{4x+1} = G_1 + G_2 + G_3 + (4x+4y+2, c, b_1, 2x+4y+3)$ and $C_{4y+3} = G_4 + G_5 + (2y, b_2, c, a_1, b_3, 1)$ where $c = 4x+4y+4$, $b_1 = 4x+6y+5$, $b_2 = 2y+1$, $a_1 = 0$, $b_3 = 4x+4y+3$, and

$$\begin{aligned}
G_1 &= P(2x+4y+3, 6x+6y+6, 2y-2),\\
G_2 &= P(2x+5y+2, 4x+7y+4, 2x),\\
G_3 &= P(3x+5y+2, 3x+9y+3, 2x-2y),\\
G_4 &= P(1, 2y+2, 2y),\\
G_5 &= P(y+1, y+3, 2y-2).
\end{aligned}$$

(Note: In the case when $y = 1$, the paths G_1 and G_5 are empty. However, this does not change the proof in any way.)

First, we show that $G_1 + G_2 + G_3 + (4x+4y+2, c, b_1, 2x+4y+3)$ is a cycle of length $4x+1$, and $G_4 + G_5 + (2y, b_2, c, a_1, b_3, 1)$ is a cycle of length $4y+3$. Note that by **P1**, the first vertex of G_1 is $2x+4y+3$, and the last is $2x+5y+2$; the first vertex of G_2 is $2x+5y+2$, and the last is $3x+5y+2$; the first vertex of G_3 is $3x+5y+2$, and the last is $4x+4y+2$; the first vertex of G_4 is 1, and the last is $y+1$; and the first vertex of G_5 is $y+1$,

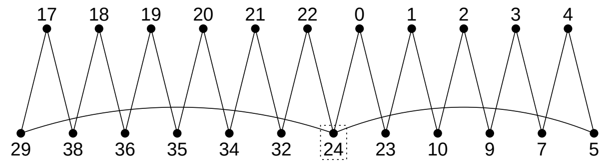

Figure 14: A σ-tripartite labeling of $C_{13} \uplus C_{11}$.

and the last is $2y$. For $1 \le i \le 5$, let A_i and B_i denote the sets labeled A' and B' in **P2** corresponding to the path G_i. Then using **P2**, we compute

$$\begin{aligned}
A_1 &= [2x+4y+3, 2x+5y+2], & B_1 &= [6x+7y+6, 6x+8y+4],\\
A_2 &= [2x+5y+2, 3x+5y+2], & B_2 &= [5x+7y+5, 6x+7y+4],\\
A_3 &= [3x+5y+2, 4x+4y+2], & B_3 &= [4x+8y+4, 5x+7y+3],\\
A_4 &= [1, y+1], & B_4 &= [3y+3, 4y+2],\\
A_5 &= [y+1, 2y], & B_5 &= [2y+3, 3y+1].
\end{aligned}$$

Thus,

$$a_1 < A_4 \le A_5 < b_2 < B_5 < B_4 < A_1 \le A_2 \le A_3 < b_3 < c < b_1 < B_3 < B_2 < B_1. \quad (6)$$

Note that $V(G_1) \cap V(G_2) = \{2x+5y+2\}$, $V(G_2) \cap V(G_3) = \{3x+5y+2\}$, and $V(G_4) \cap V(G_5) = \{y+1\}$; otherwise, G_i and G_j are vertex-disjoint for $i \ne j$. Therefore, $G_1 + G_2 + G_3 + (4x+4y+2, c, b_1, 2x+4y+3)$ is a cycle of length $4x+1$, and $G_4 + G_5 + (2y, b_2, c, a_1, b_3, 1)$ is a cycle of length $4y+3$. Furthermore, $V(C_{4x+1}) \cap V(C_{4y+3}) = \{c = 4x+4y+4\}$; therefore, G is a graph composed of two cycles that share a single vertex.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 5$. By **P3**, we have edge labels

$$\begin{aligned}
E_1 &= [4x+2y+4, 4x+4y+1],\\
E_2 &= [2x+2y+3, 4x+2y+2],\\
E_3 &= [4y+2, 2x+2y+1],\\
E_4 &= [2y+2, 4y+1],\\
E_5 &= [3, 2y].
\end{aligned}$$

Moreover, the path $(4x+4y+2, c, b_1, 2x+4y+3)$ consists of edges with edge labels 2, $2y+1$, and $2x+2y+2$; and the path $(2y, b_2, c, a_1, b_3, 1)$ consists of edges with edge labels 1, $4x+2y+3$, $4x+4y+4$, $4x+4y+3$, and $4x+4y+3$. Thus, the edge set of G has one edge of each length i where $1 \le i \le 4x+4y+4$. It is clear from (6) that $V(G) \subseteq [0, 8x+8y+8]$. Hence, the defined labeling is a σ-labeling, and condition (s1) for a σ-tripartite labeling is satisfied.

