All 2-regular graphs with uniform odd components admit $\rho$-labelings

D. I. Gannon

Department of Mathematics
University of Illinois
Urbana, Illinois 61801
U.S.A.
danitoh@aol.com

S. I. El-Zanati*

4520 Mathematics Department
Illinois State University
Normal, Illinois 61790-4520
U.S.A.
saad@ilstu.edu

Abstract

Let $G$ be a graph of size $n$ with vertex set $V(G)$ and edge set $E(G)$. A $\rho$-labeling of $G$ is a one-to-one function $h: V(G) \rightarrow \{0, 1, \ldots, 2n\}$ such that
\[\{\min\{|h(u)−h(v)|, 2n+1−|h(u)−h(v)|\}: \{u, v\} \in E(G)\} = \{1, 2, \ldots, n\} .\]
Such a labeling of $G$ yields a cyclic $G$-decomposition of $K_{2n+1}$. It is known that 2-regular bipartite graphs, the vertex-disjoint union of $C_3$’s, and the vertex-disjoint union of $C_5$’s all admit $\rho$-labelings. We show that for any odd $n \geq 7$, the vertex-disjoint union of any number of $C_n$’s admits a $\rho$-labeling.

1 Introduction

If $a$ and $b$ are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo $n$. For a graph $G$, let $V(G)$ and $E(G)$ denote the vertex set of $G$ and the edge set of $G$, respectively. Let $rG$ denote the vertex-disjoint union of $r$ copies of $G$.

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Let \( V(K_k) = \mathbb{Z}_k \) and let \( G \) be a subgraph of \( K_k \). By clicking \( G \), we mean applying the isomorphism \( i \to i + 1 \) to \( V(G) \). Let \( H \) and \( G \) be graphs such that \( G \) is a subgraph of \( H \). A \emph{G-decomposition} of \( H \) is a set \( \Gamma = \{G_1, G_2, \ldots, G_t\} \) of edge-disjoint subgraphs of \( H \) each of which is isomorphic to \( G \) and such that \( E(H) = \bigcup_{i=1}^{t} E(G_i) \). A G-decomposition of \( K_k \) is known as a \emph{G-design of order} \( k \). A G-decomposition \( \Gamma \) of \( K_k \) is \emph{cyclic} if clicking is a permutation of \( \Gamma \).

The investigation of G-designs is a popular area of research in combinatorial design theory. For example, if \( G \) is \( K_k \), then a G-design of order \( v \) is a \((v, k, 1)\)-BIBD. If \( G \) has \( n \) edges, then G-designs of order \( 2n + 1 \) are of particular interest. In 1963, Ringel [8] conjectured that there is a G-design of order \( 2n + 1 \) for every tree \( G \) with \( n \) edges. In [9], Rosa introduced graph labelings as a means of attacking Ringel’s conjecture.

For any graph \( G \), a one-to-one function \( h: V(G) \to \mathbb{N} \) is called a labeling (or a valuation) of \( G \). Let \( G \) be a graph with \( n \) edges and no isolated vertices and let \( h \) be a labeling of \( G \). Let \( h(V(G)) = \{h(u): u \in V(G)\} \). Define a function \( \bar{h}: E(G) \to \mathbb{Z}^+ \) by \( \bar{h}(e) = |h(u) - h(v)| \), where \( e = \{u, v\} \in E(G) \) and let \( (\bar{h}(e))^* = \min\{\bar{h}(e), 2n+1-\bar{h}(e)\} \). We will refer to \( \bar{h}(e) \) and \( (\bar{h}(e))^* \) as the label and the length of \( e \), respectively. If \( F \subseteq E(G) \), then \( h(F) = \{\bar{h}(e): e \in F\} \) and \( (\bar{h}(F))^* = \{\bar{h}(e))^*: e \in F\} \). We say \( h \) is a \( \rho \)-labeling of \( G \) if \( h(V(G)) \subseteq [0,2n] \) and \( \bar{h}(E(G))^* = [1,n] \). If \( h(V(G)) \subseteq [0,n] \) and \( \bar{h}(E(G)) = [1,n] \), then \( h \) is \( \beta \)-labeling or a graceful labeling of \( G \).

