# On labeling 2-regular graphs where the number of odd components is at most 2

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**Abstract**

Several graph labelings were introduced by Rosa in 1967 as means of attacking graph decomposition problems. The most basic of these labelings is what is known as a $\rho$-labeling. Rosa showed that a graph $G$ of size $n$ admits a $\rho$-labeling if and only if there is a cyclic $G$-decomposition of $K_{2n+1}$. He also showed that if $G$ is bipartite and admits what is known as an $\alpha$-labeling, then there exists a cyclic $G$-decomposition of $K_{2nx+1}$ for all positive integers $x$. Here we show that the vertex-disjoint union of a graph that admits a modified $\rho$-labeling and any number of graphs that admit $\alpha$-labelings has a $\rho$-labeling. We use this to also show that if the number of odd components in a 2-regular graph $G$ is at most two, then $G$ admits a $\rho$-labeling. This provides further evidence in support of a conjecture that every 2-regular graph admits a $\rho$-labeling.

## 1 Introduction

If $a$ and $b$ are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo $n$. For a graph $G$, let $V(G)$ and $E(G)$ denote the vertex set of $G$ and the edge set of $G$, respectively. The *order* and the *size* of a graph $G$ are $|V(G)|$ and $|E(G)|$, respectively.

Let $V(K_k) = \mathbb{Z}_k$ and let $G$ be a subgraph of $K_k$. By *clicking* $G$, we mean applying the isomorphism $i \to i + 1$ to $V(G)$. Let $H$ and $G$ be

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graphs such that $G$ is a subgraph of $H$. A $G$-*decomposition* of $H$ is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of pairwise disjoint subgraphs of $H$ each of which is isomorphic to $G$ and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. If $H$ is $K_k$, a $G$-decomposition $\Gamma$ of $H$ is *cyclic* if clicking is a permutation of $\Gamma$. If $G$ is a graph and $r$ is a positive integer, $rG$ denotes the vertex disjoint union of $r$ copies of $G$.

For any graph $G$, a one-to-one function $f : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of $G$. In [16], Rosa introduced a hierarchy of labelings. Let $G$ be a graph with $n$ edges and no isolated vertices and let $f$ be a labeling of $G$. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of $e$. Let $\bar{E}(G) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

($\ell$1) $f(V(G)) \subseteq [0, 2n]$,

($\ell$2) $f(V(G)) \subseteq [0, n]$,

($\ell$3) $\bar{E}(G) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,

($\ell$4) $\bar{E}(G) = [1, n]$.

If in addition $G$ is bipartite with bipartition $\{A, B\}$ of $V(G)$ consider also

($\ell$5) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,

($\ell$6) there exists an integer $\lambda$ (called the *boundary value* of $f$) such that $f(a) \le \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

($\ell$1), ($\ell$3) is called a $\rho$-*labeling*;

($\ell$1), ($\ell$4) is called a $\sigma$-*labeling*;

($\ell$2), ($\ell$4) is called a $\beta$-*labeling*.

A $\beta$-labeling is necessarily a $\sigma$-labeling which in turn is a $\rho$-labeling. Suppose $G$ is bipartite. If a $\rho$, $\sigma$ or $\beta$-labeling of $G$ satisfies condition ($\ell$5), then the labeling is *ordered* and is denoted by $\rho^+$, $\sigma^+$ or $\beta^+$, respectively. If in addition ($\ell$6) is satisfied, the labeling is *uniformly-ordered* and is denoted by $\rho^{++}$, $\sigma^{++}$ or $\beta^{++}$, respectively.

A $\beta$-labeling is better known as a *graceful* labeling and a uniformly-ordered $\beta$-labeling is an $\alpha$-*labeling* as introduced in [16]. Labelings of the types above are called *Rosa-type* because of Rosa's original article [16] on the topic (see [10] for a recent comprehensive survey of Rosa-type labelings). A dynamic survey on general graph labelings is maintained by Gallian [11].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [16] and [9], respectively.

**Theorem 1** *Let $G$ be a graph with $n$ edges. There exists a cyclic $G$-decomposition of $K_{2n+1}$ if and only if $G$ has a $\rho$-labeling.*

**Theorem 2** *Let $G$ be a graph with $n$ edges that has a $\rho^+$-labeling. Then there exists a cyclic $G$-decomposition of $K_{2nx+1}$ for all positive integers $x$.*

Let $G$ be a graph with Eulerian components. If $G$ admits a $\sigma$-labeling, then we have the following well-known restriction on $|E(G)|$.