Now, let $A = \bigcup_{i=1}^{5} A_i \cup \{a_1\}$, $B = \bigcup_{i=1}^{5} B_i \cup \{b_1, b_2, b_3\}$, and $C = \{c\}$. Then, $\{A, B, C\}$ is a tripartition of $V(G)$. Since all vertices in $A_1 \cup A_2 \cup A_3$ are only adjacent to vertices in $\{c, b_1\} \cup B_3 \cup B_2 \cup B_1$, condition (s2) of a σ-tripartite labeling is clear from (6). Note that $|b_1 - c| + |b_2 - c| = (2y+1) + (4x+2y+3) = 4x+4y+4$, the number of edges of G. Thus, condition (s3) is satisfied. Also, $a = v + (4x+4y+4)$, where $a \in A$ and $v \in B \cup C$, is impossible, since by (6)

$$v + (4x+4y+4) \geq b_2 + 4x + 4y + 4 = 4x + 6y + 5 > 4x + 4y + 2 = \max A.$$

Thus, condition (s4) holds.

Finally, suppose $b \in B$ and $c \in C$. The equation $|b - c| = 8x + 8y + 8$ is impossible since all vertices are in $[0, 8x+8y+8]$ and $0 \in A$. Likewise, $|b-c| = 4x+4y+4$ is impossible since $c = 4x+4y+4$ and $0 < B < 8x+8y+8$. Thus, condition (s5) holds, and we have a σ-tripartite labeling of G.

Case 3 $x \leq y < 2x$.
Let $C_{4x+1} = G_1 + G_2 + (4x+4y+2, c, b_1, 2x+4y+3)$ and $C_{4y+3} = G_3 + G_4 + G_5 + (2y, b_2, c, a_1, b_3, 1)$ where $c = 4x+4y+4$, $b_1 = 4x+6y+5$, $b_2 = 2y+1$, $a_1 = 0$, $b_3 = 4x+4y+3$, and

$$\begin{aligned}
G_1 &= P(2x+4y+3, 6x+6y+6, 2y-2),\\
G_2 &= P(2x+5y+2, 2x+9y+4, 4x-2y),\\
G_3 &= P(1, 2x+2y+3, -2x+2y),\\
G_4 &= P(-x+y+1, -x+3y+2, 2x),\\
G_5 &= P(y+1, y+3, 2y-2).
\end{aligned}$$

(Note: In the case when $y = 1$, the paths G_1, G_3, and G_5 are empty, and when $y = x$, the path G_3 is empty. However, this does not change the proof in any way.) If we proceed as in Case 2, it is easy to verify that we have a σ-tripartite labeling of G.

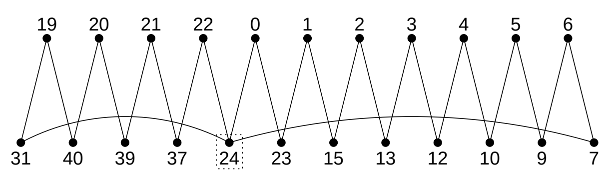

Figure 15: A σ-tripartite labeling of $C_9 \uplus C_{15}$.

Case 4 $y \geq 2x$.
Let $C_{4x+1} = G_1 + (4x+4y+2, c, b_1, 2x+4y+3)$ and $C_{4y+3} = G_2 + G_3 +$

$G_4 + G_5 + (2y, b_2, c, a_1, b_3, 1)$ where $c = 4x + 4y + 4$, $b_1 = 4x + 6y + 5$, $b_2 = 2y + 1$, $a_1 = 0$, $b_3 = 4x + 4y + 3$, and

$$\begin{aligned} G_1 &= P(2x + 4y + 3, 2x + 8y + 6, 4x - 2), \\ G_2 &= P(1, 4x + 2y + 4, -4x + 2y), \\ G_3 &= P(-2x + y + 1, 3y + 3, 2x), \\ G_4 &= P(-x + y + 1, -x + 3y + 2, 2x), \\ G_5 &= P(y + 1, y + 3, 2y - 2). \end{aligned}$$

