On γ-labeling the almost-bipartite graph $P_m + e$

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Abstract

An *almost-bipartite* graph is a non-bipartite graph with the property that the removal of a particular single edge renders the graph bipartite. A graph labeling of an almost-bipartite graph G with n edges that yields cyclic G-decompositions of the complete graph K_{2nt+1} was recently introduced by Blinco, El-Zanati, and Vanden Eynden. They called such a labeling a γ-labeling. Here we show that the class of almost-bipartite graphs obtained from a path with at least 3 edges by adding an edge joining distinct vertices of the path an even distance apart has a γ-labeling.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. The *order* and the *size* of a graph G are $|V(G)|$ and $|E(G)|$, respectively.

Let $V(K_k) = \mathbb{Z}_k$ and let G be a subgraph of K_k. By *clicking* G, we mean applying the isomorphism $i \to i + 1$ to $V(G)$. Let H and G be graphs such that G is a subgraph of H. A G-*decomposition* of H is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of pairwise edge-disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \cup_{i=1}^{t} E(G_i)$. If H is K_k, a

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G-decomposition Δ of H is *cyclic* if clicking is a permutation of Δ. For a comprehensive source on graph decompositions we refer the reader to [2].

Let $V(K_k) = \{0, 1, \ldots, k-1\}$. The *length* of an edge $\{i, j\}$ in K_k is $\min\{|i-j|, k-|i-j|\}$. Note that clicking an edge does not change its length. Also note that if k is odd, then K_k consists of k edges of length i for $i = 1, 2, \ldots, \frac{k-1}{2}$.

For any graph G, a one-to-one function $f : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [6], Rosa introduced a hierarchy of labelings. We add a few items to this hierarchy. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{E}(G) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

(ℓ1) $f(V(G)) \subseteq [0, 2n]$,

(ℓ2) $f(V(G)) \subseteq [0, n]$,

(ℓ3) $\bar{E}(G) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,

(ℓ4) $\bar{E}(G) = [1, n]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ consider also

(ℓ5) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,

(ℓ6) there exists an integer λ (called the *boundary value* of f) such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

(ℓ1), (ℓ3) is called a ρ-*labeling*;

(ℓ1), (ℓ4) is called a σ-*labeling*;

(ℓ2), (ℓ4) is called a β-*labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. If G is bipartite and a ρ, σ or β-labeling of G also satisfies (ℓ5), then the labeling is *ordered* and is denoted by ρ^+, σ^+ or β^+, respectively. If in addition (ℓ6) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++} or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an α-*labeling* as introduced in [6]. Labelings of the types above are called *Rosa-type* because of Rosa's original article [6] on the topic. A dynamic survey on graph labelings is maintained by Gallian [5].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [6] and [4], respectively.

Theorem 1 *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

Theorem 2 *Let G be a graph with n edges that has a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nt+1} for all positive integers t.*

If G with n edges is not bipartite, then the best that could be obtained up until recently from a Rosa-type labeling was a cyclic G-decomposition of K_{2n+1}. A non-bipartite graph G is *almost-bipartite* if G contains an edge e whose removal renders the remaining graph bipartite (for example, odd cycles are almost-bipartite). In [1], Blinco et al. introduced a variation of a ρ-labeling of an almost-bipartite graph G of size n that yields cyclic G-decompositions of K_{2nt+1}. They called this labeling a γ-labeling. They showed that odd cycles (other than C_3) and certain other 2-regular almost-bipartite graphs admit γ-labelings. In [3], it is shown that every 2-regular almost-bipartite graph other than C_3 and $C_3 \cup C_4$ admits a γ-labeling.

In this article, we show that the class of almost-bipartite graphs obtained from a path with at least 3 edges by adding an edge joining distinct vertices of the path an even distance apart has a γ-labeling.

2 Additional Definitions and Notation

Let G be a graph with n edges and h a labeling of the vertices of G. We call h a γ-*labeling* of G if the following conditions hold.

(g1) The function h is a ρ-labeling of G.

(g2) The graph G is tripartite with vertex tripartition A, B, C with $C = \{c\}$ and $\bar{b} \in B$ such that $\{\bar{b}, c\}$ is the unique edge joining an element of B to c.

(g3) If $\{a, v\}$ is an edge of G with $a \in A$, then $h(a) < h(v)$.

(g4) We have $h(c) - h(\bar{b}) = n$.

Note that if a nonbipartite graph G has a γ-labeling, then it is almost-bipartite as defined earlier. In this case, removing the edge $\{c, \bar{b}\}$ from G produces a bipartite graph. Figure 1 shows γ-labelings of C_5 and of C_7.

To simplify our consideration of the labelings, we will henceforth consider graphs whose vertices are named by distinct nonnegative integers,

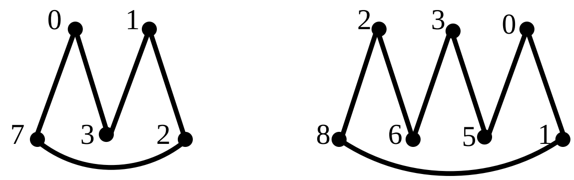

Figure 1: γ-labelings of C_5 and of C_7.

which are also their labels. Recall that by the label of the edge $\{x, y\}$ in such a graph we mean $|x - y|$. If G is a graph with n edges and if m is the label of an edge e, let $m^* = \min\{m, 2n + 1 - m\}$ (thus m^* is the length of e). If S is a set of edge labels, let $S^* = \{m^* : m \in S\}$.

