On cyclic decompositions of complete graphs into tripartite graphs

R. C. Bunge[1], A. Chantasartrassmee[2], S. I. El-Zanati*[1],
and C. Vanden Eynden[1]

[1]Department of Mathematics
Illinois State University
Normal, IL 61790-4520 USA

[2]Department of Mathematics, School of Science
University of the Thai Chamber of Commerce
Bangkok 10400, Thailand

Abstract

We introduce two new labelings for tripartite graphs and show that if a graph G with n edges admits either of these labelings, then there exists a cyclic G-decomposition of K_{2nx+1} for every positive integer x. We also show that if G is the union of two vertext-disjoint cycles of odd length, other than $C_3 \cup C_3$, then G admits one of these labelings.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. Let K_k denote the complete graph on k vertices.

Let $V(K_k) = \mathbb{Z}_k$ and let G be a subgraph of K_k. The *length* of an edge $\{i, j\} \in E(G)$ is defined as $\min\{|i-j|, k-|i-j|\}$. By *clicking* G, we mean applying the isomorphism $i \to i+1$ to $V(G)$. Let H and G be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of edgewise disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. If H is K_k, a G-decomposition Γ of H is *cyclic* if clicking induces a permutation of Γ. If G is a graph and r is a positive integer, rG denotes the vertex disjoint union of r copies of G.

For any graph G, a one-to-one function $f : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [9], Rosa introduced a hierarchy of labelings. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{f}(E(G)) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

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(ℓ1) $f(V(G)) \subseteq [0, 2n]$,

(ℓ2) $f(V(G)) \subseteq [0, n]$,

(ℓ3) $\bar{f}(E(G)) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,

(ℓ4) $\bar{f}(E(G)) = [1, n]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ consider also

(ℓ5) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,

(ℓ6) there exists an integer λ such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

(ℓ1), (ℓ3) is called a *ρ-labeling*;

(ℓ1), (ℓ4) is called a *σ-labeling*;

(ℓ2), (ℓ4) is called a *β-labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. Suppose G is bipartite. If a ρ, σ or β-labeling of G satisfies condition (ℓ5), then the labeling is *ordered* and is denoted by ρ^+, σ^+ or β^+, respectively. If in addition (ℓ6) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++} or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an *α-labeling* as introduced in [9]. Labelings of the types above are called *Rosa-type labelings* because of Rosa's original article [9] on the topic (see [7] for a recent comprehensive survey of Rosa-type labelings). A dynamic survey on general graph labelings is maintained by Gallian [8].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [9] and [6], respectively.

Theorem 1. *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

Theorem 2. *Let G be a graph with n edges that has a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

Call a connected graph G *Eulerian* if every vertex of G has even degree. If a graph G with Eulerian components admits a σ-labeling, then we have the following well-known restriction on $|E(G)|$.

Theorem 3. (Parity Condition in [9]) *If a graph G with Eulerian components and n edges has a σ-labeling, then $n \equiv 0$ or $3 \pmod 4$.*

We shall call a graph *tripartite* if its chromatic number is at most 3. A non-bipartite graph G is said to be *almost-bipartite* if $G - e$ is bipartite for some $e \in E(G)$. Note that if G is almost-bipartite with $e = \{\hat{b}, c\}$, then G is necessarily tripartite and $V(G)$ can be partitioned into three sets A, B and $C = \{c\}$ such that $\hat{b} \in B$ and e is the only edge joining an element of B to c.

Let G be an almost-bipartite graph with n edges with vertex tripartition A, B, C as above. A labeling h of the vertices of G is called a *γ-labeling* of G if the following conditions hold.

(g1) The function h is a ρ-labeling of G.

(g2) If $\{a, v\}$ is an edge of G with $a \in A$, then $h(a) < h(v)$.

(g3) We have $h(c) - h(\hat{b}) = n$.

It was shown in [3] that if a graph G with n edges admits a γ-labeling, then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x. Several classes of almost-bipartite graphs have been shown to have γ-labelings (see [7]). In particular, it was shown in [3] that every cycle of odd length at least 5 admits a γ-labeling.

Here, we generalize the concept of a γ-labeling and remove the requirement that the graph be almost-bipartite. However, we require the graph to be tripartite. We introduce two new labelings (one of them subsuming γ-labelings) and show that if a tripartite graph G with n edges has one of these labelings, then there exists a cyclic G-decomposition of K_{2nx+1} for every positive integer x. We also show that if G is the union of two cycles of odd length, other than $C_3 \cup C_3$, then G admits one of these labelings.

2 Main Results

We now define two new labelings of tripartite graphs and discuss their cyclic graph decomposition implications.

2.1 σ-tripartite Labelings

Let G be a tripartite graph with n edges having the vertex tripartition $\{A, B, C\}$. A σ-*tripartite* labeling of G is a one-to-one function $h : V(G) \to [0, 2n]$ that satisfies

(s1) h is a σ-labeling of G.

(s2) If $\{a, v\} \in E(G)$ with $a \in A$, then $h(a) < h(v)$.

(s3) If $e = \{b, c\} \in E(G)$ with $b \in B$ and $c \in C$, then there exists an edge $e' = \{b', c'\} \in E(G)$ with $b' \in B$ and $c' \in C$ such that

$$|h(c') - h(b')| + |h(c) - h(b)| = n.$$

(s4) If $a \in A$ and $v \in B \cup C$, then $h(a) - h(v) \neq n$.

(s5) If $b \in B$ and $c \in C$, then $|h(b) - h(c)| \notin \{n, 2n\}$.

Note that e and e' in (s3) need not be distinct. Figure 1 shows a σ-tripartite labeling of a graph G with 8 edges.

We also note that there are tripartite graphs G that satisfy the Parity Condition but do not admit a σ-tripartite labeling. For example, if in each tripartition of $V(G)$ the number of edges between each pair of vertex sets in the tripartition is odd (as in C_3), then G has no σ-tripartite labeling.

Theorem 4. *If a tripartite graph G with n edges has a σ-tripartite labeling, then there exists a cyclic G-decomposition of K_{2nx+1} for each positive integer x.*

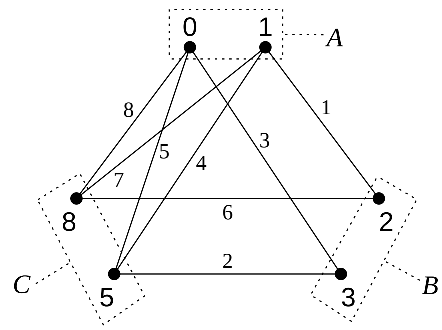

Figure 1: A σ-tripartite labeling of a graph G with 8 edges.

Proof. Let h be a σ-tripartite labeling of G, with sets A, B, and C as in the definition. Since G has no isolated vertices, $h(A) < 2n$. We can assume $x > 1$ by Theorem 1.

For $1 \le j \le x$ define $h_j : V(G) \to [0, 2nx]$ by

$$h_j(v) = \begin{cases} h(v) & v \in A, \\ h(b) + (j-1)n & v \in B, \\ h(c) + (x-j)n & v \in C. \end{cases}$$

We define a multigraph H with vertex set contained in $[0, 2nx]$ and with the nx edges $\{h_j(u), h_j(v)\}$, one for each edge $\{u, v\}$ of G and $1 \le j \le x$. We will show that the set of labels $|h_j(u) - h_j(v)|$ of edges in H is exactly $\{1, 2, \dots, nx\}$, so that actually H is a simple graph, without loops or multiple edges.

Suppose G has an edge $\{s, t\}$ between B and C. For $1 \le j \le x$, let

$$g(j) = (j-1)n - (x-j)n = (2j - x - 1)n,$$

and note that $g(x+1-j) = -g(j)$. Define $k = k(s,t,j)$ to be j or $x+1-j$ according as $s \in B$ or $s \in C$. Then in any case there exists k, $1 \le k \le x$, such that

$$h_k(s) - h_k(t) = h(s) - h(t) + g(j). \tag{1}$$

Now let $1 \le i \le nx$. We will show that H has an edge with label i. The proof will be by cases depending on q and r, where q and r are integers such that

$$i = qn + r, \quad 1 \le r \le n, \quad 0 \le q < x.$$

In the proof we will use vertices $h_j(v)$, where $v \in V(G)$. In all cases it can be checked that j is an integer and $1 \le j \le x$.

By assumption G has an edge $e = \{u, v\}$ such that $h(u) - h(v) = r$.

Case 1: $v \in A$.

If $u \in B$ then take $j = q + 1$. Then

$$h_j(u) - h_j(v) = h(u) + (j-1)n - h(v) = qn + r = i.$$

If $u \in C$ then take $j = x - q$. Then

$$h_j(u) - h_j(v) = h(u) + (x-j)n - h(v) = qn + r = i.$$

Case 2: $\{u, v\} \subseteq B \cup C$ and $q \equiv x + 1 \pmod 2$.