Labelings are critical to the study of cyclic graph decompositions as seen in the following result from [9].

**Theorem 1** Let \( G \) be a graph with \( n \) edges. There exists a cyclic G-decomposition of \( K_{2n+1} \) if and only if \( G \) has a \( \rho \)-labeling.

While a \( \rho \)-labeling is the most basic of Rosa’s labelings, \( \beta \)-labelings (i.e., graceful) are by far the most popular. Graphs that admit a graceful labeling are called graceful. A conjecture that every tree is graceful is one of the best known conjectures in design theory. Unfortunately, graceful labelings are too restrictive for many classes of graphs. For example, \( K_4 \) is the largest complete graph that is graceful and \( C_n \) is graceful if and only if \( n \equiv 0 \) or \( 3 \pmod{4} \). Kotzig [7] has shown that \( rC_5 \) is graceful only if \( r = 1 \) and that \( rC_5 \) is never graceful. For a comprehensive survey of graph labelings that lead to cyclic G-designs, we direct the reader to [5]. A dynamic survey on general graph labelings is maintained by Gallian [6].

In this paper, we will focus on \( \rho \)-labelings of \( rC_n \), where \( n \geq 7 \) is odd. A 1997 result of Dinitz and Rodney [3] on disjoint-starters in cyclic Steiner triple systems is equivalent to showing that \( rC_3 \) admits a \( \rho \)-labeling for all positive integers \( r \). From results in [1], it can be concluded that every 2-regular bipartite graph admits a \( \rho \)-labeling. More recently, it was shown in [2] that \( rC_{4x+1} \) has a \( \rho \)-labeling for \( r \leq 10 \) and \( x \geq 1 \). In [4], we showed that \( rC_5 \) admits a \( \rho \)-labeling. Here, we shall show that \( rC_n \) has a \( \rho \)-labeling for all integers \( r \geq 1 \) and all odd integers \( n \geq 7 \). We note that we arrived at these results by examining small cases and finding labeling patterns that generalize to larger values of \( n \). Our results provide further evidence in support
of a conjecture of El-Zanati and Vanden Eynden that every 2-regular graph admits a $\rho$-labeling.

2 Main Results

Let $C_n$ be the graph with vertex set $\{v_i: 1 \leq i \leq n\}$ and edge set $\{\{v_i, v_{i+1}\}: 1 \leq i \leq n - 1\} \cup \{v_n, v_1\}$. For a positive integer $r$, let $G = rC_n$, the vertex-disjoint union of $r$ copies of $C_n$. For $1 \leq j \leq r$, let the $j^{th}$ component of $G$ have vertex set $\{v_{i,j}: 1 \leq i \leq n\}$ and edge set $\{e_{i,j} = \{v_{i,j}, v_{i+1,j}\}: 1 \leq i \leq n - 1\} \cup \{e_{n,j} = \{v_{n,j}, v_{1,j}\}\}$. For $1 \leq i \leq n$, let $V_i = \{v_{i,j}: 1 \leq j \leq r\}$, and let $E_i = \{e_{i,j}: 1 \leq j \leq r\}$. Finally for $A, B \subseteq V(G)$ and $A \cap B = \emptyset$, we define $e_{A,B} = \{\{a, b\} \in E(G): a \in A, b \in B\}$.

We first show that $rC_7$ admits a $\rho$-labeling for all positive integers $r$.

Lemma 2 Let $r$ be a positive integer and let $G = rC_7$. Then $G$ admits a $\rho$-labeling.

Proof. This is already known for $r = 1$. We consider two cases depending on the parity of $r$.

Case 1: $r$ is even.

Let $r = 2t$, where $t \geq 1$. Let $h: V(G) \rightarrow \mathbb{N}$ be defined in the following 12 pieces:

$$h(v_{i,j}) = \begin{cases} 
2j - 2 & \text{if } i = 1, 1 \leq j \leq 2t, \\
28t - 2j & \text{if } i = 2, 1 \leq j \leq t, \\
22t + j - 1 & \text{if } i = 2, t < j \leq 2t, \\
2j - 1 & \text{if } i = 3, 1 \leq j \leq t, \\
2t + 2j - 2 & \text{if } i = 3, 3 \leq j \leq 2t, \\
9t + j & \text{if } i = 4, 1 \leq j \leq t, \\
10t - 2j & \text{if } i = 4, t < j \leq 2t, \\
2t + 2j - 1 & \text{if } i = 5, 1 \leq j \leq 2t, \\
14t + 4j - 2 & \text{if } i = 6, 1 \leq j \leq t, \\
17t + 3j - 1 & \text{if } i = 6, t < j \leq 2t, \\
10t + 3j - 2 & \text{if } i = 7, 1 \leq j \leq t, \\
7t + j - 1 & \text{if } i = 7, t < j \leq 2t. 
\end{cases}$$