We denote the directed path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \leq i \leq k - 1$, by $(x_0, x_1, \ldots, x_k)$. The *first vertex* of this path is x_0, the *second vertex* is x_1, and the *last vertex* is x_k. If $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$ are directed paths with $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$.

Let $P(k)$ be the path with k edges and $k + 1$ vertices $0, 1, \ldots, k$ given by $(0, k, 1, k - 1, 2, k - 2, \ldots, \lceil k/2 \rceil)$. Note that the set of vertices of this graph is $A \cup B$, where $A = [0, \lfloor k/2 \rfloor]$, $B = [\lfloor k/2 \rfloor + 1, k]$, and every edge joins a vertex of A to one of B. Furthermore the set of labels of the edges of $P(k)$ is $[1, k]$.

Now let a and b be nonnegative integers with $a \leq b$ and let us add a to all the vertices of A and b to all the vertices of B. We will denote the resulting graph by $P(a, b, k)$. Note that this graph has the following properties.

P1: $P(a, b, k)$ is a path with first vertex a and second vertex $b + k$. If k is even, its last vertex is $a + k/2$.

P2: Each edge of $P(a, b, k)$ joins a vertex of $A' = [a, \lfloor k/2 \rfloor + a]$ to a larger vertex of $B' = [\lfloor k/2 \rfloor + 1 + b, k + b]$.

P3: The set of edge labels of $P(a, b, k)$ is $[b - a + 1, b - a + k]$.

Now consider the directed path $Q(k)$ obtained from $P(k)$ replacing each vertex i with $k - i$. The new graph is the path $(k, 0, k - 1, 1, \ldots, k - \lceil k/2 \rceil)$. The set of vertices of $Q(k)$ is $A'' \cup B''$, where $A'' = k - B = [0, k - \lfloor k/2 \rfloor - 1]$ and $B'' = k - A = [k - \lfloor k/2 \rfloor, k]$, and every edge joins a vertex of A'' to one of B''. The set of edge labels is still $[1, k]$. The last vertex of $Q(k)$ is $k/2 \in B''$ if k is even and $(k - 1)/2 \in A''$ if k is odd.

We add a to the vertices of A'' and b to vertices of B'', where a and b are integers, $0 \leq a \leq b$. This graph is $(k + b, a, k + b - 1, a + 1, \ldots)$.

Let $Q(a,b,k) = (\ldots, a+1, k+b-1, a, k+b)$ be the latter graph with its orientation reversed. Note that this graph has the following properties.

Q1: $Q(a,b,k)$ is a path with last vertex $k+b$. Its first vertex is $b+k/2$ if k is even and $a+(k-1)/2$ if k is odd.

Q2: Each edge of $Q(a,b,k)$ joins a vertex of $A''' = [a, a+k-\lfloor k/2 \rfloor - 1]$ to a larger vertex of $B''' = [b+k-\lfloor k/2 \rfloor, b+k]$.

Q3: The set of edge labels of $Q(a,b,k)$ is $[b-a+1, b-a+k]$.

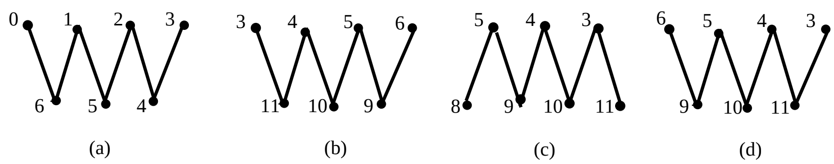

Figure 2: (a) $P(6)$, (b) $P(3,5,6)$, (c) $Q(3,5,6)$, (d) $R(3,5,6)$.

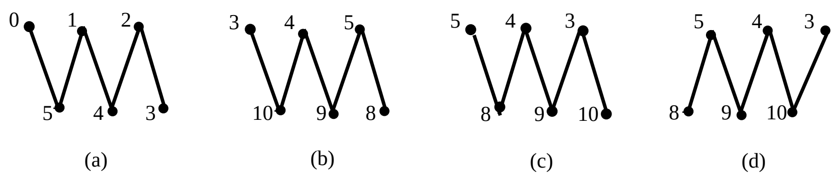

Figure 3: (a) $P(5)$, (b) $P(3,5,5)$, (c) $Q(3,5,5)$, (d) $R(3,5,5)$.

Finally let $R(a,b,k)$ be the path $P(a,b,k)$ with its orientation reversed. Note that this graph has the following properties.

R1: $R(a,b,k)$ is a path with last vertex a. If k is even, its first vertex is $a+k/2$.

R2: Each edge of $R(a,b,k)$ joins a vertex of $A' = [a, \lfloor k/2 \rfloor + a]$ to a larger vertex of $B' = [\lfloor k/2 \rfloor + 1 + b, k+b]$.

R3: The set of edge labels of $R(a,b,k)$ is $[b-a+1, b-a+k]$.