If we proceed as in Case 2, it is easy to verify that we have a σ-tripartite labeling of G. ∎

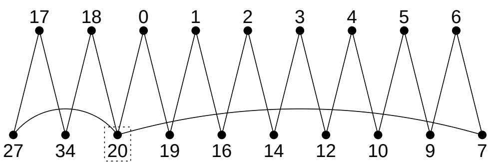

Figure 16: A σ-tripartite labeling of $C_5 \dot{\cup} C_{15}$.

Lemma 11 *If a graph G is the one-point union of C_3 and C_4 or if G is the one-point union of odd cycles C_r and C_s where $r \equiv s \pmod 4$ and $(r, s) \neq (3, 3)$, then G has a ρ-tripartite labeling.*

Proof. It is easy to verify that $C_3 \dot{\cup} C_3$ does not admit a ρ-tripartite labeling. If $G = C_3 \dot{\cup} C_4$, let $C_3 = (0, 4, 13, 0)$ and $C_4 = (0, 1, 8, 3, 0)$. It is easily checked that this is a ρ-tripartite labeling with $A = \{0\}$, $B = \{1, 3, 4\}$, and $C = \{8, 13\}$.

For the remainder of this proof, assume that G is the one-point union of odd cycles C_r and C_s where $r \equiv s \pmod 4$ and $(r, s) \neq (3, 3)$. We will consider two cases.

Case 1 $r \equiv s \equiv 1 \pmod 4$.
Let $G = C_{4x+1} \dot{\cup} C_{4y+1}$ where $x \geq y \geq 1$. We will consider two subcases.

Case 1.1 $y = 1$.
Let $C_{4x+1} = G_1 + G_2 + (2x - 1, b_1, c_1, 0)$ and $C_5 = (a_1, b_2, a_2, b_3, c_2, a_1)$ where $b_1 = 2x$, $c_1 = 6x + 7$, $a_1 = 0$, $b_2 = 2x + 4$, $a_2 = 2x + 1$, $b_3 = 2x + 3$, $c_2 = 6x + 8$, and

$$\begin{aligned} G_1 &= P(0, 6x + 9, 2x), \\ G_2 &= P(x, 3x + 6, 2x - 2). \end{aligned}$$

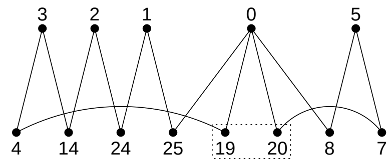

Figure 17: A ρ-tripartite labeling of $C_9 \uplus C_5$.

(Note: In the case when $x = 1$, the path G_2 is empty. However, this does not change the proof in any way.)

First, we show that $G_1 + G_2 + (2x-1, b_1, c_1, 0)$ is a cycle of length $4x+1$. Note that by **P1**, the first vertex of G_1 is 0, and the last is x; and the first vertex of G_2 is x, and the last is $2x-1$. For $1 \le i \le 2$, let A_i and B_i denote the sets labeled A' and B' in **P2** corresponding to the path G_i. Then using **P2**, we compute

$$A_1 = [0, x], \qquad B_1 = [7x+10, 8x+9],$$
$$A_2 = [x, 2x-1], \qquad B_2 = [4x+6, 5x+4].$$

Thus,

$$a_1 \le A_1 \le A_2 < b_1 < a_2 < b_3 < b_2 < B_2 < c_1 < c_2 < B_1. \tag{7}$$

Note that $V(G_1) \cap V(G_2) = \{x\}$; otherwise, G_1 and G_2 are vertex-disjoint. Therefore, $G_1 + G_2 + (2x-1, b_1, c_1, 0)$ is a cycle of length $4x+1$. Furthermore, $V(C_{4x+1}) \cap V(C_5) = \{a_1 = 0\}$; therefore, G is a graph composed of two cycles that share a single vertex.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 2$. By **P3**, we have edge labels

$$E_1 = [6x+10, 8x+9],$$
$$E_2 = [2x+7, 4x+4]$$

yielding edge lengths

$$E_1^* = \{8x+13-e : e \in E_1\} = [4, 2x+3],$$
$$E_2^* = \{e : e \in E_2\} = [2x+7, 4x+4].$$