**Theorem 3** (Parity Condition in [16]) *If a graph $G$ with Eulerian components and $n$ edges has a $\sigma$-labeling, then $n \equiv 0$ or $3 \pmod 4$. If in addition $G$ is bipartite, then $n \equiv 0 \pmod 4$.*

A non-bipartite graph $G$ is said to be *almost-bipartite* if $G-e$ is bipartite for some $e \in E(G)$. Note that if $G$ is almost-bipartite with $e = \{\hat{b}, c\}$, then $G$ is necessarily tripartite and $V(G)$ can be partitioned into three sets $A$, $B$ and $C = \{c\}$ such that $\hat{b} \in B$ and $e$ is the only edge joining an element of $B$ to $c$.

Let $G$ be an almost-bipartite graph with $n$ edges with vertex tripartition $A$, $B$, $C$ as above. A labeling $h$ of the vertices of $G$ is called a $\gamma$-*labeling* of $G$ if the following conditions hold.

(g1) The function $h$ is a $\rho$-labeling of $G$.

(g2) If $\{a, v\}$ is an edge of $G$ with $a \in A$, then $h(a) < h(v)$.

(g3) We have $h(c) - h(\hat{b}) = n$.

It was shown in [5], that if a graph $G$ with $n$ edges admits a $\gamma$-labeling, then there exists a cyclic $G$-decomposition of $K_{2nx+1}$ for all positive integers $x$.

In [16], Rosa presented $\alpha$- and $\beta$-labelings of $C_{4m}$ and of $C_{4m+3}$, respectively. It is also known that both $C_{4m+1}$ and $C_{4m+2}$ admit $\rho$-labelings. It was shown in [9] that there exists a $\rho^+$-labeling of $C_{4m+2}$, for all positive integers $m$. It can be easily checked that this labeling is actually a $\rho^{++}$-labeling. In [5], it was shown that odd cycles other than $C_3$ admit $\gamma$-labelings.

In this manuscript, we will focus on labelings of 2-regular graphs (i.e., the vertex-disjoint union of cycles). If a 2-regular graph $G$ is bipartite, then it is known that $G$ admits a $\sigma^+$-labeling if the parity condition is satisfied (see [9]) and a $\rho^{++}$-labeling otherwise (see [4]). A 2-regular graph $G$ need not admit a $\beta$-labeling even if the parity condition is satisfied. For example, it is shown in [14] that $rC_3$ does not admit a $\beta$-labeling for all $r > 1$ and $rC_5$ never admits a $\beta$-labeling. Moreover, it is known that $C_3 \bigcup C_3 \bigcup C_5$ is the smallest 2-regular graph that satisfies the parity condition, yet fails to have a $\beta$-labeling (see [1]). It is thus reasonable to focus on labelings that are less restrictive than $\beta$-labelings when studying 2-regular graphs.

Here, we shall show that if the number of odd components in a 2-regular graph $G$ is at most two, then $G$ admits a $\rho$-labeling. We do this by first showing that the vertex-disjoint union of a graph that admits a modified $\rho$-labeling and any number of graphs that admit $\alpha$-labelings has a $\rho$-labeling. This parallels a result of Hevia and Ruiz [12] who showed that vertex-disjoint union of a graph that admits a modified $\beta$-labeling and any number of graphs that admit $\alpha$-labelings has a $\sigma$-labeling. Our results on labelings of 2-regular graphs provide further evidence in support of a conjecture of El-Zanati and Vanden Eynden that every 2-regular graph admits a $\rho$-labeling.

# 2 Summary of Some of the Known Results

As stated in the previous section, the following is known for cycles (see [15], [16] and [9]).

**Theorem 4** *Let $m \geq 3$ be an integer. Then, $C_m$ admits an $\alpha$-labeling if $m \equiv 0 \pmod 4$, a $\rho$-labeling if $m \equiv 1 \pmod 4$, a $\rho^{++}$-labeling if $m \equiv 2 \pmod 4$, and a $\beta$-labeling if $m \equiv 3 \pmod 4$.*

For 2-regular graphs with two components, we have the following important result from Abrham and Kotzig [1].