3 Main Result

Theorem 3 *Let $G(x,y,z)$ denote the graph formed by adding the edge $\{v_x, v_{x+2y}\}$ to the path $(v_0, v_1, \ldots, v_{x+2y+z})$, where x, y, and z are non-negative integers with $y \geq 1$. Then $G(x,y,z)$ has a γ-labeling unless $(x,y,z) = (0,1,0)$.*

Proof. The graph $G(x, y, z)$ is not bipartite, since it contains a cycle of length $2y + 1$, but it is clearly almost-bipartite. Without loss of generality we can assume that $x \geq z$. Note that $G(0, y, 0)$ is the odd cycle C_{2y+1} which was shown in [1] to admit a γ-labeling unless $y = 1$. We break the rest of the problem into 5 cases. Set $t = -x + y + z - 2$.

Case 1 $y = 1$ and $z = 0$.

Note that $x > 0$ since our path has at least 3 edges. We will take our path to be $F + Q(4, 6, x - 1) + (x + 5, 0, 2)$ and the added edge to be $(x + 5, 2)$. Here F is an edge that will be defined below. This graph has $n = x + 3$ edges, which is the length of the added edge $(x + 5, 2)$. Note that by Q1 and Q3 the path $Q(4, 6, x - 1)$ has last vertex $x + 5$ and edge label set $[3, x + 1]$. The labels of the edges in $(x + 5, 0, 2)$ are $x + 5$ and 2, and $(x + 5)^* = x + 2$. Thus if S is the set of labels of the edges other than F, then $S^* = [2, x + 3] = [2, n]$.

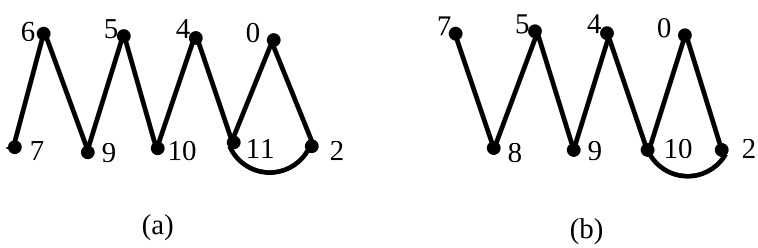

Figure 4: γ-labelings of: (a) $G(6, 1, 0)$ and of: (b) $G(5, 1, 0)$.

Now if x is even we take $F = (4 + x/2, 3 + x/2)$, which has label 1. Note that since $x - 1$ is odd, the first vertex of $Q(4, 6, x - 1)$ is $4 + (x - 2)/2 = 3 + x/2$. The vertex sets of $Q(4, 6, x - 1)$ are $A''' = [4, 4 + x - 1 - (x - 2)/2 - 1] = [4, 3 + x/2]$ and $B''' = [6 + x - 1 - (x - 2)/2, 6 + x - 1] = [6 + x/2, x + 5]$. The additional vertices are $4 + x/2$ from F and 0 and 2 from $(x + 5, 0, 2)$. Since $0 < 2 < [4, 3 + x/2] < 4 + x/2 < [6 + x/2, x + 5]$, the vertices are distinct and we have a ρ-labeling. In fact we have a γ-labeling with $c = x + 5$ and $\bar{b} = 2$.

Likewise if x is odd we take $F = ((9 + x)/2, (11 + x)/2)$, which has label 1. Since $x - 1$ is even, the first vertex of $Q(4, 6, x - 1)$ is $6 + (x - 1)/2 = (11 + x)/2$. The vertex sets of $Q(4, 6, x - 1)$ are $A''' = [4, 4 + x - 1 - (x - 1)/2 - 1] = [4, (5 + x)/2]$ and $B''' = [6 + x - 1 - (x - 1)/2, 6 + x - 1] = [(11 + x)/2, x + 5]$. The additional vertices are $(9 + x)/2$, 0, and 2. Since $0 < 2 < [4, (5 + x)/2] < (9 + x)/2 < [(11 + x)/2, x + 5]$, the vertices are distinct. Again we have a γ-labeling with $c = x + 5$ and $\bar{b} = 2$.

Case 2 t is even and $t \geq 0$.

Note that since t is even $\pm x \pm y \pm z$ is even for any choice of signs. We will take our graph to be $G_1 + (x+3y+2z+1, 0) + G_2 + G_3 + G_4$ plus the edge $(x+3y+2z+1, y+z)$, where (recalling that $x \geq z$)

$$\begin{aligned} G_1 &= Q(y+z+1, 3y+2z+1, x), \\ G_2 &= P(0, x+3y+2z+1, x+y-z), \\ G_3 &= P(\frac{x+y-z}{2}, \frac{5x+5y+z+4}{2}, -x+y+z-2), \\ G_4 &= P(y-1, y-1, z+1). \end{aligned}$$

Notice that by Q1, P1, and the assumption that t is even the last vertex of G_1 is $x+3y+2z+1$, the first vertex of G_2 is 0 and the last $(x+y-z)/2$, the first vertex of G_3 is $(x+y-z)/2$ and the last is $y-1$, and the first vertex of G_4 is $y-1$ and the second is $y+z$. Thus $G_1+(x+3y+2z+1,0)+G_2+G_3+G_4$ is a path of length $x+2y+z$ and in it $v_x = x+3y+2z+1$ and $v_{x+2y} = y+z$.