Moreover, the path $(2x-1, b_1, c_1, 0)$ consists of edges with lengths 1, $(4x+7)^* = 4x+6$, and $(6x+7)^* = 2x+6$, and the path $(a_1, c_2, b_3, a_2, b_2, a_1)$ consists of edges with lengths $(6x+8)^* = 2x+5$, $4x+5$, 2, 3, and $2x+4$. Thus, the edge set of G has one edge of each length i where $1 \le i \le 4x+6$. It is clear from (7) that $V(G) \subseteq [0, 8x+12]$. Hence, the defined labeling is a ρ-labeling, and condition (r1) for a ρ-labeling is satisfied.

Finally, let $A = A_1 \cup A_2 \cup \{a_1, a_2\}$, $B = B_1 \cup B_2 \cup \{b_1, b_2, b_3\}$, and $C = \{c_1, c_2\}$. Then, $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (r2) of a ρ-tripartite labeling is clear from (7). Note that $|b_1 - c_1| + |b_3 - c_2| = (4x + 7) + (4x + 5) = 8x + 12$, twice the number of edges of G. Thus condition (r3) is satisfied. Also, $|b - c| = 8x + 12$, where $b \in B$ and $c \in C$, is impossible since all vertices are in $[0, 8x + 12]$ and $0 \in A$. Thus, condition (r4) holds, and we have a ρ-tripartite labeling of G.

Case 1.2 $y > 1$.
Let $C_{4x+1} = G_1 + G_2 + G_3 + (2x - 1, b_1, c_1, 0)$ and $C_{4y+1} = G_4 + (2x + 4y - 3, b_2, a_1, c_2, b_3, a_2, b_4, 2x + 2y)$ where $b_1 = 2x$, $c_1 = 6x + 4y + 3$, $b_2 = 2x + 4y$, $a_1 = 0$, $c_2 = 6x + 4y + 4$, $b_3 = 2x + 3$, $a_2 = 2x + 1$, $b_4 = 2x + 6y - 2$, and

$$\begin{aligned} G_1 &= P(0, 8x + 2y + 8, 2y - 2), \\ G_2 &= P(y - 1, 6x + 5y + 4, 2x - 2y + 2), \\ G_3 &= P(x, 3x + 4y + 2, 2x - 2), \\ G_4 &= P(2x + 2y, 2x + 2y + 3, 4y - 6). \end{aligned}$$

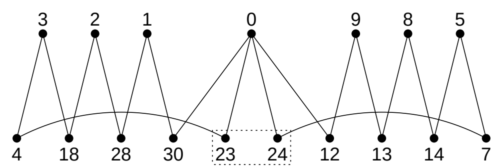

Figure 18: A ρ-tripartite labeling of $C_9 \mathbin{\dot{\cup}} C_9$.

First, we show that $G_1 + G_2 + G_3 + (2x - 1, b_1, c_1, 0)$ is a cycle of length $4x + 1$, and $G_4 + (2x + 4y - 3, b_2, a_1, c_2, b_3, a_2, b_4, 2x + 2y)$ is a cycle of length $4y + 1$. Note that by **P1**, the first vertex of G_1 is 0, and the last is $y - 1$; the first vertex of G_2 is $y - 1$, and the last is x; the first vertex of G_3 is x, and the last is $2x - 1$; and the first vertex of G_4 is $2x + 2y$, and the last is $2x + 4y - 3$. For $1 \le i \le 4$, let A_i and B_i denote the sets labeled A' and B' in **P2** corresponding to the path G_i. Then using **P2**, we compute

$$\begin{aligned} A_1 &= [0, y - 1], & B_1 &= [8x + 3y + 8, 8x + 4y + 6], \\ A_2 &= [y - 1, x], & B_2 &= [7x + 4y + 6, 8x + 3y + 6], \\ A_3 &= [x, 2x - 1], & B_3 &= [4x + 4y + 2, 5x + 4y], \\ A_4 &= [2x + 2y, 2x + 4y - 3], & B_4 &= [2x + 4y + 1, 2x + 6y - 3]. \end{aligned}$$