An example of the $\rho$-labeling of $4C_7$ obtained from $h$ is shown in Figure 1. Note that
We can see from the difference in parities of their elements that \( h(V_1) \cap h(V_3) = \emptyset \), \( h(V_1) \cap h(V_5) = \emptyset \), and \( h(V_3) \cap h(V_5) = \emptyset \). Therefore, \( h \) is one-to-one and \( h(V(G)) \subseteq [0, 28t] = [0, 2|E(G)|] \).

We now compute the resulting edge labels.

\[
\begin{align*}
\bar{h}(E_1) &= \{28t - 4j + 2: 1 \leq j \leq t\} \cup \{22t - j + 1: t < j \leq 2t\}, \\
\bar{h}(E_2) &= \{28t - 4j + 1: 1 \leq j \leq t\} \cup \{20t - j + 1: t < j \leq 2t\}, \\
\bar{h}(E_3) &= \{9t - j + 1: 1 \leq j \leq t\} \cup \{8t - 4j + 2: t < j \leq 2t\}, \\
\bar{h}(E_4) &= \{7t - j + 1: 1 \leq j \leq t\} \cup \{8t - 4j + 1: t < j \leq 2t\}, \\
\bar{h}(E_5) &= \{12t + 2j - 1: 1 \leq j \leq t\} \cup \{15t + j: t < j \leq 2t\}, \\
\bar{h}(E_6) &= \{4t + j: 1 \leq j \leq t\} \cup \{10t + 2j: t < j \leq 2t\}, \\
\bar{h}(E_7) &= \{10t + j: 1 \leq j \leq t\} \cup \{7t - j + 1: t < j \leq 2t\}.
\end{align*}
\]
Since the edge labels in $\tilde{h}(E_1)$ and $\tilde{h}(E_2)$ all exceed $14t$, we have

\[
(\tilde{h}(E_1))^* = \{2(14t) + 1 - (28t - 4j + 2): 1 \leq j \leq t\}
\cup \{2(14t) + 1 - (22t - j + 1): t < j \leq 2t\}
= \{4j - 1: 1 \leq j \leq t\} \cup \{6t + j: t < j \leq 2t\},
\]

\[
(\tilde{h}(E_2))^* = \{2(14t) + 1 - (28t - 4j + 1): 1 \leq j \leq t\}
\cup \{2(14t) + 1 - (20t - j + 1): t < j \leq 2t\}
= \{4j: 1 \leq j \leq t\} \cup \{8t + j: t < j \leq 2t\}.
\]

Additionally, we have $\tilde{h}(E_5) > 14t$ only if $j \in [t + 1, 2t]$. Thus,

\[
(\tilde{h}(E_5))^* = \{12t + 2j - 1: 1 \leq j \leq t\} \cup \{13t - j + 1: t < j \leq 2t\}.
\]

Otherwise, the label of an edge under $\tilde{h}$ is also the length of the edge. Therefore,

\[
(\tilde{h}(E_1))^* = \{4k - 1: 1 \leq k \leq t\} \cup [7t + 1, 8t],
(\tilde{h}(E_2))^* = \{4k: 1 \leq k \leq t\} \cup [9t + 1, 10t],
(\tilde{h}(E_3))^* = [8t + 1, 9t] \cup \{4k - 2: 1 \leq k \leq t\},
(\tilde{h}(E_4))^* = [6t + 1, 7t] \cup \{4k - 3: 1 \leq k \leq t\},
(\tilde{h}(E_5))^* = \{2k + 1: 6t \leq k \leq 7t - 1\} \cup [11t + 1, 12t],
(\tilde{h}(E_6))^* = [4t + 1, 5t] \cup \{2k: 6t + 1 \leq k \leq 7t\},
(\tilde{h}(E_7))^* = [10t + 1, 11t] \cup [5t + 1, 6t].
\]

Hence, $(\tilde{h}(E(G)))^* = [1, 14t]$ and $h$ is a $\rho$-labeling of $G$.