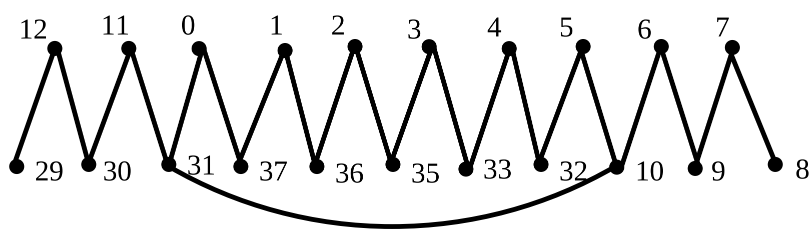

Figure 5: A γ-labeling $G(4,6,4)$.

We start by showing that the vertices in our graph are distinct. For $1 \leq i \leq 4$ let A_i and B_i denote the sets labeled A' or B' in P2 or A''' or B''' in Q2, as appropriate, corresponding to the path G_i. Then using Q2, P2, and the assumption that t is even we compute

$$A_1 = [y+z+1, x - \left\lfloor \tfrac{x}{2} \right\rfloor + y + z],$$

$$B_1 = [x - \left\lfloor \tfrac{x}{2} \right\rfloor + 3y + 2z + 1, x + 3y + 2z + 1],$$

$$\begin{aligned} A_2 &= [0, \frac{x+y-z}{2}], & B_2 &= [\frac{3x+7y+3z+4}{2}, 2x+4y+z+1], \\ A_3 &= [\frac{x+y-z}{2}, y-1], & B_3 &= [2x+3y+z+2, \frac{3x+7y+3z}{2}], \\ A_4 &= [y-1, y + \left\lfloor \frac{z+1}{2} \right\rfloor - 1], & B_4 &= [y + \left\lfloor \frac{z+1}{2} \right\rfloor, y+z]. \end{aligned}$$

Using the assumptions that $x \geq z$ and $y > 0$ we can check that $A_2 \leq A_3 \leq A_4 < B_4 < A_1 < B_1 < B_3 < B_2$. (Note that G_2 and G_3 share the vertex $(x+y-z)/2 \in A_2 \bigcap A_3$ and G_3 and G_4 share the vertex $y-1 \in A_3 \bigcap A_4$.) Thus the vertices of our graph are distinct.

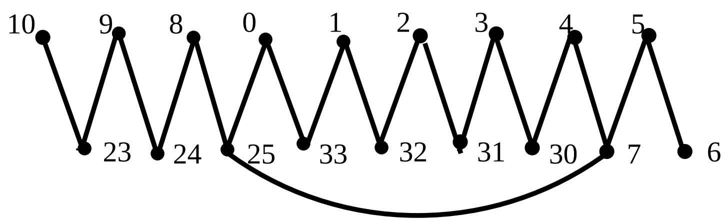

Figure 6: A γ-labeling $G(5,5,2)$.

Now let E_i denote the set of edge labels of G_i, $1 \leq i \leq 4$. Note that our graph has $n = x+2y+z+1$ edges, and $2n+1 = 2x+4y+2z+3$. Using Q3, P3, and the assumption that t is even we compute

$$E_1^* = [2y+z+1, x+2y+z]^* = [2y+z+1, x+2y+z],$$
$$E_2^* = [x+3y+2z+2, 2x+4y+z+1]^* = [z+2, x+y+1],$$
$$E_3^* = [2x+2y+z+3, x+3y+2z]^* = [x+y+3, 2y+z],$$
$$E_4^* = [1, z+1]^* = [1, z+1].$$

Note that the edges $\{x+3y+2z+1, 0\}$ and $\{x+3y+2z+1, y+z\}$ have labels $x+3y+2z+1$ and $x+2y+z+1 = n$, respectively, and $(x+3y+2z+1)^* = x+y+2$. Ordering these sets as

$[1, z+1], [z+2, x+y+1], \{x+y+2\}, [x+y+3, 2y+z], [2y+z+1, x+2y+z], \{x+2y+z+1\}$,

we see that our graph has a ρ-labeling.

If we take $c = x+3y+2z+1$ and $\bar{b} = y+z$, we easily check that the other conditions for a γ-labeling are satisfied.

Case 3 t is even, $t < 0$, and $(y,z) \neq (1,0)$.

Notice that $y+z-2 \geq 0$ by the assumption that $(y,z) \neq (1,0)$. We will take the path $G_1 + G_2 + (x+3y+2z+1, 0) + G_3 + G_4$, plus the edge $(x+3y+2z+1, y+z)$, where G_1 will be a path with $-t = x-y-z+2$

edges depending on the parity of x,

$$G_2 = Q(y+z+1, x+2y+z+3, y+z-2),$$
$$G_3 = P(0, 2x+2y+z+3, 2y-2),$$
$$G_4 = P(y-1, y-1, z+1).$$

Note that the last vertex of G_2 is $x+3y+2z+1$, the first vertex of G_3 is 0 and the last is $y-1$, the first vertex of G_4 is $y-1$ and the second is $y+z$. Thus, assuming that the last vertex of G_1 is the first vertex of G_2, $G_1+G_2+(x+3y+2z+1,0)+G_3+G_4$ will be a path of length $x+2y+z$ and in it $v_x = x+3y+2z+1$ and $v_{x+2y} = y+z$.