Take $j = (q + x + 1)/2$. Then for some k

$$h_k(u) - h_k(v) = h(u) - h(v) + g(j) = r + (2j - x - 1)n = r + qn = i.$$

Case 3: $\{u, v\} \subseteq B \cup C$ and $q \equiv x \pmod 2$.

By condition (s3) then G has an edge $\{u', v'\} \subseteq B \cup C$ such that $h(u') - h(v') = n - r$. Take $j = (x - q)/2$. Then

$$|h_k(u') - h_k(v')| = |h(u') - h(v') + g(j)| = |n - r + (2j - x - 1)n| = |-r - nq| = i.$$

Thus H is a simple graph.

Now we claim that h_j is one-to-one on $V(G)$ for $1 \leq j \leq x$. It is clear that h_j is one-to-one on each set A, B, and C. First we show that h_j is one-to-one on $A \cup B$. Suppose $h_j(a) = h_j(b)$, $a \in A$, $b \in B$. Then $h(a) = h(b) + (j - 1)n$. Clearly $j > 1$ since h is one-to-one. If $j > 2$, then $h(a) < 2n \leq h(b) + 2n \leq h_j(b)$, so we must have $j = 2$, and this case is excluded by condition (s4).

A similar proof shows that h_j is one-to-one on $A \cup C$.

Finally we show h_j is one-to-one on $B \cup C$. Suppose $h_j(b) = h_j(c)$, $b \in B$, $c \in C$. Then $h(c) - h(b) = (2j - x - 1)n$. Clearly $|2j - x - 1|$ must be 1 or 2. This contradicts condition (s5).

Thus for fixed j, $1 \leq j \leq x$, the edges $\{h_j(u), h_j(v)\}$ as $\{u, v\}$ runs through $E(G)$ form an isomorphic copy of G. Thus G divides H. Furthermore, taking each vertex of H as its label gives a σ-labeling of H, producing a cyclic H-decomposition of K_{2nx+1}, and thus a cyclic G-decomposition of K_{2nx+1}. □

Figure 2 shows the three copies of the graph G (from Figure 1) that can be used to produce a cyclic G-decomposition of K_{49}.

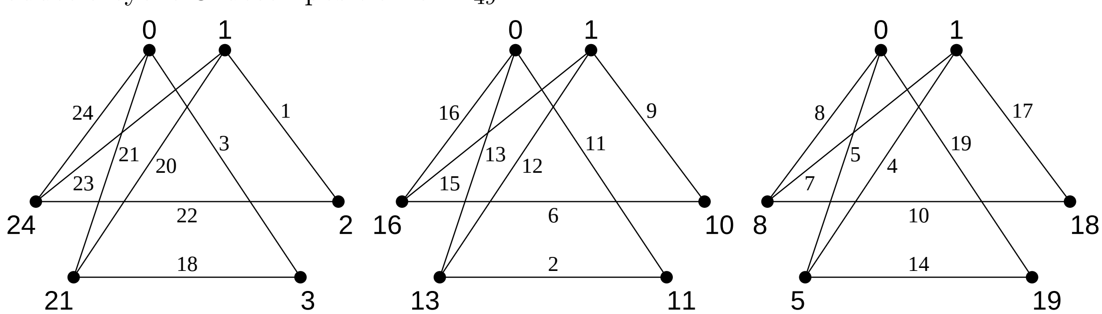

Figure 2: The three copies of G (from Figure 1) in a cyclic G-decomposition of K_{49}.

It is easy to see that if a graph G with n edges admits a σ-labeling, then there exists a cyclic G-decomposition of $K_{2n+2} - I$, where I is a 1-factor in K_{2n+2}. (The edges of length $n+1$ constitute the 1-factor in K_{2n+2}.) Since the graph H in the proof of Theorem 4 admits a σ-labeling, we have the following corollary.

Corollary 5. *Let G be a graph with n edges that admits a σ-tripartite labeling and let x be a positive integer. Then there exists a cyclic G-decomposition of $K_{2nx+2} - I$, where I is a 1-factor of K_{2nx+2}.*

2.2 ρ-tripartite Labelings

Let G be a tripartite graph with n edges having the vertex tripartition $\{A, B, C\}$. A ρ-*tripartite* labeling of G is a one-to-one function $h : V(G) \to [0, 2n]$ that satisfies (r1) h is a ρ-labeling of G.

(r2) If $\{a, v\} \in E(G)$ with $a \in A$, then $h(a) < h(v)$.

(r3) If $e = \{b, c\} \in E(G)$ with $b \in B$ and $c \in C$, then there exists an edge $e' = \{b', c'\} \in E(G)$ with $b' \in B$ and $c' \in C$ such that

$$|h(c') - h(b')| + |h(c) - h(b)| = 2n.$$

(r4) If $b \in B$ and $c \in C$, then $|h(b) - h(c)| \neq 2n$.

Note that e and e' in (r3) need not be distinct. Figure 3 shows a ρ-tripartite labeling of the Petersen graph P.

We note that although a σ-labeling is necessarily a ρ-labeling, a σ-tripartite labeling is not a ρ-tripartite labeling. We also note that a γ-labeling is necessarily a ρ-tripartite labeling.

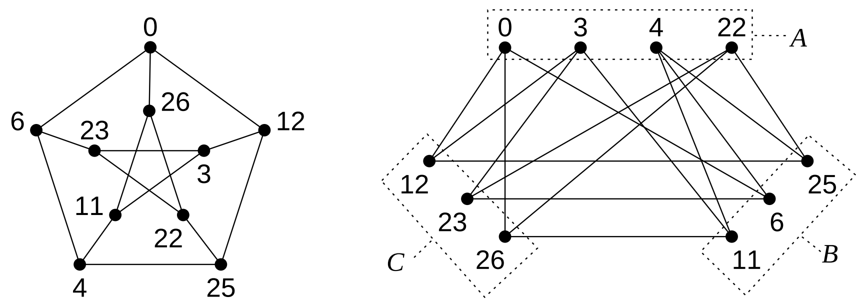

Figure 3: A ρ-tripartite labeling of the Petersen graph P.

Theorem 6. *If a tripartite graph G with n edges has a ρ-tripartite labeling, then there exists a cyclic G-decomposition of K_{2nx+1} for each positive integer x.*

Proof. Let h be a ρ-tripartite labeling of G, with sets A, B, and C as in the definition. Since G has no isolated vertices, $h(A) < 2n$. We can assume $x > 1$ by Theorem 1.

Let $B_1, B_2, \ldots, B_x$ and $C_1, C_2, \ldots, C_x$ be x vertex-disjoint copies of B and C, respectively. The vertices in B_i and C_i corresponding to $b \in B$ and $c \in C$ will be denoted b_i and c_i, respectively. Let $\tilde{B} = \bigcup_{i=1}^{x} B_i$ and $\tilde{C} = \bigcup_{i=1}^{x} C_i$. We define a new graph $\tilde{G}$ with vertex set $A \cup \tilde{B} \cup \tilde{C}$ with edges $\{a, v_i\}, 1 \leq i \leq x$, whenever $a \in A$ and $\{a, v\}$ is an edge of G, and edges $\{b_i, c_i\}, 1 \leq i \leq x$, whenever $b \in B$, $c \in C$, and $\{b, c\}$ is an edge of G. Clearly $\tilde{G}$ has nx edges and G divides $\tilde{G}$.

The plan of the proof is to show that $\tilde{G}$ has a ρ-labeling, so that $\tilde{G}$ divides K_{2nx+1} cyclicly, and so does G. We define a labeling $\tilde{h}$ on $\tilde{G}$ by

$$\tilde{h}(v) = \begin{cases} h(v) & v \in A, \\ h(b) + (i-1)2n & v = b_i \in B_i, \\ h(c) + (x-i)2n & v = c_i \in C_i. \end{cases}$$

Note that $\tilde{h}(A) = h(A) \subseteq [0, 2n)$, while

$$\tilde{h}(B_i \cup C_{x+1-i}) = h(B \cup C) + (i-1)2n \subseteq [0, 2n] + (i-1)2n.$$

The sets on the right do not intersect by condition (r4). Thus $\tilde{h}$ is one-to-one from $V(\tilde{G})$ to $[0, 2nx]$.

Suppose G has an edge $\{s,t\}$ between B and C. For $1 \leq j \leq x$, let

$$f(j) = (j-1)2n - (x-j)2n = (2j - x - 1)2n,$$

and note that $f(x+1-j) = -f(j)$. Define $k = k(s,t,j)$ to be j or $x+1-j$ according as $s \in B$ or $s \in C$. Then in any case there exists k, $1 \leq k \leq x$, such that

$$\tilde{h}(s_k) - \tilde{h}(t_k) = h(s) - h(t) + f(j). \tag{2}$$

Now let $1 \leq i \leq nx$. We will show that $\tilde{G}$ has an edge with label either i or $2nx+1-i$. The proof will be by cases depending on q and r, where q and r are integers such that

$$i = qn + r, \quad 1 \leq r \leq n, \quad 0 \leq q < x.$$

In the proof we will use vertices v_j, where $v \in B \cup C$. In all cases it can be checked that j is an integer and $1 \leq j \leq x$.