Thus,

$$a_1 \le A_1 \le A_2 \le A_3 < b_1 < a_2 < b_3 < A_4 < b_2 < B_4 < b_4 < B_3 < c_1 < c_2 < B_2 < B_1. \quad (8)$$

Note that $V(G_1) \cap V(G_2) = \{y-1\}$ and $V(G_2) \cap V(G_3) = \{x\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1 + G_2 + G_3 + (2x-1, b_1, c_1, 0)$ is a cycle of length $4x+1$, and $G_4 + (2x+4y-3, b_2, a_1, c_2, b_3, a_2, b_4, 2x+2y)$ is a cycle of length $4y+1$. Furthermore, $V(C_{4x+1}) \cap V(C_{4y+1}) = \{a_1 = 0\}$; therefore, G is a graph composed of two cycles that share a single vertex.

Next, let E_i denote the set of edge labels in G_i for $1 \leq i \leq 4$. By **P3**, we have edge labels

$$\begin{aligned}
E_1 &= [8x+2y+9, 8x+4y+6], \\
E_2 &= [6x+4y+6, 8x+2y+7], \\
E_3 &= [2x+4y+3, 4x+4y], \\
E_4 &= [4, 4y-3]
\end{aligned}$$

yielding edge lengths

$$\begin{aligned}
E_1^* &= \{8x+8y+5-e : e \in E_1\} = [4y-1, 6y-4], \\
E_2^* &= \{8x+8y+5-e : e \in E_2\} = [6y-2, 2x+4y-1], \\
E_3^* &= \{e : e \in E_3\} = [2x+4y+3, 4x+4y], \\
E_4^* &= \{e : e \in E_4\} = [4, 4y-3].
\end{aligned}$$

Moreover, the path $(2x-1, b_1, c_1, 0)$ consists of edges with lengths 1, $(4x+4y+3)^* = 4x+4y+2$, and $(6x+4y+3)^* = 2x+4y+2$, and the path $(2x+4y-3, b_2, a_1, c_2, b_3, a_2, b_4, 2x+2y)$ consists of edges with lengths 3, $2x+4y$, $(6x+4y+4)^* = 2x+4y+1$, $4x+4y+1$, 2, $6y-3$, and $4y-2$. Thus, the edge set of G has one edge of each length i where $1 \leq i \leq 4x+4y+2$. It is clear from (8) that $V(G) \subseteq [0, 8x+4y+4]$. Hence, the defined labeling is a ρ-labeling, and condition (r1) for a ρ-tripartite labeling is satisfied.

Finally, let $A = \bigcup_{i=1}^4 A_i \cup \{a_1, a_2\}$, $B = \bigcup_{i=1}^4 B_i \cup \{b_1, b_2, b_3, b_4\}$, and $C = \{c_1, c_2\}$. Then, $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (r2) of a ρ-tripartite labeling is clear from (8). Note that $|b_1 - c_1| + |b_3 - c_2| = (4x+4y+3) + (4x+4y+1) = 8x+8y+4$, twice the number of edges of G. Thus, condition (r3) is satisfied. Also, $|b-c| = 8x+8y+4$, where $b \in B$ and $c \in C$, is impossible since all vertices are in $[0, 8x+8y+4]$ and $0 \in A$. Thus, condition (r4) holds, and we have a ρ-tripartite labeling of G.

Case 2 $r \equiv s \equiv 3 \mod 4$.
Let $G = C_{4x+3} \,\dot{\cup}\, C_{4y+3}$ where $y \geq x \geq 0$ but $(x, y) \neq (0, 0)$. We will consider three subcases.

Case 2.1 $x = 0$ and $y \geq 1$.
Let $C_3 = (a_1, b_1, c_1, a_1)$ and $C_{4y+3} = G_1 + G_2 + (2y-1, b_2, a_2, c_2, b_3, 0)$

where $a_1 = 0$, $b_1 = 4y+3$, $c_1 = 8y+10$, $b_2 = 2y+1$, $a_2 = 2y$, $c_2 = 8y+9$, $b_3 = 4y+4$, and

$$G_1 = P(0, 2y+4, 2y-2),$$
$$G_2 = P(y-1, y+2, 2y).$$

(Note: In the case when $y = 1$, the path G_1 is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a ρ-tripartite labeling of G.