For $1 \le i \le 4$ let A_i and B_i denote the sets of vertices of G_i labeled A' and B' in P2 or R2 or A''' and B''' in Q2, as appropriate, and let E_i be the set of edge labels of G_i. Then we compute

$$A_2 = [y+z+1, 2y+2z-2-\lfloor \tfrac{y+z-2}{2} \rfloor],$$
$$B_2 = [x+3y+2z+1-\lfloor \tfrac{y+z-2}{2} \rfloor x+3y+2z+1],$$

$$A_3 = [0, y-1], \qquad B_3 = [2x+3y+z+3, 2x+4y+z+1],$$
$$A_4 = [y-1, y+\left\lfloor \frac{z+1}{2} \right\rfloor - 1], \quad B_4 = [y+\left\lfloor \frac{z+1}{2} \right\rfloor, y+z].$$

It can be checked that $A_3 \le A_4 < B_4 < A_2$ (note that G_3 and G_4 share the vertex $y-1$) and $B_2 < B_3$ (recall the assumption $x \ge z$). Thus to show that the vertices are distinct it suffices to show that $A_2 \le A_1 < B_1 \le B_2$ and that G_1 and G_2 intersect only in the last vertex of G_1, which is also the first vertex of G_2.

Furthermore

$$E_2^* = [x+y+3, x+2y+z]^* = [x+y+3, x+2y+z],$$
$$E_3^* = [2x+2y+z+4, 2x+4y+z+1]^* = [z+2, 2y+z-1],$$
$$E_4^* = [1, z+1]^* = [1, z+1].$$

The edges $(x+3y+2z+1, 0)$ and $(x+3y+2z+1, y+z)$ have the labels $x+3y+2z+1$ and $x+2y+z+1$, respectively, and $(x+3y+2z+1)^* = x+y+2$. Thus if S is the set of edge labels not in G_1, we have $S^* = [1, 2y+z-1] \cup [x+y+2, x+2y+z+1]$. We see that we need that if T_1 is the set of edge labels of G_1, then $T_1^* = [2y+z, x+y+1]$.

We finish this case by defining G_1 according as x is even or odd. If x is even, then let

$$G_1 = Q(\frac{3y+3z+2}{2}, \frac{7y+5z}{2}, x-y-z+2).$$

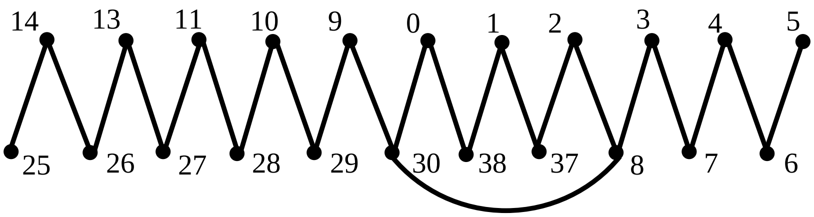

Figure 7: A γ-labeling of $G(10, 3, 5)$.

Since $t = -x + y + z - 2 < 0$, this path has the positive length $-t$, and since t and x are even, y and z have the same parity. We compute

$$A_1 = \left[\frac{3y+3z+2}{2}, \frac{x+2y+2z+2}{2}\right],$$
$$B_1 = \left[\frac{x+6y+4z+2}{2}, \frac{2x+5y+3z+4}{2}\right],$$

and $E_1 = [2y + z, x + y + 1]$, which is the desired set of edge labels. Furthermore, the inequalities $A_2 < A_1 < B_1 \leq B_2$ are easily checked, where B_1 and B_2 overlap only in the vertex $(2x + 5y + 3z + 4)/2$. Note that by Q1 this is the last vertex of G_1 and, since $y + z - 2$ is even, also the first vertex of G_2.

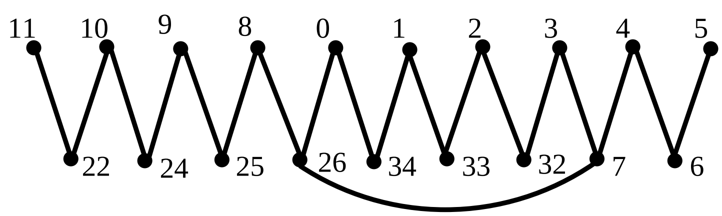

Figure 8: A γ-labeling of $G(7, 4, 3)$.

Now suppose x is odd. We let

$$G_1 = R\left(\frac{3y+3z-1}{2}, \frac{7y+5z-3}{2}, x-y-z+2\right).$$

This path has the positive length $-t$, and since $t = -x + y + z - 2$ is even and x odd, y and z have opposite parities. We compute

$$A_1 = \left[\frac{3y+3z-1}{2}, \frac{x+2y+2z+1}{2}\right],$$
$$B_1 = \left[\frac{x+6y+4z+1}{2}, \frac{2x+5y+3z+1}{2}\right],$$

and $E_1 = [2y+z, x+y+1]$, which is the desired set of edge labels. Furthermore, the inequalities $A_2 \leq A_1 < B_1 < B_2$ are easily checked, where A_1 and A_2 overlap only in the vertex $(3y+3z-1)/2$. Note that by R1 and $Q1$ this is the last vertex of G_1, and, since $y+z-2$ is odd, also the first vertex of G_2.