Case 1: q is even.

Note that since $1 \leq r \leq n$, G has an edge e with label r or $2n+1-r$.

Case 1a: e has label r.

If $e = \{a, v\}$, $a \in A$ and $v \in B \cup C$, then note that if $v \in B$, then

$$\tilde{h}(v_{1+q/2}) - \tilde{h}(a) = r + (q/2)2n = i,$$

while if $v \in C$, then

$$\tilde{h}(v_{x-q/2}) - \tilde{h}(a) = r + (x - (x - q/2))2n = i.$$

There remains the case $e = \{s,t\}$ with $s,t \in B \cup C$ and $h(s) - h(t) = r$. Let e' be as in the definition of a ρ-tripartite labeling. Then $e' = \{s', t'\} \subseteq B \cup C$ and $|h(s) - h(t)| + |h(s') - h(t')| = 2n$. First assume $q/2 \equiv x+1 \pmod 2$. Set $j = (q/2 + x + 1)/2$. Then for some k

$$|\tilde{h}(s_k) - \tilde{h}(t_k)| = |h(s) - h(t) + f(j)| = |i| = i.$$

If $q/2 \equiv x \pmod 2$, then we must have $|h(s') - h(t')| = 2n - r$, and we can assume that $h(s') - h(t') = 2n - r$. Set $j = (x - q/2)/2$. Then for some k

$$|\tilde{h}(s'_k) - \tilde{h}(t'_k)| = |h(s') - h(t') + f(j)| = |-qn - r| = i.$$

Case 1b: e has label $2n+1-r$.

First assume $e = \{a, v\}$ for $a \in A$. Take $j = x - q/2$. Then we compute that $\tilde{h}(v_j) - \tilde{h}(a) = 2nx + 1 - i$ for $v \in B$, while $\tilde{h}(v_{x+1-j}) - \tilde{h}(a) = 2nx + 1 - i$ for $v \in C$.

Now assume that G has edges $e = \{s,t\}$ and $e' = \{s', t'\}$ with $s, t, s', t' \in B \cup C$ such that $h(s) - h(t) = 2n+1-r$ and $h(s') - h(t') = 2n - (2n+1-r) = r - 1$. If $q \equiv 0 \pmod 4$, set $j = x - q/4$. Then for some k

$$\tilde{h}(s_k) - \tilde{h}(t_k) = 2n + 1 - r + f(j) = 2nx + 1 - i.$$

On the other hand, if $q \equiv 2 \pmod 4$, set $j = (q+2)/4$. Then for some k

$$|\tilde{h}(s'_k) - \tilde{h}(t'_k)| = |r - 1 + f(j)| = |i - 1 - 2nx| = 2nx + 1 - i.$$

Case 2: q is odd.

Thus $q \geq 1$. Note that $1 \leq n+1-r \leq n$. Thus G has an edge e with label either $n+1-r$ or $2n+1-(n+1-r) = n+r$.

Case 2a: e has label $n+1-r$.

First suppose $e = \{a, v\}$ with $a \in A$. If $v \in B$ we take $j = x-(q-1)/2$ and find that

$$\tilde{h}(v_j) - \tilde{h}(a) = n+1-r+(j-1)2n = 2nx+1-i,$$

while if $v \in C$ we take $j = (q+1)/2$ and find

$$\tilde{h}(v_j) - \tilde{h}(a_j) = n+1-r+(x-j)2n = 2nx+1-i.$$

Otherwise we can assume G contains edges $e = \{s,t\}$ and $e' = \{s',t'\}$ between B and C such that $h(s)-h(t) = n+1-r$ and $h(s')-h(t') = 2n-(n+1-r) = n+r-1$. If $q \equiv 1 \pmod 4$ take $j = x-(q-1)/4$. Then for some k

$$\tilde{h}(s_k) - \tilde{h}(t_k) = n+1-r+f(j) = 2nx+1-i.$$

If $q \equiv 3 \pmod 4$ take $j = (q+1)/4$. Then for some k

$$|\tilde{h}(s'_k) - \tilde{h}(t'_k) = |n+r-1+f(j)| = |i-(2nx+1)| = 2nx+1-i.$$

Case 2b: e has label $n+r$.

First suppose $e = \{a, v\}$ with $a \in A$. If $v \in B$ we take $j = (q+1)/2$, and find that

$$\tilde{h}(v_j) - \tilde{h}(a) = n+r+(j-1)2n = i,$$

while if $v \in C$ we take $j = x+(1-q)/2$, making

$$\tilde{h}(v_j) - \tilde{h}(a) = n+r+(x-j)2n = i.$$

Otherwise we can assume G contains edges $e = \{s,t\}$ and $e' = \{s',t'\}$ between B and C such that $h(s)-h(t) = n+r$ and $h(s')-h(t') = 2n-(n+r) = n-r$. If $(q-1)/2 \equiv x \pmod 2$ take $j = (2x+1-q)/4$. Then for some k

$$|\tilde{h}(s'_k) - \tilde{h}(t'_k)| = |n-r+f(j)| = |-i| = i.$$

If $(q-1)/2 \equiv x+1 \pmod 2$ take $j = (2x+1+q)/4$. Then for some k

$$\tilde{h}(s_k) - \tilde{h}(t_k) = n+r+f(j) = i.$$

This concludes the proof. □

Figure 4 shows the three copies of the Petersen graph P (from Figure 3) that can be used to produce a cyclic P-decomposition of K_{91}.

We note that there are tripartite graphs with n edges that cyclically decompose K_{2nx+1} for every positive integer x, but do not admit a ρ-tripartite labeling. The complete graph K_3 is one such graph.

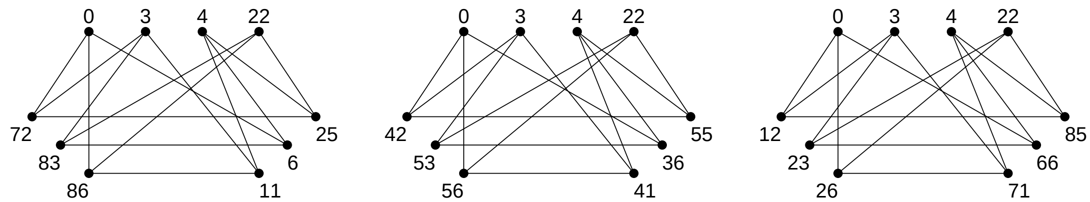

Figure 4: The three copies of P in a cyclic P-decomposition of K_{91}.

3 Labeling the Union of Two Odd Cycles

In this section, we show that the graph consisting of the union of two cycles of odd length (other than $C_3 \cup C_3$) admits a σ-tripartite labeling if the parity condition is satisfied, and a ρ-tripartite labeling, otherwise. We focus on labelings of 2-regular graphs because of the strong interest in cycle related designs. For comprehensive recent surveys on graph designs in general and on cycle designs in particular, we direct the reader to [1] and [4], respectively. We note that if G is a 2-regular bipartite graph, then G admits a σ^+-labeling if the Parity Condition is satisfied (see [2]) and a ρ^+-labeling otherwise (see [6]).

We will henceforth consider graphs whose vertices are (distinct) nonnegative integers. Each vertex will be its own label, so the label of the edge $\{x, y\}$ in such a graph will be simply $|x - y|$.

We denote the directed path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \le i \le k-1$, by $(x_0, x_1, \ldots, x_k)$. The *first vertex* of this path is x_0, the *second vertex* is x_1, and the *last vertex* is x_k. If $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$ are directed paths with $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$.

Let $P(k)$ be the path with k edges and $k+1$ vertices $0, 1, \ldots, k$ given by $(0, k, 1, k-1, 2, k-2, \ldots, \lceil k/2 \rceil)$. Note that the set of vertices of this graph is $A \cup B$, where $A = [0, \lfloor k/2 \rfloor]$, $B = [\lfloor k/2 \rfloor + 1, k]$, and every edge joins a vertex of A to one of B. Furthermore the set of labels of the edges of $P(k)$ is $[1, k]$.

Now let a and b be nonnegative integers with $a \le b$ and let us add a to all the vertices of A and b to all the vertices of B. We will denote the resulting graph by $P(a, b, k)$. Note that this graph has the following properties.

P1 $P(a, b, k)$ is a path with first vertex a and second vertex $b + k$. If k is even, its last vertex is $a + k/2$.

P2 Each edge of $P(a, b, k)$ joins a vertex of $A' = [a, \lfloor k/2 \rfloor + a]$ to a larger vertex of $B' = [\lfloor k/2 \rfloor + 1 + b, k + b]$.

P3 The set of edge labels of $P(a, b, k)$ is $[b - a + 1, b - a + k]$.

Figure 5 shows $P(6)$ and $P(4, 7, 6)$.

Theorem 7. *Suppose C_r and C_s are odd cycles with $r \not\equiv s \pmod 4$. Then $G = C_r \cup C_s$ has a σ-tripartite labeling.*

Proof. We assume $r = 4x + 1$ and $s = 4y + 3$, where $x > 0$ and $y \ge 0$.