**Case 2: $r \geq 3$ is odd.**

Let $r = 2t - 1$. Let $h: V(G) \to \mathbb{N}$ be defined as follows:

\[
h(v_{i,j}) = \begin{cases} 
2j - 2 & \text{if } i = 1, 1 \leq j \leq t, \\
2j - 1 & \text{if } i = 1, t < j \leq 2t - 1, \\
28t - 2j - 12 & \text{if } i = 2, 1 \leq j \leq t, \\
22t + j - 11 & \text{if } i = 2, t < j \leq 2t - 1, \\
2j - 1 & \text{if } i = 3, 1 \leq j \leq t, \\
2t + 2j - 2 & \text{if } i = 3, t < j \leq 2t - 1, \\
22t + j - 1 & \text{if } i = 4, 1 \leq j \leq t, \\
10t - 2j - 2 & \text{if } i = 4, t < j \leq 2t - 1, \\
2t + 2j - 2 & \text{if } i = 5, 1 \leq j \leq 2t, \\
2t + 2j - 1 & \text{if } i = 5, t < j \leq 2t - 1, \\
14t + 4j - 9 & \text{if } i = 6, 1 \leq j \leq t, \\
14t + 4j - 8 & \text{if } i = 6, t < j \leq 2t - 1, \\
10t + 3j - 7 & \text{if } i = 7, 1 \leq j \leq t, \\
4t + 3j - 3 & \text{if } i = 7, t < j \leq 2t - 1.
\end{cases}
\]
Figure 2: A $\rho$-labeling of $5C_7$.

An example of the $\rho$-labeling of $5C_7$ obtained from $h$ is shown in Figure 2. If we proceed as in Case 1, it is easy to verify that the given labeling is a $\rho$-labeling of $rC_7$.

**Theorem 3** Let $r$ be a positive integer and let $n \geq 9$ be odd. Let $G = rC_n$. Then $G$ admits a $\rho$-labeling.

**Proof.** This is already known to be true if $r = 1$. We consider four cases.

**Case 1a:** $n \equiv 3 \pmod{4}$, $n \geq 11$, and $r$ is even.

Let $n = 4m + 3$ and $r = 2t$, where $m \geq 2$ and $t \geq 1$. Partition $V(G)$ into the following sets:

- $S_1 = \{v_{i,j} : i \text{ odd}, 1 \leq i < 4m + 3, 1 \leq j \leq 2t\}$,
- $S_2 = \{v_{i,j} : i \text{ even}, 1 \leq i \leq 2m - 2, 1 \leq j \leq t\}$,
- $S_3 = \{v_{i,j} : i \text{ even}, 2m - 2 < i < 4m + 2, 1 \leq j \leq t\}$,
- $S_4 = \{v_{i,j} : i \text{ even}, 1 \leq i \leq 2m + 2, t < j \leq 2t\}$,
- $S_5 = \{v_{i,j} : i \text{ even}, 2m + 2 < i < 4m + 2, t < j \leq 2t\}$,
- $S_6 = \{v_{i,j} : i = 4m + 2, 1 \leq j \leq 2t\}$,
- $S_7 = \{v_{i,j} : i = 4m + 3, 1 \leq j \leq 2t\}$.

Let $h : V(G) \to \mathbb{N}$ be defined as follows:

$$h(v_{i,j}) = \begin{cases} -t + (ti + j) - 1 & \text{if } v_{i,j} \in S_1, \\ 16tm + 14t - (ti + j) + 1 & \text{if } v_{i,j} \in S_2, \\ 16tm + 12t - (ti + j) + 1 & \text{if } v_{i,j} \in S_3, \\ 8tm + 6t - (ti + j) + 1 & \text{if } v_{i,j} \in S_4, \\ 8tm + 4t - (ti + j) + 1 & \text{if } v_{i,j} \in S_5, \\ 12tm + 2t + 2j - 1 & \text{if } v_{i,j} \in S_6, \\ 8tm + 4t + 3j - 2 & \text{if } v_{i,j} \in S_7. \end{cases}$$
An example of the \( \rho \)-labeling of \( 4C_{11} \) obtained from \( h \) is shown in Figure 3.