**Theorem 5** *Let $m \geq 3$ and $n \geq 3$ be integers. Then the graph $C_m \bigcup C_n$ has a $\beta$-labeling if and only if $m+n \equiv 0$ or $3 \pmod 4$. Moreover, $C_m \bigcup C_n$ has an $\alpha$-labeling if and only if both $m$ and $n$ are even and $m+n \equiv 0 \pmod 4$.*

If the parity condition is not satisfied, then we have the following from [4] and [8].

**Theorem 6** *Let $m \geq 3$ and $n \geq 3$ be integers such that $m+n \equiv 1$ or $2 \pmod 4$. Then $C_m \bigcup C_n$ has a $\rho^{++}$-labeling if both $m$ and $n$ are even and a $\rho$-labeling otherwise.*

For 2-regular graphs with more than two components, the following is known. In [13], Kotzig shows that if $r > 1$, then $rC_3$ does not admit a $\beta$-labeling. Similarly, he shows that $rC_5$ does not admit a $\beta$-labeling for any $r$. In [14], Kotzig shows that $3C_{4k+1}$ admits a $\beta$-labeling for all $k \geq 2$.

From results in [7], it can be shown that $rC_3$ admits a $\rho$-labeling for all $r \geq 1$. An additional result follows by combining results from [9] and from [4].

**Theorem 7** *Let $G$ be a 2-regular bipartite graph of order $n$. Then $G$ has a $\sigma^+$-labeling if $n \equiv 0 \pmod 4$ and a $\rho^{++}$-labeling if $n \equiv 2 \pmod 4$.*

A result of Hevia and Ruiz [12] proves very useful.

**Theorem 8** *The disjoint union of a graph with a $\beta$-labeling, together with a collection of graphs with $\alpha$-labelings, has a $\sigma$-labeling.*

When applied to 2-regular graphs and combined with the results of Abrham and Kotzig [1], Theorem 8 yields the following.

**Corollary 9** *Let $x \geq 0$, $y \geq 1$, and $z \geq 1$ be integers and let $G_1 \in \{C_{4x+3}, C_{4x+3} \bigcup C_{4y+1}, C_{4x+1} \bigcup C_{4y+2}\}$. If $G_2$ is a 2-regular bipartite graph of order $4z$, then $G_1 \bigcup G_2$ admits a $\sigma$-labeling.*

In [5], it is shown that if $G$ admits an $\alpha$-labeling and $j > 1$, then $G \bigcup C_{2j+1}$ admits a $\gamma$-labeling. Thus for example, both $C_{4x} \bigcup C_{4y} \bigcup C_{4z+1}$ and $C_{4x+1} \bigcup C_{4y+2} \bigcup C_{4z+2}$ admit $\gamma$-labelings. These results are generalized in [6], where the following is shown.

**Theorem 10** *Every 2-regular almost-bipartite graph $G$, other than $C_3$ or $C_3 \bigcup C_4$, has a $\gamma$-labeling.*

# 3 Main Results

Let $H$ be a graph of size $n$ with a $\rho$-labeling $h$ such that $\min(\{h(v) : v \in V(H)\}) = 0$. Let $G$ be a bipartite graph of size $m$. We say $h$ *$\alpha$-accommodates* $G$ if the following conditions hold:

(a1) $G$ admits an $\alpha$-labeling.

(a2) If $H$ is not bipartite, then $V(H)$ can be partitioned into two sets $A$ and $B$ such that the labels of all edges $\{u, v\}$ where $u, v \in A$ or $u, v \in B$ comprise the set $\{1, 2, \ldots, \ell\}$ for some integer $\ell$. If $H$ is bipartite, let $\{A, B\}$ be a bipartition of $V(H)$, and let $\ell = 0$.

(a3) If $a \in A$ and $b \in B$, then $h(a) < h(b)$.

(a4) If $d_1 = 2n + 1 - \max(\{h(v) : v \in B\})$ and $d_2 = \min(\{h(v) : v \in B\}) - \max(\{h(v) : v \in A\})$, then $\max(\{d_1, d_2\}) \geq \ell + 2$ and $\min(\{d_1, d_2\}) \geq \ell + 2 - m$.