Case 1: $y < \frac{x-1}{2}$.
Let $C_{4x+1} = G_1 + G_2 + G_3 + G_4 + (2x - 1, b_1, c_1, 0)$ and $C_{4y+3} = G_5 + (4x + 6y + 5, b_2, c_2, 4x +$

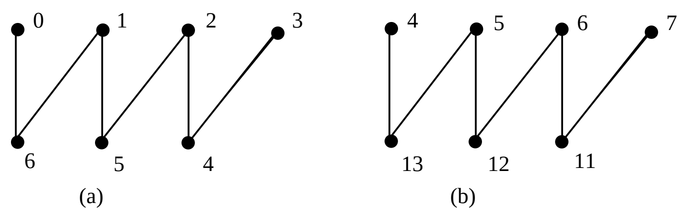

Figure 5: (a) $P(6)$, (b) $P(4,7,6)$.

$4y+5)$ where $b_1 = 2x+1$, $c_1 = 4x+4y+4$, $b_2 = 4x+6y+6$, $c_2 = 6x+6y+7$, and

$$\begin{aligned}
G_1 &= P(0, 2x+4y+3, 2x),\\
G_2 &= P(x, 3x+2y+2, 2y),\\
G_3 &= P(x+y, 3x+y+1, 2y),\\
G_4 &= P(x+2y, x+6y+2, 2x-4y-2),\\
G_5 &= P(4x+4y+5, 4x+4y+7, 4y).
\end{aligned}$$

(Note: In the case when $y = 0$, the paths G_2, G_3, and G_5 are empty. However this does not change the proof in any way.)

First, we show that $G_1 + G_2 + G_3 + G_4 + (2x-1, b_1, c_1, 0)$ is a cycle of length $4x+1$ and $G_5 + (4x+6y+5, b_2, c_2, 4x+4y+5)$ is a cycle of length $4y+3$. Note that by **P1**, the first vertex of G_1 is 0 and the last is x, the first vertex of G_2 is x and the last is $x+y$, the first vertex of G_3 is $x+y$ and the last is $x+2y$, the first vertex of G_4 is $x+2y$ and the last is $2x-1$, and the first vertex of G_5 is $4x+4y+5$ and the last is $4x+6y+5$. For $1 \le i \le 5$, let A_i and B_i denote the sets labeled A' and B' in **P2**, corresponding to the path G_i. Then using **P2**, we compute

$$\begin{aligned}
A_1 &= [0, x], & B_1 &= [3x+4y+4, 4x+4y+3],\\
A_2 &= [x, x+y], & B_2 &= [3x+3y+3, 3x+4y+2],\\
A_3 &= [x+y, x+2y], & B_3 &= [3x+2y+2, 3x+3y+1],\\
A_4 &= [x+2y, 2x-1], & B_4 &= [2x+4y+2, 3x+2y],\\
A_5 &= [4x+4y+5, 4x+6y+5], & B_5 &= [4x+6y+8, 4x+8y+7].
\end{aligned}$$

Thus,

$$A_1 \le A_2 \le A_3 \le A_4 < b_1 < B_4 < B_3 < B_2 < B_1 < c_1 < A_5 < b_2 < B_5 < c_2, \tag{3}$$

where the last inequality follows from the condition $y < \frac{x-1}{2}$. Note that $V(G_1) \cap V(G_2) = \{x\}$, $V(G_2) \cap V(G_3) = \{x+y\}$, and $V(G_3) \cap V(G_4) = \{x+2y\}$; otherwise, G_i and G_j are vertex-disjoint for $i \ne j$. Therefore, $G_1 + G_2 + G_3 + G_4 + (2x-1, b_1, c_1, 0)$ is a cycle of length $4x+1$ and $G_5 + (4x+6y+5, b_2, c_2, 4x+4y+5)$ is a cycle of length $4y+3$.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 5$. By **P3**, we have edge labels

$$\begin{aligned}
E_1 &= [2x+4y+4, 4x+4y+3],\\
E_2 &= [2x+2y+3, 2x+4y+2],\\
E_3 &= [2x+2, 2x+2y+1],\\
E_4 &= [4y+3, 2x],\\
E_5 &= [3, 4y+2].
\end{aligned}$$

Moreover, the path $(2x-1, b_1, c_1, 0)$ consists of edges with labels 2, $2x+4y+3$, and $4x+4y+4$, and the path $(4x+6y+5, b_2, c_2, 4x+4y+5)$ consists of edges with labels 1, $2x+1$, and $2x+2y+2$. Thus the edge set of G has one edge of each label i where $1 \le i \le 4x+4y+4$. Hence the defined labeling is a σ-labeling, and condition (s1) for a σ-tripartite labeling is satisfied.

Case 1a: $y = 0$.
Now, let $A = \bigcup_{i=1}^{5} A_i$, $B = \{b_1, b_2\}$, and $C = B_1 \cup B_4 \cup \{c_1, c_2\}$. (Recall that when $y = 0$, $B_2 = B_3 = B_5 = \emptyset$.) Thus, $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (s2) of a σ-tripartite labeling is clear from (3) since all vertices in the C_{4x+1} do not exceed c_1, while all vertices in the $C_{4y+3} = C_3$ do. Note that $|b_1-c_1|+|b_2-c_2| = (2x+3)+(2x+1) = 4x+4 = n$, the number of edges of G. Thus condition (s3) is satisfied. Also $a = v + n$, where $a \in A$ and $v \in B \cup C$, is impossible, since by (3) and the assumption $y = 0$ we have

$$v + n \ge b_1 + n = 6x + 5 > 4x + 5 = \max A.$$

Thus condition (s4) holds.

Finally, suppose $b \in B$ and $c \in C$. The equation $|b - c| = 2n$ is impossible since all vertices are in $[0, 2n]$ and $0 \in A$. If $c - b_1 = n$, then $c = 6x + 5$. This contradicts (3), since B_5 is empty and $B_4 < B_1 < c_1 < 6x + 5 < c_2$. Also if $b_2 - c = n$, then $c = 2$. This contradicts (3), since $2 \le B_4 \le C$. Thus condition (s5) holds, and we have a σ-tripartite labeling for $C_{4x+1} \cup C_3$.

Case 1b: $y > 0$.
Now, let $A = \bigcup_{i=1}^{5} A_i$, $B = \bigcup_{i=1}^{5} B_i \cup \{b_1, b_2\}$, and $C = \{c_1, c_2\}$. Thus, $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (s2) of a σ-tripartite labeling is clear from (3) since all vertices in the C_{4x+1} do not exceed c_1, while all vertices in the C_{4y+3} do. Note that $|b_1 - c_1| + |b_2 - c_2| = (2x + 4y + 3) + (2x + 1) = 4x + 4y + 4 = n$, the number of edges of G. Thus condition (s3) is satisfied. Also $a = v + n$, where $a \in A$ and $v \in B \cup C$, is impossible, since by (3) and the assumption $x > 2y + 1$ we have

$$v+n \ge b_1+n = 6x+4y+5 > 4x+2(2y+1)+4y+5 = 4x+8y+7 > 4x+6y+5 = \max A.$$

Thus condition (s4) holds.

Finally, suppose $b \in B$ and $c \in C$. The equation $|b - c| = 2n$ is impossible since all vertices are in $[0, 2n]$ and $0 \in A$. Likewise $|b - c_1| = n$ is impossible since $c_1 = n$ and $0 < B < 2n$. The case remains that $c_2 - b = n$, which gives $b = 2x + 2y + 3$. This contradicts (3), since $2x + 2y + 3$ exceeds b_1 but is less than everything in B_4. Thus condition (s5) holds, and we have a σ-tripartite labeling for $C_{4x+1} \cup C_{4y+3}$.

Case 2: $\frac{x-1}{2} \le y \le x - 1$.
Let $C_{4x+1} = G_1+G_2+G_3+(2x-1, b_1, c_1, 0)$ and $C_{4y+3} = G_4+G_5+(4x+6y+5, b_2, c_2, 4x+4y+5)$ where $b_1 = 2x + 1$, $c_1 = 4x + 4y + 4$, $b_2 = 4x + 6y + 6$, $c_2 = 6x + 6y + 7$, and

$$\begin{aligned}
G_1 &= P(0, 2x + 4y + 3, 2x),\\
G_2 &= P(x, 3x + 2y + 2, 2y),\\
G_3 &= P(x + y, x + 5y + 3, 2x - 2y - 2),\\
G_4 &= P(4x + 4y + 5, 6x + 4y + 6, -2x + 4y + 2),\\
G_5 &= P(3x + 6y + 6, 3x + 6y + 8, 2x - 2).
\end{aligned}$$

(Note: In the case when $x = y+1$, the path G_3 is empty; when $x = 2y+1$, the path G_4 is empty; and when $y = 0$, the paths G_2 and G_5 are empty. However this does not change the proof in any way.)