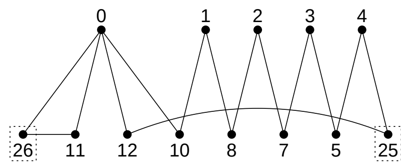

Figure 19: A ρ-tripartite labeling of $C_3 \uplus C_{11}$.

Case 2.2 $1 \le x \le y < x+2$.
Let $C_{4x+3} = G_1 + G_2 + (2x, c_1, b_1, 0)$ and $C_{4y+3} = G_3 + G_4 + (2x + 2y, b_2, c_2, 2x)$ where $c_1 = 8x+8y+11$, $b_1 = 4x+4y+4$, $b_2 = 4x+4y+7$, $c_2 = 8x+8y+12$, and

$$G_1 = P(0, 2x+2y+7, 2x+2y-4),$$
$$G_2 = P(x+y-2, x+5y, 2x-2y+4),$$
$$G_3 = P(2x, 4x+2, 4y-2x),$$
$$G_4 = P(x+2y, x+2y, 2x).$$

(Note: In the case when $x = y = 1$, the path G_1 are empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a ρ-tripartite labeling of G.

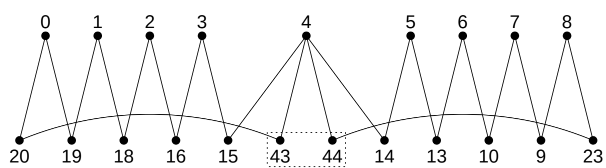

Figure 20: A ρ-tripartite labeling of $C_{11} \uplus C_{11}$.

Case 2.3 $x \ge 1$ and $y \ge x+2$.
Let $C_{4x+3} = G_1 + (2x, c_1, b_1, 0)$ and $C_{4y+3} = G_2 + G_3 + G_4 + (2x +$

$2y, b_2, c_2, 2x)$ where $c_1 = 8x + 8y + 11$, $b_1 = 4x + 4y + 4$, $b_2 = 4x + 4y + 7$, $c_2 = 8x + 8y + 12$, and

$$\begin{aligned} G_1 &= P(0, 4y + 3, 4x), \\ G_2 &= P(2x, 4x + 2y + 7, 2y - 2x - 4), \\ G_3 &= P(x + y - 2, 3x + y, 2y + 4), \\ G_4 &= P(x + 2y, x + 2y, 2x). \end{aligned}$$

(Note: In the case when $y = x + 2$, the path G_2 is empty. However, this does not change the proof in any way.) If we proceed as in Case 1, it is easy to verify that we have a ρ-tripartite labeling of G. ∎

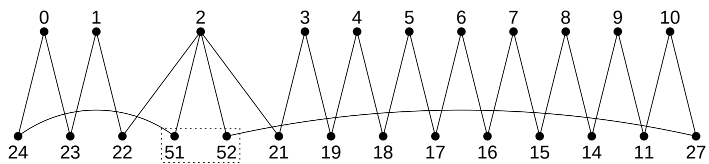

Figure 21: A ρ-tripartite labeling of $C_7 \dot{\cup} C_{19}$.

Although the graph $G = C_3 \dot{\cup} C_3$ does not admit a labeling that yields cyclic G-decompositions of K_{12t+1}, it is easy to verify that such a cyclic decomposition does exist. It is well known that there exists a cyclic C_3-decomposition of K_{6t+1} for all positive integers t. (Such a decomposition is better known as a *cyclic Steiner Triple System* of *order* $6t + 1$.) The t starters in such a decomposition are usually obtained from the partition of $[1, 3t]$ into t triples $\{a, b, c\}$ such that $a+b = c$ or $a+b+c \equiv 0 \pmod{6t+1}$. (This is known as the *First Heffter difference problem.*) Then the set of all triples $\{0, a, a + b\}$ gives the t starter blocks in the triple system. In other words, the triples $\{0, a, a + b\}$ give a ρ-labeling of the graph $C_3^{(t)}$. Thus there exists a cyclic $C_3^{(t)}$-decomposition of K_{6nt+1} for all positive integers t. Since $C_3 \dot{\cup} C_3$ divides $C_3^{(2t)}$, there exists a cyclic $(C_3 \dot{\cup} C_3)$-decomposition of K_{12t+1}. We can now state our main theorem.

Theorem 12 *If G with n edges is the one-point union of any two cycles, then there exists a cyclic G-decomposition of K_{2nt+1} for all positive integers t.*

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