A $\rho$-labeling $h$ of $C_5$ is given in Figure 1 below. In this case, $A = \{0, 1, 2\}$, $B = \{3, 7\}$, $\ell = 1$, $d_1 = 4$ and $d_2 = 1$. Thus $h$ $\alpha$-accommodates any bipartite graph $G$ of size at least 2 (provided that $G$ admits an $\alpha$-labeling). We also give $\alpha$-labelings of $C_4$ and of $C_8$.

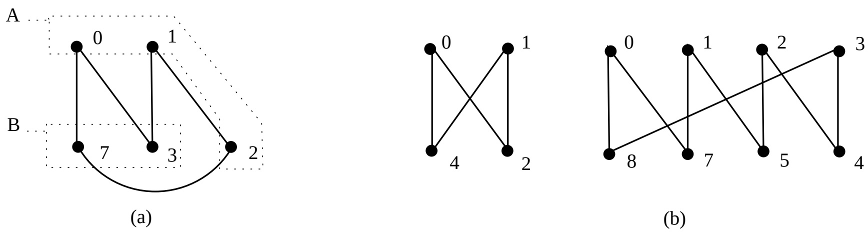


Figure 1: (a) A $\rho$-labeling of $C_5$ that $\alpha$-accommodates all graphs of size at least 2. (b) $\alpha$-labelings of $C_4$ and of $C_8$.

**Theorem 11** *For $t \geq 1$, let $G_1, G_2, \ldots, G_t$ be graphs with $\alpha$-labelings and let $H$ be a graph with a $\rho$-labeling that $\alpha$-accommodates $G_1$. Then the disjoint union of $G_1, G_2, \ldots, G_t$, and $H$ has a $\rho$-labeling.*

*Proof.* For $1 \leq i \leq t$, let $G_i$ have $m_i$ edges and the $\alpha$-labeling $g_i$, with the vertex bipartition $\{A_i, B_i\}$, so that $g_i(A_i) < g_i(B_i)$. Let $H$ have $n$ edges and the $\rho$-labeling $h_0$ that $\alpha$-accommodates $G_1$, with $A$, $B$, $\ell$, $d_1$, and $d_2$ as in the definition (of $\alpha$-accommodates). Let the following variables be defined as such:

$$\begin{aligned} M_i &= \sum_{j=i+1}^{t} m_j \quad \text{for } 0 \leq i \leq t-1, \\ M_t &= 0, \\ \lambda_i &= \max(\{g_i(v) : v \in A_i\}) \quad \text{for } 1 \leq i \leq t. \end{aligned}$$

**Case 1:** Consider when $d_1 \geq d_2$.
Then let the following variables be defined as such:

$$\begin{aligned} \lambda_{-1} &= \max(\{h_0(v) : v \in B\}) + M_0, \\ \lambda_0 &= \max(\{h_0(v) : v \in A\}). \end{aligned}$$

Let $G^* = (\bigcup_{i=1}^{t} G_i) \bigcup H$ and define a labeling $h$ on $V(G^*)$ by

$$h(v) = \begin{cases} h_0(v) & \text{for } v \in A, \\ h_0(v) + M_0 & \text{for } v \in B, \\ g_i(v) + \sum_{j=1}^{\lceil \frac{i}{2} \rceil} (\lambda_{i-2j} + 1) & \text{for } v \in A_i, \\ g_i(v) + M_i + \ell + \sum_{j=1}^{\lceil \frac{i}{2} \rceil} (\lambda_{i-2j} + 1) & \text{for } v \in B_i. \end{cases}$$

Note that because $d_2 \geq \ell + 2 - m_1$ and $d_1 \geq \ell + 2$, we have $h(A) < h(A_2) < h(A_4) < \cdots < h(A_{2\lfloor \frac{t}{2} \rfloor}) < h(B_{2\lfloor \frac{t}{2} \rfloor}) < h(B_{2\lfloor \frac{t}{2} \rfloor - 2}) < \cdots < h(B_2) \leq m_2 + M_2 + \ell + \lambda_0 + 1 = M_1 + \ell + \min(\{h_0(v) : v \in B\}) - d_2 + 1 \leq M_0 + \min(\{h_0(v) : v \in B\}) - 1 < h(B) < h(A_1) < h(A_3) < \cdots < h(A_{2\lceil \frac{t}{2} \rceil - 1}) < h(B_{2\lceil \frac{t}{2} \rceil - 1}) < h(B_{2\lceil \frac{t}{2} \rceil - 3}) < \cdots < h(B_1) \leq m_1 + M_1 + \ell + \lambda_{-1} + 1 = 2M_0 + 2n + \ell + 2 - d_1 \leq 2(M_0 + n)$. Thus $h$ is one-to-one.