Case 4 t is odd and $t > 0$.

Notice that since t is odd $\pm x \pm y \pm z$ is odd for any choice of signs. We will take our graph to be the path $G_1 + (x - \lfloor x/2 \rfloor - 1, 2x - \lfloor x/2 \rfloor + 3y + 2z + 1, x - \lfloor x/2 \rfloor) + G_2 + G_3$ plus the edge $(2x - \lfloor x/2 \rfloor + 3y + 2z + 1, x - \lfloor x/2 \rfloor + y + z)$, where G_1 will be a path with $x - 1$ edges depending on the parity of x,

$$G_2 = P\left(x - \left\lfloor\frac{x}{2}\right\rfloor, 2x - \left\lfloor\frac{x}{2}\right\rfloor + y + 2, -x + y + z - 1\right),$$
$$G_3 = P\left(\frac{x+y+z-1}{2} - \left\lfloor\frac{x}{2}\right\rfloor, \frac{x+y+z-1}{2} - \left\lfloor\frac{x}{2}\right\rfloor, x+y\right).$$

Note that by P1 and the assumption that t is odd the first vertex of G_2 is $x - \lfloor x/2 \rfloor$ and the last is $(x+y+z-1)/2 - \lfloor x/2 \rfloor$, the first vertex of G_3 is $(x+y+z-1)/2 - \lfloor x/2 \rfloor$. Thus, assuming that the last vertex of G_1 is $x - \lfloor x/2 \rfloor - 1$ and G_3 contains the vertex $x - \lfloor x/2 \rfloor + y + z$, $G_1 + (x - \lfloor x/2 \rfloor - 1, 2x - \lfloor x/2 \rfloor + 3y + 2z + 1, x - \lfloor x/2 \rfloor) + G_2 + G_3$ will be a path of length $x + 2y + z$ and in it $v_x = 2x - \lfloor x/2 \rfloor + 3y + 2z + 1$ and $v_{x+2y} = x - \lfloor x/2 \rfloor + y + z$.

For $1 \leq i \leq 3$ let A_i and B_i denote the sets of vertices of G_i labeled A' and B' in P2 respectively and let E_i be the set of edge labels of G_i. Then we compute

$$A_2 = \left[x - \left\lfloor\frac{x}{2}\right\rfloor, \frac{x+y+z-1}{2} - \left\lfloor\frac{x}{2}\right\rfloor\right],$$
$$B_2 = \left[\frac{3x+3y+z+5}{2} - \left\lfloor\frac{x}{2}\right\rfloor, x - \left\lfloor\frac{x}{2}\right\rfloor + 2y + z + 1\right],$$
$$A_3 = \left[\frac{x+y+z-1}{2} - \left\lfloor\frac{x}{2}\right\rfloor, \frac{x+y+z-1}{2} - \left\lfloor\frac{x}{2}\right\rfloor + \left\lfloor\frac{x+y}{2}\right\rfloor\right],$$
$$B_3 = \left[\frac{x+y+z+1}{2} - \left\lfloor\frac{x}{2}\right\rfloor + \left\lfloor\frac{x+y}{2}\right\rfloor, \frac{3x+3y+z-1}{2} - \left\lfloor\frac{x}{2}\right\rfloor\right].$$

It can be checked that $A_2 \leq A_3 < B_3 < B_2$ (note that G_2 and G_3 share the vertex $(x+y+z-1)/2 - \lfloor x/2 \rfloor$).

Furthermore

$$E_2^* = [x+y+3, 2y+z+1]^* = [x+y+3, 2y+z+1],$$
$$E_3^* = [1, x+y]^* = [1, x+y].$$

The edges $(x - \lfloor x/2 \rfloor - 1, 2x - \lfloor x/2 \rfloor + 3y + 2z + 1)$, $(2x - \lfloor x/2 \rfloor + 3y + 2z + 1, x - \lfloor x/2 \rfloor)$, and $(2x - \lfloor x/2 \rfloor + 3y + 2z + 1, x - \lfloor x/2 \rfloor + y + z)$ have the labels $(x + 3y + 2z + 2)^* = x + y + 1$, $(x + 3y + 2z + 1)^* = x + y + 2$, and $x + 2y + z + 1$, respectively. Thus if S is the set of edge labels not in G_1, we have $S^* = [1, 2y + z + 1] \cup \{x + 2y + z + 1\}$. We see that we need that if T_1 is the set of edge labels of G_1, then $T_1^* = [2y + z + 2, x + 2y + z]$.

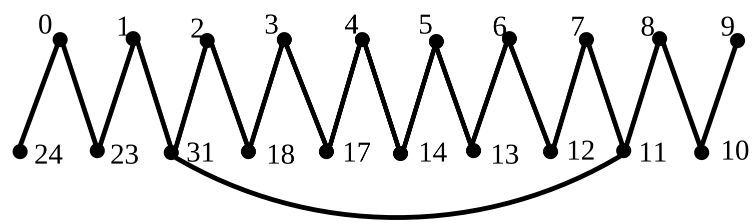

Figure 9: A γ-labeling of $G(4, 6, 3)$.