First, we show that $G_1 + G_2 + G_3 + (2x-1, b_1, c_1, 0)$ is a cycle of length $4x+1$ and $G_4 + G_5 + (4x+6y+5, b_2, c_2, 4x+4y+5)$ is a cycle of length $4y+3$. Note that by **P1**, the first vertex of G_1 is 0 and the last is x, the first vertex of G_2 is x and the last is $x+y$, the first vertex of G_3 is $x+y$ and the last is $2x-1$, the first vertex of G_4 is $4x+4y+5$ and the last is $3x+6y+6$, and the first vertex of G_5 is $3x+6y+6$ and the last is $4x+6y+5$. For $1 \le i \le 5$, let A_i and B_i denote the sets labeled A' and B' in **P2**, corresponding to the path G_i. Then using **P2**, we compute

$$
\begin{aligned}
A_1 &= [0, x], & B_1 &= [3x+4y+4, 4x+4y+3],\\
A_2 &= [x, x+y], & B_2 &= [3x+3y+3, 3x+4y+2],\\
A_3 &= [x+y, 2x-1], & B_3 &= [2x+4y+3, 3x+3y+1],\\
A_4 &= [4x+4y+5, 3x+6y+6], & B_4 &= [5x+6y+8, 4x+8y+8],\\
A_5 &= [3x+6y+6, 4x+6y+5], & B_5 &= [4x+6y+8, 5x+6y+6].
\end{aligned}
$$

Thus,

$$A_1 \le A_2 \le A_3 < b_1 < B_3 < B_2 < B_1 < c_1 < A_4 \le A_5 < b_2 < B_5 < B_4 < c_2, \qquad (4)$$

where the last inequality follows from the condition $y \le x-1$. Note that $V(G_1) \cap V(G_2) = \{x\}$, $V(G_2) \cap V(G_3) = \{x+y\}$, and $V(G_4) \cap V(G_5) = \{3x+6y+6\}$; otherwise, G_i and G_j are vertex-disjoint for $i \ne j$. Therefore, $G_1 + G_2 + G_3 + (2x-1, b_1, c_1, 0)$ is a cycle of length $4x+1$ and $G_4 + G_5 + (4x+6y+5, b_2, c_2, 4x+4y+5)$ is a cycle of length $4y+3$.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 5$. By **P3**, we have edge labels

$$
\begin{aligned}
E_1 &= [2x+4y+4, 4x+4y+3],\\
E_2 &= [2x+2y+3, 2x+4y+2],\\
E_3 &= [4y+4, 2x+2y+1],\\
E_4 &= [2x+2, 4y+3],\\
E_5 &= [3, 2x].
\end{aligned}
$$

Moreover, the path $(2x-1, b_1, c_1, 0)$ consists of edges with labels 2, $2x+4y+3$, and $4x+4y+4$, and the path $(4x+6y+5, b_2, c_2, 4x+4y+5)$ consists of edges with labels 1, $2x+1$, and $2x+2y+2$. Thus the edge set of G has one edge of each label i where $1 \le i \le 4x+4y+4$. Hence the defined labeling is a σ-labeling, and condition (s1) for a σ-tripartite labeling is satisfied.

Now, let $A = \bigcup_{i=1}^{5} A_i$, $B = \bigcup_{i=1}^{5} B_i \cup \{b_1, b_2\}$, and $C = \{c_1, c_2\}$. Then $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (s2) of a σ-tripartite labeling is clear from (4) since all vertices in the C_{4x+1} do not exceed c_1, while all vertices in the C_{4y+3} do. Note that $|b_1 - c_1| + |b_2 - c_2| = (2x+4y+3) + (2x+1) = 4x+4y+4 = n$, the number of edges of G. Thus condition (s3) is satisfied. Also $a = v+n$, where $a \in A$ and $v \in B \cup C$, is impossible, since by (4) and the assumption $x \ge y+1$ we have

$$v + n \ge b_1 + n = 6x+4y+5 \ge 4x + 2(y+1) + 4y + 5 = 4x+6y+7 > 4x+6y+5 = \max A.$$

Thus condition (s4) holds.

Finally, suppose $b \in B$ and $c \in C$. The equation $|b - c| = 2n$ is impossible since all vertices are in $[0, 2n]$ and $0 \in A$. Likewise $|b - c_1| = n$ is impossible since $c_1 = n$ and $0 < B < 2n$. The case remains that $c_2 - b = n$, which gives $b = 2x + 2y + 3$. This contradicts (4), since $2x + 2y + 3$ exceeds b_1 but is less than everything in B_3 unless $y = 0$. In the latter case $x \leq 2y + 1$ implies $x = 1$ and it can be checked directly that this construction gives a σ-tripartite labeling for $C_5 \cup C_3$. Thus condition (s5) holds, and we have a σ-tripartite labeling for $C_{4x+1} \cup C_{4y+3}$.

Case 3: $x - 1 < y \leq 2x$.
Let $C_{4x+1} = G_1 + G_2 + (6x+4y+4, b_1, c_1, 4x+4y+5)$ and $C_{4y+3} = G_3 + G_4 + G_5 + (2y, b_2, c_2, 0)$ where $b_1 = 6x + 4y + 6$, $c_1 = 6x + 6y + 7$, $b_2 = 2y + 1$, $c_2 = 4x + 4y + 4$, and

$$\begin{aligned}
G_1 &= P(4x + 4y + 5, 4x + 6y + 6, 4x - 2y), \\
G_2 &= P(6x + 3y + 5, 6x + 3y + 7, 2y - 2), \\
G_3 &= P(0, 4x + 2y + 3, 2y), \\
G_4 &= P(y, 2x + 3y + 2, 2x), \\
G_5 &= P(x + y, 5x + y + 1, 2y - 2x).
\end{aligned}$$

(Note: In the case when $y = 1$, the path G_2 is empty; when $y = x$, the path G_5 is empty; and when $y = 2x$, the path G_1 is empty. However this does not change the proof in any way.)

First, we show that $G_1 + G_2 + (6x + 4y + 4, b_1, c_1, 4x + 4y + 5)$ is a cycle of length $4x + 1$ and $G_3 + G_4 + G_5 + (2y, b_2, c_2, 0)$ is a cycle of length $4y + 3$. Note that by **P1**, the first vertex of G_1 is $4x + 4y + 5$ and the last is $6x + 3y + 5$, the first vertex of G_2 is $6x + 3y + 5$ and the last is $6x + 4y + 4$, the first vertex of G_3 is 0 and the last is y, the first vertex of G_4 is y and the last is $x + y$, and the first vertex of G_5 is $x + y$ and the last is $2y$. For $1 \leq i \leq 5$, let A_i and B_i denote the sets labeled A' and B' in **P2**, corresponding to the path G_i. Then using **P2**, we compute

$$\begin{aligned}
A_1 &= [4x + 4y + 5, 6x + 3y + 5], & B_1 &= [6x + 5y + 7, 8x + 4y + 6], \\
A_2 &= [6x + 3y + 5, 6x + 4y + 4], & B_2 &= [6x + 4y + 7, 6x + 5y + 5], \\
A_3 &= [0, y], & B_3 &= [4x + 3y + 4, 4x + 4y + 3], \\
A_4 &= [y, x + y], & B_4 &= [3x + 3y + 3, 4x + 3y + 2], \\
A_5 &= [x + y, 2y], & B_5 &= [4x + 2y + 2, 3x + 3y + 1].
\end{aligned}$$

Thus,

$$A_3 \leq A_4 \leq A_5 < b_2 < B_5 < B_4 < B_3 < c_2 < A_1 \leq A_2 < b_1 < B_2 < B_1 < c_1, \tag{5}$$

where the last inequality follows from the condition $x \leq y$. Note that $V(G_1) \cap V(G_2) = \{6x + 3y + 5\}$, $V(G_3) \cap V(G_4) = \{y\}$, and $V(G_4) \cap V(G_5) = \{x + y\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1 + G_2 + (6x + 4y + 4, b_1, c_1, 4x + 4y + 5)$ is a cycle of length $4x + 1$ and $G_3 + G_4 + G_5 + (2y, b_2, c_2, 0)$ is a cycle of length $4y + 3$.

Next, let E_i denote the set of edge labels in G_i for $1 \leq i \leq 5$. By **P3**, we have edge labels

$$\begin{aligned}
E_1 &= [2y+2, 4x+1],\\
E_2 &= [3, 2y],\\
E_3 &= [4x+2y+4, 4x+4y+3],\\
E_4 &= [2x+2y+3, 4x+2y+2],\\
E_5 &= [4x+2, 2x+2y+1].
\end{aligned}$$

Moreover, the path $(6x+4y+4, b_1, c_1, 4x+4y+5)$ consists of edges with labels 2, $2y+1$, and $2x+2y+2$, and the path $(2y, b_2, c_2, 0)$ consists of edges with labels 1, $4x+2y+3$, and $4x+4y+4$. Thus the edge set of G has one edge of each label i where $1 \leq i \leq 4x+4y+4$. Hence the defined labeling is a σ-labeling, and condition (s1) for a σ-tripartite labeling is satisfied.