The edge labels (also lengths) of $G_i$ using $g_i$ were $[1, m_i]$. In $G^*$, using $h$, these edge labels (also lengths) become $[1 + M_i + \ell, m_i + M_i + \ell]$. Recall that all edges in $H$ with both endvertices in $A$ or both endvertices in $B$, have labels (also lengths) $[1, \ell]$ using $h_0$. Since the same quantity is added to both endvertices of such edges, they maintain the same lengths in $G^*$. Moreover, if $\ell < k \leq n$, then there is an edge $e = \{a, b\} \in E(H)$ with $a \in A$ and $b \in B$ such that $\bar{h}_0(e) = k$ or $\bar{h}_0(e) = 2n + 1 - k$. Thus, in $G^*$, we have $\bar{h}(e) = k + M_0$ or $\bar{h}(e) = 2n + 1 - k + M_0$. In either case, the length of $e$ is $k + M_0$. Thus the $n - \ell$ edges of $H$ that had lengths in $[\ell + 1, n]$ under $h_0$ now have edge lengths $[\ell + 1 + M_0, n + M_0]$ in $G^*$.

Thus in $G^*$, the edges with lengths in $[1, \ell] \bigcup [\ell + M_0 + 1, n + M_0]$ are in $H$. The edges with lengths in $[\ell + 1, \ell + m_t] = [\ell + 1, \ell + M_{t-1}]$ are in $G_t$, $[\ell + M_{t-1} + 1, \ell + M_{t-1} + m_{t-1}] = [\ell + M_{t-1} + 1, \ell + M_{t-2}]$ are in $G_{t-1}$, $[\ell + M_{t-2} + 1, \ell + M_{t-2} + m_{t-2}] = [\ell + M_{t-2} + 1, \ell + M_{t-3}]$ are in $G_{t-2}$, $\ldots$, $[\ell + M_1 + 1, \ell + M_1 + m_1] = [\ell + M_1 + 1, \ell + M_0]$ are in $G_1$. Therefore, the edge lengths $[1, n + M_0]$ are all accounted for in $G^*$. Thus $G^*$ has a $\rho$-labeling.

**Case 2:** Consider when $d_1 < d_2$.
Define a new labeling $h'_0$ of $H$ by $h'_0(v) = h_0(v) - \min(\{h_0(u) : u \in B\})$. Since $h'_0$ was obtained from $h_0$ by subtracting the same quantity modulo $2n + 1$, the edge labels are preserved. Thus, $h'_0$ is a $\rho$-labeling of $H$. Moreover, $A' = B$ and $B' = A$ partition $V(H)$. With this new partition of $V(H)$, if we calculate the parameters $\ell'$, $d'_1$, and $d'_2$, corresponding to $\ell$, $d_1$, and $d_2$, respectively, we see that $\ell' = \ell$, $d'_1 = d_2$, and $d'_2 = d_1$, and thus $h'_0$ $\alpha$-accommodates $G$. Since $d'_1 > d'_2$, **Case 1** applies. □

Figure 2 below shows a $\rho$-labeling of $C_5 \bigcup C_8 \bigcup C_4 \bigcup C_4$ obtained using a $\rho$-labeling of $C_5$ (see Figure 1) that $\alpha$-accommodates $C_8$.

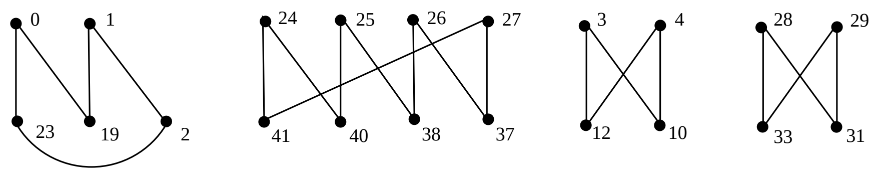


Figure 2: A $\rho$-labeling of $C_5 \bigcup C_8 \bigcup C_4 \bigcup C_4$

## 4 Labelings of 2-regular Graphs with at Most Two Odd Components

We use Theorem 11 to show that every 2-regular graph with at most two odd components has a $\rho$-labeling. First, we introduce some additional notation.