We finish this case by defining G_1 according as x is even or odd. If x even, then let

$$G_1 = (2x + 2y + z + 1, 0) + P(0, x + 2y + z + 2, x - 2).$$

Since $t = -x+y+z-2$ is odd and x is even, y and z have opposite parities. We compute

$$\begin{aligned}
A_1 &= \{0\} \cup \left[0, \frac{x-2}{2}\right] = \left[0, \frac{x-2}{2}\right], \\
B_1 &= \{2x + 2y + z + 1\} \cup \left[\frac{3x + 4y + 2z + 4}{2}, 2x + 2y + z\right] \\
&= \left[\frac{3x + 4y + 2z + 4}{2}, 2x + 2y + z + 1\right], \\
E_1^* &= \{2x + 2y + z + 1\}^* \cup [x + 2y + z + 3, 2x + 2y + z]^* \\
&= [2y + z + 2, x + 2y + z].
\end{aligned}$$

We see that this is the desired set of edge labels. Furthermore, note that $A_1 < A_2 < B_2 < B_1$ and that the path $(x - \lfloor x/2 \rfloor - 1, 2x - \lfloor x/2 \rfloor + 3y +$

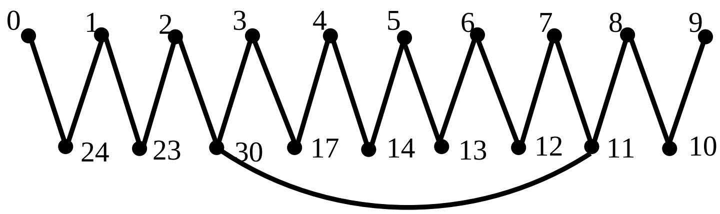

Figure 10: A γ-labeling of $G(5,5,3)$.

$2z+1, x-\lfloor x/2\rfloor)$ of length 2 starts at the last vertex of G_1 and ends at the first of G_2.

Now suppose x is odd. We let

$$G_1 = P(0, x+2y+z+2, x-1).$$

Since t and x are odd, y and z have the same parity. We compute

$$\begin{aligned}
A_1 &= \left[0, \frac{x-1}{2}\right], \\
B_1 &= \left[\frac{3x+4y+2z+5}{2}, 2x+2y+z+1\right], \\
E_1^* &= [x+2y+z+3, 2x+2y+z+1]^* = [2y+z+2, x+2y+z].
\end{aligned}$$

We see that this is the desired set of edge labels. Furthermore, note that $A_1 < A_2 < B_2 < B_1$ and that the path $(x-\lfloor x/2\rfloor-1, 2x-\lfloor x/2\rfloor+3y+2z+1, x-\lfloor x/2\rfloor)$ of length 2 starts at the last vertex of G_1 and ends at the first of G_2.

Case 5 t is odd and $t<0$.

We will take our graph to be the path $G_1+G_2+G_3$ plus the edge $(2x-\lfloor x/2\rfloor+3y+2z+1, x-\lfloor x/2\rfloor+y+z)$, where G_1 will be a path with $y+z-2$ edges depending on the parity of x,

$$\begin{aligned}
G_2 &= P\left(\frac{x+y+z-3}{2} - \left\lfloor\frac{x}{2}\right\rfloor, \frac{3x+7y+5z-3}{2} - \left\lfloor\frac{x}{2}\right\rfloor, x-y-z+3\right), \\
G_3 &= P\left(x - \left\lfloor\frac{x}{2}\right\rfloor, x - \left\lfloor\frac{x}{2}\right\rfloor, 2y+z-1\right).
\end{aligned}$$

Note that by P1 and the assumption that t is odd the first vertex of G_2 is $(x+y+z-3)/2-\lfloor x/2\rfloor$ and the last is $x-\lfloor x/2\rfloor$, the first vertex of G_3 is $x-\lfloor x/2\rfloor$. Thus, assuming that the last vertex of G_1 is $(x+y+z-3)/2-\lfloor x/2\rfloor$ and G_2 and G_3 contain the vertices $2x-\lfloor x/2\rfloor+3y+2z+1$ and

$x-\lfloor x/2\rfloor+y+z$, respectively, $G_1+G_2+G_3$ will be a path of length $x+2y+z$ and in it $v_x = 2x-\lfloor x/2\rfloor+3y+2z+1$ and $v_{x+2y} = x-\lfloor x/2\rfloor+y+z$.

For $1 \le i \le 3$ let A_i and B_i denote the sets of vertices of G_i labeled A' and B' in P2 respectively and let E_i be the set of edge labels of G_i. Then we compute

$$\begin{aligned}
A_2 &= \left[\frac{x+y+z-3}{2} - \left\lfloor\frac{x}{2}\right\rfloor, x - \left\lfloor\frac{x}{2}\right\rfloor\right],\\
B_2 &= \left[2x+3y+2z+1-\left\lfloor\frac{x}{2}\right\rfloor, \frac{5x+5y+3z+3}{2} - \left\lfloor\frac{x}{2}\right\rfloor\right],\\
A_3 &= \left[x-\left\lfloor\frac{x}{2}\right\rfloor, x-\left\lfloor\frac{x}{2}\right\rfloor + \left\lfloor\frac{2y+z-1}{2}\right\rfloor\right],\\
B_3 &= \left[x+1-\left\lfloor\frac{x}{2}\right\rfloor + \left\lfloor\frac{2y+z-1}{2}\right\rfloor, x+2y+z-1-\left\lfloor\frac{x}{2}\right\rfloor\right].
\end{aligned}$$

It can be checked that $A_2 \le A_3 < B_3 < B_2$ (note that G_2 and G_3 share the vertex $x-\lfloor x/2\rfloor$).