![Graph Image]  

**Figure 3:** A \( \rho \)-labeling of \( 4C_{11} \).

Let \( h_k \) be the restriction of \( h \) on \( S_k \). Since \( t \) and \( m \) are constants, \( h_6 \) and \( h_7 \) are strictly increasing, and thus one-to-one. For \( k \in [1, 5] \) and \( v_{i,j}, v_{i',j'} \in S_k \), if \( h_k(v_{i,j}) = h_k(v_{i',j'}) \), then \( t(i - i') = j' - j \). Since \( i - i' \) is even and \( j, j' \in [1, 2t] \), if \( t(i - i') = j' - j \), then \( i = i' \) and \( j = j' \). Thus, \( h_k \) is one-to-one for \( k \in [1, 5] \).

Moreover,

\[
\begin{align*}
    h(S_1) &\subseteq [0, 4tm + 2t - 1], \\
    h(S_2) &\subseteq [14tm + 15t + 1, 16tm + 12t], \\
    h(S_3) &\subseteq [12tm + 11t + 1, 14tm + 12t], \\
    h(S_4) &\subseteq [6tm + 2t + 1, 8tm + 3t], \\
    h(S_5) &\subseteq [4tm + 2t + 1, 6tm - t], \\
    h(S_6) &\subseteq [12tm + 2t + 1, 12tm + 6t - 1], \\
    h(S_7) &\subseteq [8tm + 4t + 1, 8tm + 10t - 2].
\end{align*}
\]

We see that \( 0 \leq h(S_1) < h(S_5) < h(S_4) < h(S_7) < h(S_6) < h(S_3) < h(S_2) \leq 16tm + 12t \). Therefore, \( h \) is one-to-one and \( h(V(G)) \subseteq [0, 2|E(G)|] \).

We now compute the resulting edge labels. If \( i \in [1, 2m - 2] \) and \( j \in [1, t] \), then \( e_{i,j} \in e_{S_1,S_2} \). Additionally, if \( i \) is odd, then

\[
h(e_{i,j}) = (16tm + 14t - (t(i + 1) + j) + 1) - (-t + (ti + j) - 1),
\]

and if \( i \) is even, then

\[
h(e_{i,j}) = (16tm + 14t - (ti + j) + 1) - (-t + (ti + 1) + j) - 1).
\]

Therefore,

\[
h(e_{S_1,S_2}) = \{16tm + 14t - 2(ti + j) + 2: 1 \leq i \leq 2m - 2, 1 \leq j \leq t\}.
\]
Similarly, we have

\[
\begin{align*}
\bar{h}(e_{s_1,s_3}) &= \{16tm + 12t - 2(ti + j) + 2: 2m - 2 < i < 4m + 1, 1 \leq j \leq t\}, \\
\bar{h}(e_{s_1,s_4}) &= \{8tm + 6t - 2(ti + j) + 2: 1 \leq i \leq 2m + 2, t < j \leq 2t\}, \\
\bar{h}(e_{s_1,s_5}) &= \{8tm + 4t - 2(ti + j) + 2: 2m + 2 < i < 4m + 1, t < j \leq 2t\}, \\
\bar{h}(e_{s_1,s_6}) &= \bar{h}(E_{4m+1}) = \{8tm + 2t + j: 1 \leq j \leq 2t\}, \\
\bar{h}(e_{s_6,s_7}) &= \bar{h}(E_{4m+2}) = \{4tm - 2t - j + 1: 1 \leq j \leq 2t\}, \\
\bar{h}(e_{s_1,s_7}) &= \bar{h}(E_{4m+3}) = \{8tm + 4t + 2j - 1: 1 \leq j \leq 2t\}.
\end{align*}
\]