Now, let $A = \bigcup_{i=1}^{5} A_i$, $B = \bigcup_{i=1}^{5} B_i \cup \{b_1, b_2\}$, and $C = \{c_1, c_2\}$. Then $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (s2) of a σ-tripartite labeling is clear from (5) since all vertices in the C_{4y+3} do not exceed c_2, while all vertices in the C_{4x+1} do. Note that $|b_1 - c_1| + |b_2 - c_2| = (2y+1) + (4x+2y+3) = 4x+4y+4 = n$, the number of edges of G. Thus condition (s3) is satisfied. Also $a = v + n$, where $a \in A$ and $v \in B \cup C$, is impossible, since by (5) and the assumption $x - 1 < y$ we have

$$v + n \geq b_2 + n = 4x+6y+5 \geq 4x+4y+2(x)+5 = 6x+4y+5 > 6x+4y+4 = \max A.$$

Thus condition (s4) holds.

Finally, suppose $b \in B$ and $c \in C$. The equation $|b - c| = 2n$ is impossible since all vertices are in $[0, 2n]$ and $0 \in A$. Likewise $|b - c_2| = n$ is impossible since $c_2 = n$ and $0 < B < 2n$. The case remains that $c_1 - b = n$, which gives $b = 2x+2y+3$. This contradicts (5), since $2x+2y+3$ exceeds b_2 but is less than everything in B_5. Thus condition (s5) holds, and we have a σ-tripartite labeling for $C_{4x+1} \cup C_{4y+3}$.

Case 4: $2x < y$.
Let $C_{4x+1} = G_1 + (6x+4y+4, b_1, c_1, 4x+4y+5)$ and $C_{4y+3} = G_2+G_3+G_4+G_5+(2y, b_2, c_2, 0)$ where $b_1 = 6x+4y+6$, $c_1 = 6x+6y+7$, $b_2 = 2y+1$, $c_2 = 4x+4y+4$,

$$\begin{aligned}
G_1 &= P(4x+4y+5, 4x+4y+7, 4x-2),\\
G_2 &= P(0, 4x+2y+3, 2y),\\
G_3 &= P(y, 2x+3y+2, 2x),\\
G_4 &= P(x+y, x+3y+1, 2x),\\
G_5 &= P(2x+y, 6x+y, 2y-4x).
\end{aligned}$$

First, we show that $G_1 + (6x+4y+4, b_1, c_1, 4x+4y+5)$ is a cycle of length $4x+1$ and $G_2+G_3+G_4+G_5+(2y, b_2, c_2, 0)$ is a cycle of length $4y+3$. Note that by **P1**, the first vertex of G_1 is $4x+4y+5$ and the last is $6x+4y+4$, the first vertex of G_2 is 0 and the last is y, the first vertex of G_3 is y and the last is $x+y$, the first vertex of G_4 is $x+y$ and the last is $2x+y$, and the first vertex of G_5 is $2x+y$ and the last is $2y$. For $1 \leq i \leq 5$, let A_i and B_i denote the sets labeled A' and B' in **P2**, corresponding to the path G_i. Then using **P2**, we compute

$$\begin{aligned}
A_1 &= [4x+4y+5, 6x+4y+4], & B_1 &= [6x+4y+7, 8x+4y+5],\\
A_2 &= [0, y], & B_2 &= [4x+3y+4, 4x+4y+3],\\
A_3 &= [y, x+y], & B_3 &= [3x+3y+3, 4x+3y+2],\\
A_4 &= [x+y, 2x+y], & B_4 &= [2x+3y+2, 3x+3y+1],\\
A_5 &= [2x+y, 2y], & B_5 &= [4x+2y+1, 2x+3y].
\end{aligned}$$

Thus,

$$A_2 \le A_3 \le A_4 \le A_5 < b_2 < B_5 < B_4 < B_3 < B_2 < c_2 < A_1 < b_1 < B_1 < c_1, \tag{6}$$

where the last inequality follows from the condition $2x < y$. Note that $V(G_2) \cap V(G_3) = \{y\}$, $V(G_3) \cap V(G_4) = \{x+y\}$, and $V(G_4) \cap V(G_5) = \{2x+y\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1 + (6x+4y+4, b_1, c_2, 4x+4y+5)$ is a cycle of length $4x+1$ and $G_2 + G_3 + G_4 + G_5 + (2y, b_2, c_2, 0)$ is a cycle of length $4y+3$.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 5$. By **P3**, we have edge labels

$$\begin{aligned}
E_1 &= [3, 4x],\\
E_2 &= [4x+2y+4, 4x+4y+3],\\
E_3 &= [2x+2y+3, 4x+2y+2],\\
E_4 &= [2y+2, 2x+2y+1],\\
E_5 &= [4x+1, 2y].
\end{aligned}$$

Moreover, the path $(6x+4y+4, b_1, c_1, 4x+4y+5)$ consists of edges with labels 2, $2y+1$, and $2x+2y+2$, and the path $(2y, b_2, c_2, 0)$ consists of edges with labels 1, $4x+2y+3$, and $4x+4y+4$. Thus the edge set of G has one edge of each label i where $1 \le i \le 4x+4y+4$. Hence the defined labeling is a σ-labeling, and condition (s1) for a σ-tripartite labeling is satisfied.

Case 4a: $x = 1$.
Now, let $A = \bigcup_{i=1}^{5} A_i$, $B = \{b_1, b_2\}$, and $C = \bigcup_{i=1}^{5} B_i \cup \{c_1, c_2\}$. Then $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (s2) of a σ-tripartite labeling is clear from (5) since all vertices in the C_{4y+3} do not exceed c_2, while all vertices in the $C_{4x+1} = C_5$ do. Note that $|b_1 - c_1| + |b_2 - c_2| = (2y+1) + (2y+7) = 4y+8 = n$, the number of edges of G. Thus condition (s3) is satisfied. Also $a = v + n$, where $a \in A$ and $v \in B \cup C$, is impossible, since by (6) and the assumptions $y > 2x = 2$ we have

$$v + n \ge b_2 + n = 6y + 9 > 4y + 10 = \max A.$$

Thus condition (s4) holds.

Finally, suppose $b \in B$ and $c \in C$. The equation $|b - c| = 2n$ is impossible since all vertices are in $[0, 2n]$ and $0 \in A$. If $c - b_2 = n$, then $c = 6y + 9$. This contradicts (6), since $6y + 9$ exceeds everything in B_1 but is less than c_1. Also if $b_1 - c = n$, then $c = 4$. This contradicts (6), since 4 is less than everything in A_5. Thus condition (s5) holds, and we have a σ-tripartite labeling for $C_5 \cup C_{4y+3}$.

Case 4b: $x > 1$.
Now, let $A = \bigcup_{i=1}^{5} A_i$, $B = \bigcup_{i=1}^{5} B_i \cup \{b_1, b_2\}$, and $C = \{c_1, c_2\}$. Then $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (s2) of a σ-tripartite labeling is clear from (5) since all vertices in the C_{4y+3} do not exceed c_2, while all vertices in the C_{4x+1} do. Note that $|b_1 - c_1| + |b_2 - c_2| = (2y+1) + (4x+2y+3) = 4x+4y+4 = n$, the number of edges of G. Thus condition (s3) is satisfied. Also $a = v + n$, where $a \in A$ and $v \in B \cup C$, is impossible, since by (6) and the assumption $y > 2x$ we have

$$v + n \geq b_2 + n = 4x + 6y + 5 > 4x + 5y + (2x) + 5 = 6x + 5y + 5 > 6x + 4y + 4 = \max A.$$

Thus condition (s4) holds.

Finally, suppose $b \in B$ and $c \in C$. The equation $|b - c| = 2n$ is impossible since all vertices are in $[0, 2n]$ and $0 \in A$. Likewise $|b - c_2| = n$ is impossible since $c_2 = n$ and $0 < B < 2n$. The case remains that $c_1 - b = n$, which gives $b = 2x + 2y + 3$. This contradicts (6), since $2x + 2y + 3$ exceeds b_2 but is less than everything in B_5. Thus condition (s5) holds, and we have a σ-tripartite labeling for $C_{4x+1} \cup C_{4y+3}$. □

Let G be a graph with n edges. If m is the label of an edge, let $m^* = \min\{m, 2n+1-m\}$ be the *length* of the edge, and if S is a set of edge labels, let $S^* = \{m^* : m \in S\}$ be the corresponding set of edge lengths. Thus if the set of vertices of G is a subset of $[0, 2n]$, and the set E of edge labels of G satisfies $E^* = [1, n]$, then G has a ρ-labeling.

Theorem 8. *Let C_r and C_s be odd cycles with $r \equiv s \pmod 4$. Then $G = C_r \cup C_s$ has a ρ-tripartite labeling unless $r = s = 3$.*

Proof. It is easy to verify that $C_3 \cup C_3$ does not admit a ρ-tripartite labeling. Since a ρ-tripartite labeling must be a ρ-labeling, in our constructions the vertex set for $C_r \cup C_s$ will be a subset of $[0, 2(r+s)]$.