Furthermore

$$\begin{aligned}
E_2^* &= [x+3y+2z+1, 2x+2y+z+3]^* = [2y+z, x+y+2],\\
E_3^* &= [1, 2y+z-1]^* = [1, 2y+z-1].
\end{aligned}$$

The edge $(2x-\lfloor x/2\rfloor+3y+2z+1, x-\lfloor x/2\rfloor+y+z)$ has a label of $x+2y+z+1$. Thus if S is the set of edge labels not in G_1, we have $S^* = [1, x+y+2] \cup \{x+2y+z+1\}$. We see that we need that if T_1 is the set of edge labels of G_1, then $T_1^* = [x+y+3, x+2y+z]$.

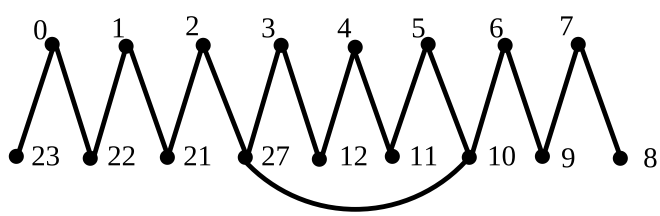

Figure 11: A γ-labeling of $G(6,3,4)$.

We finish this case by defining G_1 according as x is even or odd. If x even, then let

$$G_1 = (x+3y+2z, 0) + P(0, x+2y+z+2, y+z-3).$$

Since $t = -x+y+z-2$ is odd and x is even, y and z have opposite parities. We compute

$$\begin{aligned} A_1 &= \{0\} \cup \left[0, \frac{y+z-3}{2}\right] = \left[0, \frac{y+z-3}{2}\right], \\ B_1 &= \{x+3y+2z\} \cup \left[\frac{2x+5y+3z+3}{2}, x+3y+2z-1\right] \\ &= \left[\frac{2x+5y+3z+3}{2}, x+3y+2z\right], \\ E_1^* &= \{x+3y+2z\}^* \cup [x+2y+z+3, x+3y+2z-1]^* \\ &= [x+y+3, x+2y+z]. \end{aligned}$$

We see that this is the desired set of edge labels. Furthermore, the inequalities $A_1 \leq A_2$ and $B_3 < B_1 < B_2$ are easily checked, where A_1 and A_2 overlap only in the vertex $\frac{y+z-3}{2}$. Note that since $y+z-3$ is even, $\frac{y+z-3}{2}$ is the last vertex in G_1 and it is also the first vertex of G_2.

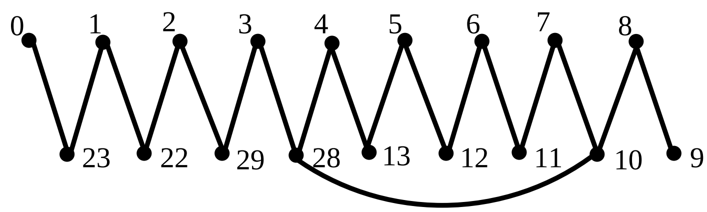

Figure 12: A γ-labeling of $G(7, 4, 2)$.

Now suppose x is odd. We let

$$G_1 = P(0, x+2y+z+2, y+z-2).$$

Since t and x are odd, y and z have the same parity. We compute

$$\begin{aligned} A_1 &= \left[0, \frac{y+z-2}{2}\right], \\ B_1 &= \left[\frac{2x+5y+3z+4}{2}, x+3y+2z\right], \\ E_1^* &= [x+2y+z+3, x+3y+2z]^* = [x+y+3, x+2y+z]. \end{aligned}$$

We see that this is the desired set of edge labels. Furthermore, the inequalities $A_1 \leq A_2$ and $B_3 < B_1 < B_2$ are easily checked, where A_1 and A_2 overlap only in the vertex $\frac{y+z-2}{2}$. Note that since $y+z-2$ is even, $\frac{y+z-3}{2}$ is the last vertex in G_1 and it is also the first vertex of G_2.

Thus, in each of the cases the given labeling satisfies the conditions for a γ-labeling. □

Although C_3 does not admit a γ-labeling, it is known that there exists a cyclic C_3-decomposition of K_{6t+1} for all positive integers t (see [2]). Therefore we have the following corollary.

Corollary 4 *Let $G(x, y, z)$ denote the graph with n edges formed by adding the edge $\{v_x, v_{x+2y}\}$ to the path $(v_0, v_1, \ldots, v_{x+2y+z})$, where x, y, and z are nonnegative integers with $y \geq 1$. Then there exists a cyclic $G(x, y, z)$-decomposition of K_{2nt+1} for all positive integers t.*

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