Thus,

\[
\begin{align*}
\bar{h}(e_{s_1,s_2}) &= \{2k: k \in [6tm + 8t + 1, 8tm + 6t]\} \subseteq [12tm + 16t + 2, 16tm + 12t], \\
\bar{h}(e_{s_1,s_3}) &= \{2k: k \in [4tm + 5t + 1, 6tm + 7t]\} \subseteq [8tm + 10t + 2, 12tm + 14t], \\
\bar{h}(e_{s_1,s_4}) &= \{2k: k \in [2tm - t + 1, 4tm + t]\} \subseteq [4tm - 2t + 2, 8tm + 2t], \\
\bar{h}(e_{s_1,s_5}) &= \{2k: k \in [1, 2tm - 2t]\} \subseteq [2, 4tm - 4t], \\
\bar{h}(e_{s_1,s_6}) &= [8tm + 2t + 1, 8tm + 4t], \\
\bar{h}(e_{s_6,s_7}) &= [4tm - 4t + 1, 4tm - 2t], \\
\bar{h}(e_{s_1,s_7}) &= \{2k + 1: k \in [4tm + 2t, 4tm + 4t - 1]\} \subseteq [8tm + 4t + 1, 8tm + 8t - 1].
\end{align*}
\]

Since the edge labels in \(\bar{h}(e_{s_1,s_2})\) and \(\bar{h}(e_{s_1,s_3})\) all exceed \(8tm + 6t\), we have

\[
\begin{align*}
\bar{h}(e_{s_1,s_2})^{*} &= \{2(8tm + 6t) + 1 - (16tm + 14t - 2(ti + j) + 2): 1 \leq i \leq 2m - 2, 1 \leq j \leq t\} \\
&= \{-2t + 2(ti + j) - 1: 1 \leq i \leq 2m - 2, 1 \leq j \leq t\} \\
&= \{2k + 1: k \in [0, 2tm - 2t - 1]\} \subseteq [1, 4tm - 4t - 1], \\
\bar{h}(e_{s_1,s_3})^{*} &= \{2(8tm + 6t) + 1 - (16tm + 12t - 2(ti + j) + 2): 2m - 2 < i < 4m + 1, \\
&\quad 1 \leq j \leq t\} \\
&= \{2(ti + j) - 1: 2m - 2 < i < 4m + 1, 1 \leq j \leq t\} \\
&= \{2k + 1: k \in [2tm - t, 4tm + t - 1]\} \subseteq [4tm - 2t + 1, 8tm + 2t - 1].
\end{align*}
\]

Additionally, for \(e_{i,j} \in e_{s_1,s_7}\), we have \(\bar{h}(e_{i,j}) > 8tm + 6t\) only if \(j \in [t + 1, 2t]\). Thus,

\[
\bar{h}(e_{s_1,s_7})^{*} = \{8tm + 4t + 2j - 1: 1 \leq j \leq t\} \cup \{8tm + 8t - 2j + 2: t < j \leq 2t\} \\
= [8tm + 4t + 1, 8tm + 6t].
\]

Otherwise, the label of an edge under \(\bar{h}\) is also the length of the edge. Finally, we can see from the difference in parities of their elements that \((\bar{h}(e_{s_1,s_2})^{*} \cap (\bar{h}(e_{s_1,s_5}))^{*} = \emptyset\) and \((\bar{h}(e_{s_1,s_4}))^{*} \cap (\bar{h}(e_{s_1,s_5}))^{*} = \emptyset\). Thus,

\[
\begin{align*}
(\bar{h}(e_{s_1,s_2})^{*} \cup (\bar{h}(e_{s_1,s_5}))^{*} &= [1, 4tm - 4t], \\
(\bar{h}(e_{s_1,s_4}))^{*} \cup (\bar{h}(e_{s_1,s_5}))^{*} &= [4tm - 2t + 1, 8tm + 2t], \\
(\bar{h}(e_{s_1,s_5}))^{*} &= [8tm + 2t + 1, 8tm + 4t], \\
(\bar{h}(e_{s_6,s_7}))^{*} &= [4tm - 4t + 1, 4tm - 2t], \\
(\bar{h}(e_{s_1,s_7}))^{*} &= [8tm + 4t + 1, 8tm + 6t].
\end{align*}
\]
Hence, \((\bar{h}(E(G))\})^* = [1, 8tm + 6t]\) and \(h\) is a \(\rho\)-labeling of \(G\).