Case 1: $G = C_{4x+1} \cup C_5$ where $x \geq 1$.
If $x = 1$, let $G = (0, 16, 1, 9, 20, 0) \cup (3, 6, 4, 8, 17, 3)$. Using the tripartition $A = \{0, 1, 3, 4\}$, $B = \{6, 8, 9, 16\}$, and $C = \{17, 20\}$, it is easy to see that the conditions for a ρ-tripartite labeling are satisfied.

If $x > 1$, let $C_{4x+1} = G_1 + G_2 + (4x+2, b_1, c_1, 2x+3)$ and $C_5 = (a_1, b_2, a_2, b_3, c_2, a_1)$ where $b_1 = 4x+4$, $c_1 = 8x+9$, $a_1 = 0$, $b_2 = 8x+10$, $a_2 = 1$, $b_3 = 4x+5$, $c_2 = 8x+12$, and

$$G_1 = P(2x+3, 4x+10, 2x-4),$$
$$G_2 = P(3x+1, 3x+5, 2x+2).$$

(Note: In the case when $x = 2$, the path G_1 is empty. However this does not change the proof in any way.)

First, we show that $G_1 + G_2 + (4x+2, b_1, c_1, 2x+3)$ is a cycle of length $4x+1$. Note that by **P1**, the first vertex of G_1 is $2x+3$ and the last is $3x+1$, and the first vertex of G_2 is $3x+1$ and the last is $4x+2$. For $1 \leq i \leq 2$, let A_i and B_i denote the sets labeled A' and B' in **P2**, corresponding to the path G_i. Then using **P2**, we compute

$$A_1 = [2x+3, 3x+1], \qquad B_1 = [5x+9, 6x+6],$$
$$A_2 = [3x+1, 4x+2], \qquad B_2 = [4x+7, 5x+7].$$

Thus,

$$a_1 < a_2 < A_1 \leq A_2 < b_1 < b_3 < B_2 < B_1 < c_1 < b_2 < c_2. \tag{7}$$

Note that $V(G_1) \cap V(G_2) = \{3x+1\}$; otherwise, G_1 and G_2 are vertex-disjoint. Therefore, $G_1 + G_2 + (4x+2, b_1, c_1, 2x+3)$ is a cycle of length $4x+1$.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 2$. By **P3**, we have edge labels

$$E_1 = [2x+8, 4x+3],$$
$$E_2 = [5, 2x+6],$$

yielding edge lengths

$$E_1^* = [2x+8, 4x+3],$$
$$E_2^* = [5, 2x+6].$$

Moreover, the path $(4x+2, b_1, c_1, 2x+3)$ consists of edges with lengths 2, $4x+5$, and $(6x+6)^* = 2x+7$, and the cycle $(a_1, b_2, a_2, b_3, c_2, a_1)$ consists of edges with lengths $(8x+10)^* = 3$, $(8x+9)^* = 4$, $4x+4$, $(4x+7)^* = 4x+6$, and $(8x+12)^* = 1$. Thus the edge set of G has one edge of each length i where $1 \le i \le 4x+6$. Hence the defined labeling is a ρ-labeling, and condition (r1) for a ρ-tripartite labeling is satisfied.

Finally, let $A = A_1 \cup A_2 \cup \{a_1, a_2\}$, $B = B_1 \cup B_2 \cup \{b_1, b_2, b_3\}$, and $C = \{c_1, c_2\}$. Then $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (r2) of a ρ-tripartite labeling is clear from (7). Note that $|b_1 - c_1| + |b_3 - c_2| = (4x+5) + (4x+7) = 8x+12 = 2n$, twice the number of edges of G. Thus condition (r3) is satisfied. Also $|b - c| = 2n$, where $b \in B$ and $c \in C$, is impossible since all vertices are in $[0, 2n]$ and $0 \in A$. Thus condition (r4) holds, and we have a ρ-tripartite labeling of G.

Case 2: $G = C_{4x+1} \cup C_{4y+1}$ where $x \ge y > 1$.
Let $C_{4x+1} = G_1 + G_2 + G_3 + (2x-1, b_1, c_1, 0)$ and $C_{4y+1} = G_4 + (4x+4y-2, b_2, c_2, 4x+2y-1)$ where $b_1 = 4x+4y$, $c_1 = 8x+8y+3$, $b_2 = 4x+4y-1$, $c_2 = 8x+8y$, and

$$G_1 = P(0, 6x+4y+4, 2x),$$
$$G_2 = P(x, 5x+6y+1, 2x-2y+2),$$
$$G_3 = P(2x-y+1, 6x+3y+5, 2y-4),$$
$$G_4 = P(4x+2y-1, 4x+2y+1, 4y-2).$$

(Note: In the case when $y = 2$, the path G_3 is empty. However this does not change the proof in any way.)

First, we show that $G_1 + G_2 + G_3 + (2x-1, b_1, c_1, 0)$ is a cycle of length $4x+1$ and $G_4 + (4x+4y-2, b_2, c_2, 4x+2y-1)$ is a cycle of length $4y+1$. Note that by **P1**, the first vertex of G_1 is 0 and the last is x, the first vertex of G_2 is x and the last is $2x-y+1$, the first vertex of G_3 is $2x-y+1$ and the last is $2x-1$, and the first vertex of G_4 is $4x+2y-1$ and the last is $4x+4y-2$. For $1 \le i \le 4$, let A_i and B_i denote the sets labeled A' and B' in **P2**, corresponding to the path G_i. Then using **P2**, we compute

$$A_1 = [0, x], \qquad B_1 = [7x+4y+5, 8x+4y+4],$$
$$A_2 = [x, 2x-y+1], \qquad B_2 = [6x+5y+3, 7x+4y+3],$$
$$A_3 = [2x-y+1, 2x-1], \qquad B_3 = [6x+4y+4, 6x+5y+1],$$
$$A_4 = [4x+2y-1, 4x+4y-2], \qquad B_4 = [4x+4y+1, 4x+6y-1].$$

Thus,

$$A_1 \le A_2 \le A_3 < A_4 < b_2 < b_1 < B_4 < B_3 < B_2 < B_1 < c_2 < c_1. \tag{8}$$

Note that $V(G_1) \cap V(G_2) = \{x\}$ and $V(G_2) \cap V(G_3) = \{2x - y + 1\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1 + G_2 + G_3 + (2x - 1, b_1, c_1, 0)$ is a cycle of length $4x + 1$ and $G_4 + (4x + 4y - 2, b_2, c_2, 4x + 2y - 1)$ is a cycle of length $4y + 1$.

Next, let E_i denote the set of edge labels in G_i for $1 \leq i \leq 4$. By **P3**, we have edge labels

$$\begin{aligned} E_1 &= [6x + 4y + 5, 8x + 4y + 4], \\ E_2 &= [4x + 6y + 2, 6x + 4y + 3], \\ E_3 &= [4x + 4y + 5, 4x + 6y], \\ E_4 &= [3, 4y], \end{aligned}$$

yielding edge lengths

$$\begin{aligned} E_1^* &= [4y + 1, 2x + 4y], \\ E_2^* &= [2x + 4y + 2, 4x + 2y + 3], \\ E_3^* &= [4x + 2y + 5, 4x + 4y], \\ E_4^* &= [3, 4y]. \end{aligned}$$

Moreover, the path $(2x-1, b_1, c_1, 0)$ consists of edges with lengths $2x+4y+1$, $(4x+4y+3)^* = 4x + 4y + 2$, and $(8x + 8y + 3)^* = 2$, and the path $(4x + 4y - 2, b_2, c_2, 4x + 2y - 1)$ consists of edges with lengths 1, $4x + 4y + 1$, and $(4x + 6y + 1)^* = 4x + 2y + 4$. Thus the edge set of G has one edge of each length i where $1 \leq i \leq 4x + 4y + 2$. Hence the defined labeling is a ρ-labeling, and condition (r1) for a ρ-tripartite labeling is satisfied.

Finally, let $A = \bigcup_{i=1}^4 A_i$, $B = \bigcup_{i=1}^4 B_i \cup \{b_1, b_2\}$, and $C = \{c_1, c_2\}$. Then, $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (r2) of a ρ-tripartite labeling is clear from (8). Note that $|b_1 - c_1| + |b_2 - c_2| = (4x + 4y + 3) + (4x + 4y + 1) = 8x + 8y + 4 = 2n$, twice the number of edges of G. Thus condition (r3) is satisfied. Also $|b - c| = 2n$, where $b \in B$ and $c \in C$, is impossible since all vertices are in $[0, 2n]$ and $0 \in A$. Thus condition (r4) holds, and we have a ρ-tripartite labeling of G.