Henceforth, we will consider graphs whose vertices are (distinct) nonnegative integers. By the *label* of the edge $\{x, y\}$ in such a graph we mean $|x - y|$.

Let $G$ be a graph with $n$ edges. If $m$ is the label of an edge, let $m^* = \min\{m, 2n + 1 - m\}$ be the *length* of the edge, and if $S$ is a set of edge labels, let $S^* = \{m^* : m \in S\}$ be the set of all edge lengths. Thus if the set of vertices of $G$ is a subset of $[0, 2n]$, and the set $E$ of edge labels of $G$ satisfies $E^* = [1, n]$, then $G$ has a $\rho$-labeling.

We denote the directed path with vertices $x_0, x_1, \ldots, x_k$, where $x_i$ is adjacent to $x_{i+1}$, $0 \leq i \leq k - 1$, by $(x_0, x_1, \ldots, x_k)$. The *first vertex* of this path is $x_0$, the *second vertex* is $x_1$, and the *last vertex* is $x_k$. If $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$ are directed paths with $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$.

Let $P(k)$ be the path with $k$ edges and $k + 1$ vertices $0, 1, \ldots, k$ given by $(0, k, 1, k - 1, 2, k - 2, \ldots, \lceil k/2 \rceil)$. Note that the set of vertices of this graph is $A \bigcup B$, where $A = [0, \lfloor k/2 \rfloor]$, $B = [\lfloor k/2 \rfloor + 1, k]$, and every edge joins a vertex of $A$ to one of $B$. Furthermore the set of labels of the edges of $P(k)$ is $[1, k]$.

Now let $a$ and $b$ be nonnegative integers with $a \leq b$ and let us add $a$ to all the vertices of $A$ and $b$ to all the vertices of $B$. We will denote the resulting graph by $P(a, b, k)$. Note that this graph has the following properties. Figure 3 shows $P(6)$ and $P(3, 5, 6)$.


Figure 3: The paths $P(6)$ and $P(3,5,6)$.

**P1** $P(a,b,k)$ is a path with first vertex $a$ and second vertex $b+k$. If $k$ is even, its last vertex is $a+k/2$.

**P2** Each edge of $P(a,b,k)$ joins a vertex of $A' = [a, \lfloor k/2 \rfloor + a]$ to a larger vertex of $B' = [\lfloor k/2 \rfloor + 1 + b, k+b]$.

**P3** The set of edge labels of $P(a,b,k)$ is $[b-a+1, b-a+k]$.

**Theorem 12** *If the number of odd components of a 2-regular graph $G$ is at most two, then $G$ has a $\rho$-labeling.*

*Proof.* Let $G$ be a 2-regular graph where the number of odd components is at most two. If $G$ is bipartite, then $G$ has either a $\sigma^+$-labeling or a $\rho^{++}$-labeling by Theorem 7. If $G$ has exactly one odd component, then $G$ has a $\gamma$-labeling by Theorem 10 unless $G \in \{C_3, C_3 \bigcup C_4\}$. Since both $C_3$ and $C_3 \bigcup C_4$ are graceful, they have a $\rho$-labeling. Thus it remains to show that if the number of odd components in $G$ is two, then $G$ has a $\rho$-labeling.

Let $G = C_{m_1} \bigcup C_{m_2} \bigcup C_{4x_1+2} \bigcup \cdots \bigcup C_{4x_r+2} \bigcup C_{4y_1} \bigcup \cdots \bigcup C_{4y_s}$, where $m_1$ and $m_2$ are both odd and at least 3 and each of the $x_i$'s and $y_i$'s is positive. If $r$ is even (i.e., if the number of cycles in $G$ of length $\equiv 2 \pmod 4$ is even), then the $r$ cycles of length $2 \pmod 4$ can paired into $r/2$ graphs that admit $\alpha$-labelings (by Theorem 5). Thus it would suffice to show that $G' = C_{m_1} \bigcup C_{m_2}$ has a $\rho$-labeling that $\alpha$-accommodates graphs of size at least 4. If $r$ is odd, then we show that $C_{m_1} \bigcup C_{m_2} \bigcup C_{4x_1+2}$ has a $\rho$-labeling that $\alpha$-accommodates graphs of size at least 4. This results in the examination of nine cases which are established below. The first case is established using Hevia and Ruiz's results [12] (see Theorem 8).