**Case 1b:** \(n \equiv 3 \pmod{4}, n \geq 11, \) and \(r\) is odd.

Let \(n = 4m + 3\) and \(r = 2t + 1\) where \(m \geq 2\) and \(t \geq 1\). Partition \(V(G)\) into the following sets:

\[
S_1 = \{v_{i,j}: i \text{ odd}, 1 \leq i \leq 4m + 3, 1 \leq j \leq 2t + 1\}, \\
S_2 = \{v_{i,j}: i \text{ even}, 1 \leq i \leq 2m - 2, 1 \leq j \leq t + 1\}, \\
S_3 = \{v_{i,j}: i \text{ even}, 2m \leq i \leq 4m, 1 \leq j \leq t + 1\}, \\
S_4 = \{v_{i,j}: i \text{ even}, 1 \leq i \leq 2m + 2, t + 2 \leq j \leq 2t + 1\}, \\
S_5 = \{v_{i,j}: i \text{ even}, 2m + 4 \leq i \leq 4m, t + 2 \leq j \leq 2t + 1\}, \\
S_6 = \{v_{i,j}: i = 4m + 2, 1 \leq j \leq 2t + 1\}, \\
S_7 = \{v_{i,j}: i = 4m + 3, 1 \leq j \leq 2t + 1\}.
\]

Let \(h: V(G) \to \mathbb{N}\) be defined as follows:

\[
h(v_{i,j}) = \begin{cases} 
-t + (ti + j) - 1 + \frac{i-1}{2} & \text{if } v_{i,j} \in S_1, \\
16tm + 14t + 8m - (ti + j) + 8 - \frac{i}{2} & \text{if } v_{i,j} \in S_2, \\
16tm + 12t + 8m - (ti + j) + 7 - \frac{i}{2} & \text{if } v_{i,j} \in S_3, \\
8tm + 6t + 4m - (ti + j) + 4 - \frac{i}{2} & \text{if } v_{i,j} \in S_4, \\
8tm + 4t + 4m - (ti + j) + 3 - \frac{i}{2} & \text{if } v_{i,j} \in S_5, \\
12tm + 2t + 6m + 2j & \text{if } v_{i,j} \in S_6, \\
8tm + 4t + 4m + 3j & \text{if } v_{i,j} \in S_7.
\end{cases}
\]

An example of the \(\rho\)-labeling of \(5C_{11}\) obtained from \(h\) is shown in Figure 4. If we proceed as in Case 1a, it is easy to verify that the given labeling is a \(\rho\)-labeling of \(rC_n\).

**Case 2a:** \(n \equiv 1 \pmod{4}, n \geq 9, \) and \(r\) even.

Let \(n = 4m + 1\) and \(r = 2t\) where \(m \geq 2\) and \(t \geq 1\). Partition \(V(G)\) into the following sets:

\[
S_1 = \{v_{i,j}: i \text{ odd}, 1 \leq i \leq 4m + 1, 1 \leq j \leq 2t\}, \\
S_2 = \{v_{i,j}: i \text{ even}, 1 \leq i \leq 2m, 1 \leq j \leq t\}, \\
S_3 = \{v_{i,j}: i \text{ even}, 2m \leq i \leq 4m - 2, 1 \leq j \leq t\}, \\
S_4 = \{v_{i,j}: i \text{ even}, 1 \leq i \leq 2m - 2, t \leq j \leq 2t\}, \\
S_5 = \{v_{i,j}: i \text{ even}, 2m \leq i \leq 4m - 2, t \leq j \leq 2t\}, \\
S_6 = \{v_{i,j}: i = 4m, 1 \leq j \leq t\}, \\
S_7 = \{v_{i,j}: i = 4m + t + 1 \leq j \leq 2t\}, \\
S_8 = \{v_{i,j}: i = 4m + 1, 1 \leq j \leq t\}, \\
S_9 = \{v_{i,j}: i = 4m + 1, t + 1 \leq j \leq 2t\}.
\]
Let $h: V(G) \to \mathbb{N}$ be defined as follows:

$$h(v_{i,j}) = \begin{cases} 
-t + (ti + j) - 1 & \text{if } v_{i,j} \in S_1, \\
16tm + 6t - (ti + j) + 1 & \text{if } v_{i,j} \in S_2, \\
16tm + 4t - (ti + j) + 1 & \text{if } v_{i,j} \in S_3, \\
8tm + 2t - (ti + j) + 1 & \text{if } v_{i,j} \in S_4, \\
8tm - (ti + j) + 1 & \text{if } v_{i,j} \in S_5, \\
12tm + 2t - 3j + 3 & \text{if } v_{i,j} \in S_6, \\
12tm + 6t - j + 1 & \text{if } v_{i,j} \in S_7, \\
8tm - j + 1 & \text{if } v_{i,j} \in S_8, \\
8tm + 8t - 3j + 2 & \text{if } v_{i,j} \in S_9.
\end{cases}$$

An example of the $\rho$-labeling of $4C_9$ obtained from $h$ is shown in Figure 5. If we proceed as in Case 1a, it is easy to verify that the given labeling is a $\rho$-labeling of $rC_n$.

**Case 2b:** $n \equiv 1 \pmod{4}$, $n \geq 9$, and $r$ odd.
Let \( n = 4m + 1 \) and \( r = 2t + 1 \). Partition \( V(G) \) into the following sets:

\[
S_1 = \{v_{i,j}: i \text{ odd}, 1 \leq i < 4m + 1, 1 \leq j \leq 2t + 1\},
\]

\[
S_2 = \{v_{i,j}: i \text{ even}, 1 \leq i \leq 2m, 1 \leq j \leq t + 1\},
\]

\[
S_3 = \{v_{i,j}: i \text{ even}, 2m < i \leq 4m - 2, 1 \leq j \leq t + 1\},
\]

\[
S_4 = \{v_{i,j}: i \text{ even}, 1 \leq i \leq 2m - 2, t + 1 < j \leq 2t + 1\},
\]

\[
S_5 = \{v_{i,j}: i \text{ even}, 2m < i \leq 4m - 2, t + 1 < j \leq 2t + 1\},
\]

\[
S_6 = \{v_{i,j}: i = 4m, 1 \leq j \leq t + 1\},
\]

\[
S_7 = \{v_{i,j}: i = 4m, t + 1 < j \leq 2t + 1\},
\]

\[
S_8 = \{v_{i,j}: i = 4m + 1, 1 \leq j \leq t + 1\},
\]

\[
S_9 = \{v_{i,j}: i = 4m + 1, t + 1 < j \leq 2t + 1\}.
\]

Let \( h: V(G) \to \mathbb{N} \) be defined as follows:

\[
h(v_{i,j}) = \begin{cases} 
-t + (ti + j) - 1 + \frac{i-1}{2} & \text{if } v_{i,j} \in S_1, \\
16tm + 6t + 8m - (ti + j) + 4 - \frac{i}{2} & \text{if } v_{i,j} \in S_2, \\
16tm + 4t + 8m - (ti + j) + 3 - \frac{i}{2} & \text{if } v_{i,j} \in S_3, \\
8tm + 2t + 4m - (ti + j) + 2 - \frac{i}{2} & \text{if } v_{i,j} \in S_4, \\
8tm + 4m - (ti + j) + 1 - \frac{i}{2} & \text{if } v_{i,j} \in S_5, \\
12tm + 2t + 6m - 3j + 4 & \text{if } v_{i,j} \in S_6, \\
12tm + 6t + 6m - j + 4 & \text{if } v_{i,j} \in S_7, \\
8tm + 4m - j + 1 & \text{if } v_{i,j} \in S_8, \\
8tm + 8t + 4m - 3j + 6 & \text{if } v_{i,j} \in S_9.
\end{cases}
\]

An example of the \( \rho \)-labeling of \( 5C_9 \) obtained from \( h \) is shown in Figure 6. If we proceed as in Case 1a, it is easy to verify that the given labeling is a \( \rho \)-labeling of \( rC_n \). \( \Box \)
In light of Theorem 1 and of the results from [1], [3], and [4], we have the following.

**Corollary 4** Let $r \geq 1$ and $n \geq 3$ be integers and let $G = rC_n$. Then there exists a cyclic $G$-decomposition of $K_{2rn+1}$.

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### References


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