Case 3: $G = C_{4x+3} \cup C_3$ where $x \geq 1$.
Let $C_{4x+3} = G_1 + G_2 + (2x - 1, b_1, a_1, b_2, c_1, 0)$ and $C_3 = (a_2, b_3, c_2, a_2)$ where $b_1 = 2x + 2$, $a_1 = 2x$, $b_2 = 4x + 5$, $c_1 = 8x + 12$, $a_2 = 4x + 2$, $b_3 = 4x + 6$, $c_2 = 8x + 11$, and

$$\begin{aligned} G_1 &= P(0, 6x + 8, 2x), \\ G_2 &= P(x, 5x + 9, 2x - 2). \end{aligned}$$

(Note: In the case when $x = 1$, the path G_2 is empty. However this does not change the proof in any way.)

First, we show that $G_1 + G_2 + (2x - 1, b_1, a_1, b_2, c_1, 0)$ is a cycle of length $4x + 3$. Note that by **P1**, the first vertex of G_1 is 0 and the last is x, and the first vertex of G_2 is x and the last is $2x - 1$. For $1 \leq i \leq 2$, let A_i and B_i denote the sets labeled A' and B' in **P2**, corresponding to the path G_i. Then using **P2**, we compute

$$\begin{aligned} A_1 &= [0, x], & B_1 &= [7x + 9, 8x + 8], \\ A_2 &= [x, 2x - 1], & B_2 &= [6x + 9, 7x + 7]. \end{aligned}$$

Thus,

$$A_1 \leq A_2 < a_1 < b_1 < a_2 < b_2 < b_3 < B_2 < B_1 < c_2 < c_1. \tag{9}$$

Note that $V(G_1) \cap V(G_2) = \{x\}$; otherwise, G_1 and G_2 are vertex-disjoint. Therefore, $G_1 + G_2 + (2x-1, b_1, a_1, b_2, c_1, 0)$ is a cycle of length $4x+3$.

Next, let E_i denote the set of edge labels in G_i for $1 \leq i \leq 2$. By **P3**, we have edge labels

$$E_1 = [6x+9, 8x+8],$$
$$E_2 = [4x+10, 6x+7],$$

yielding edge lengths

$$E_1^* = [5, 2x+4],$$
$$E_2^* = [2x+6, 4x+3].$$

Moreover, the path $(2x-1, b_1, a_1, b_2, c_1, 0)$ consists of edges with lengths 3, 2, $2x+5$, $(4x+7)^* = 4x+6$, and $(8x+12)^* = 1$, and the cycle (a_2, b_3, c_2, a_2) consists of edges with lengths 4, $4x+5$, and $(4x+9)^* = 4x+4$. Thus the edge set of G has one edge of each length i where $1 \leq i \leq 4x+6$. Hence the defined labeling is a ρ-labeling, and condition (r1) for a ρ-tripartite labeling is satisfied.

Finally, let $A = A_1 \cup A_2 \cup \{a_1, a_2\}$, $B = B_1 \cup B_2 \cup \{b_1, b_2, b_3\}$, and $C = \{c_1, c_2\}$. Then, $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (r2) of a ρ-tripartite labeling is clear from (9). Note that $|b_2 - c_1| + |b_3 - c_2| = (4x+7) + (4x+5) = 8x+12 = 2n$, twice the number of edges of G. Thus condition (r3) is satisfied. Also $|b-c| = 2n$, where $b \in B$ and $c \in C$, is impossible since all vertices are in $[0, 2n]$ and $0 \in A$. Thus condition (r4) holds, and we have a ρ-tripartite labeling of G.

Case 4: $G = C_{4x+3} \cup C_{4y+3}$ where $x \geq y \geq 1$.
Let $C_{4x+3} = G_1 + G_2 + G_3 + (2x, b_1, c_1, 0)$ and $C_{4y+3} = G_4 + (4x+4y+4, b_2, c_2, 4x+2y+4)$ where $b_1 = 4x+4y+5$, $c_1 = 8x+8y+12$, $b_2 = 4x+4y+6$, $c_2 = 8x+8y+11$, and

$$G_1 = P(0, 6x+4y+8, 2x+2),$$
$$G_2 = P(x+1, 5x+6y+8, 2x-2y),$$
$$G_3 = P(2x-y+1, 6x+3y+9, 2y-2),$$
$$G_4 = P(4x+2y+4, 4x+2y+6, 4y).$$

(Note: In the case when $x = y$, the path G_2 is empty, and when $y = 1$, the path G_3 is empty. However this does not change the proof in any way.)

First, we show that $G_1 + G_2 + G_3 + (2x, b_1, c_1, 0)$ is a cycle of length $4x+3$ and $G_4 + (4x+4y+4, b_2, c_2, 4x+2y+4)$ is a cycle of length $4y+3$. Note that by **P1**, the first vertex of G_1 is 0 and the last is $x+1$, the first vertex of G_2 is $x+1$ and the last is $2x-y+1$, the first vertex of G_3 is $2x-y+1$ and the last is $2x$, and the first vertex of G_4 is $4x+2y+4$ and the last is $4x+4y+4$. For $1 \leq i \leq 4$, let A_i and B_i denote the sets labeled A' and B' in **P2**, corresponding to the path G_i. Then using **P2**, we compute

$$A_1 = [0, x+1], \qquad B_1 = [7x+4y+10, 8x+4y+10],$$
$$A_2 = [x+1, 2x-y+1], \qquad B_2 = [6x+5y+9, 7x+4y+8],$$
$$A_3 = [2x-y+1, 2x], \qquad B_3 = [6x+4y+9, 6x+5y+7],$$
$$A_4 = [4x+2y+4, 4x+4y+4], \qquad B_4 = [4x+4y+7, 4x+6y+6].$$

Thus,

$$A_1 \leq A_2 \leq A_3 < A_4 < b_1 < b_2 < B_4 < B_3 < B_2 < B_1 < c_2 < c_1. \tag{10}$$

Note that $V(G_1) \cap V(G_2) = \{x+1\}$ and $V(G_2) \cap V(G_3) = \{2x-y+1\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1 + G_2 + G_3 + (2x, b_1, c_1, 0)$ is a cycle of length $4x+3$ and $G_4 + (4x+4y+4, b_2, c_2, 4x+2y+4)$ is a cycle of length $4y+3$.

Next, let E_i denote the set of edge labels in G_i for $1 \leq i \leq 4$. By **P3**, we have edge labels

$$\begin{aligned} E_1 &= [6x+4y+9, 8x+4y+10], \\ E_2 &= [4x+6y+8, 6x+4y+7], \\ E_3 &= [4x+4y+9, 4x+6y+6], \\ E_4 &= [3, 4y+2], \end{aligned}$$

yielding edge lengths

$$\begin{aligned} E_1^* &= [4y+3, 2x+4y+4], \\ E_2^* &= [2x+4y+6, 4x+2y+5], \\ E_3^* &= [4x+2y+7, 4x+4y+4], \\ E_4^* &= [3, 4y+2]. \end{aligned}$$

Moreover, the path $(2x, b_1, c_1, 0)$ consists of edges with lengths $2x+4y+5$, $(4x+4y+7)^* = 4x+4y+6$, and $(8x+8y+12)^* = 1$, and the path $(4x+4y+4, b_2, c_2, 4x+2y+4)$ consists of edges with lengths 2, $4x+4y+5$, and $(4x+6y+7)^* = 4x+2y+6$. Thus the edge set of G has one edge of each length i where $1 \leq i \leq 4x+4y+6$. Hence the defined labeling is a ρ-labeling, and condition (r1) for a ρ-tripartite labeling is satisfied.

Finally, let $A = \bigcup_{i=1}^4 A_i$, $B = \bigcup_{i=1}^4 B_i \cup \{b_1, b_2\}$, and $C = \{c_1, c_2\}$. Then, $\{A, B, C\}$ is a tripartition of $V(G)$. Condition (r2) of a ρ-tripartite labeling is clear from (10). Note that $|b_1 - c_1| + |b_2 - c_2| = (4x+7) + (4x+5) = 8x+12 = 2n$, twice the number of edges of G. Thus condition (r3) is satisfied. Also $|b-c| = 2n$, where $b \in B$ and $c \in C$, is impossible since all vertices are in $[0, 2n]$ and $0 \in A$. Thus condition (r4) holds, and we have a ρ-tripartite labeling of G. □

It is known that if G is the union of r vertex-disjoint C_3's, then G admits a ρ-labeling (see [5]), and thus there exists a cyclic G-decomposition of K_{6r+1}. Therefore, if $G = C_3 \cup C_3$, then there exists a cyclic G-decomposition of K_{12x+1} for every positive integer x. In light of this and Theorems 7 and 8, we have the following corollary.

Corollary 9. *If G is the union of two vertex-disjoint cycles of odd length and n is the number of edges in G, then there exists a cyclic G-decomposition of K_{2nx+1} for every positive integer x.*

We note that if $n = 2^t$, then Corollary 9 can be strengthened as follows.

Corollary 10. *If G is the union of two vertex-disjoint cycles of odd length C_r and C_s and if $r+s = 2^t$, then there exists a G-decomposition of K_m if and only if $m \equiv 1 \pmod{2^{t+1}}$.*

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