**Case 1:** $G' = C_{4x+1} \bigcup C_{4y+3}$ where $x \geq 1$ and $y \geq 0$.
By Theorem 5, $G'$ has a $\beta$-labeling. Thus by Corollary 9, $G$ has a $\sigma$-labeling.

**Case 2:** $G' = C_{4x+1} \bigcup C_{4y+1}$ where $y \geq x \geq 1$.
First, we consider when $y = 1$. Let $G' = (4,9,5,7,10,4) \bigcup (0,13,3,15,14,0)$. Letting $A = \{0,3,4,5,7\}$ and $B = \{9,10,13,14,15\}$ would yield $\ell = 2$,

$d_1 = 6$, and $d_2 = 2$. Thus we have a $\rho$-labeling of $G^{'}$ that $\alpha$-accommodates graphs of size at least 2.

Next, we consider when $y \geq 2$. Let $C_{4x+1} = G_1 + G_2 + (6x+2y-2, 6x+2y-1, 8x+6y+2, 4x+2y-1)$ and $C_{4y+1} = G_3 + (y-2, 8x+7y+3, y) + G_4 + (2y, 8x+8y+4, 8x+8y+2, 0)$, where

$$\begin{aligned} G_1 &= P(4x+2y-1, 6x+6y+2, 2x-2), \\ G_2 &= P(5x+2y-2, 5x+6y, 2x), \\ G_3 &= P(0, 8x+6y+5, 2y-4), \\ G_4 &= P(y, 8x+5y+2, 2y). \end{aligned}$$

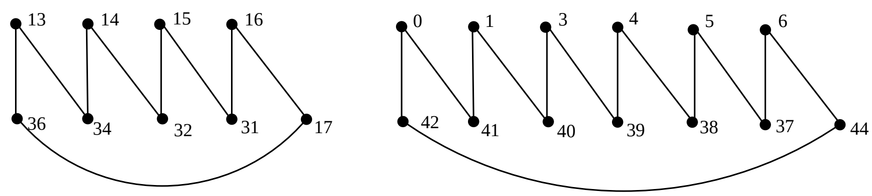


Figure 4: A $\rho$-labeling of $C_9 \bigcup C_{13}$ that $\alpha$-accommodates graphs of size at least 3.

Note that when $x = 1$, $G_1$ is an empty path consisting of a single vertex labeled $2y+3$, and when $y = 2$, $G_3$ is an empty path consisting of a single vertex labeled 0. However, this does not change the proof.

We start by showing that $G_1 + G_2 + (6x+2y-2, 6x+2y-1, 8x+6y+2, 4x+2y-1)$ is a cycle of length $4x+1$, and $G_3 + (y-2, 8x+7y+3, y) + G_4 + (2y, 8x+8y+4, 8x+8y+2, 0)$ is a cycle of length $4y+1$. Note that by P1, the first vertex of $G_1$ is $4x+2y-1$ and the last is $5x+2y-2$, the first vertex of $G_2$ is $5x+2y-2$ and the last is $6x+2y-2$, the first vertex of $G_3$ is 0 and the last is $y-2$, the first vertex of $G_4$ is $y$ and the last is $2y$. For $1 \leq i \leq 4$, let $A_i$ and $B_i$ denote the sets labeled $A'$ or $B'$ in P2, corresponding to the path $G_i$. Then using P2, we compute

$$\begin{aligned} A_1 &= [4x+2y-1, 5x+2y-2], & B_1 &= [7x+6y+2, 8x+6y], \\ A_2 &= [5x+2y-2, 6x+2y-2], & B_2 &= [6x+6y+1, 7x+6y], \\ A_3 &= [0, y-2], & B_3 &= [8x+7y+4, 8x+8y+1], \\ A_4 &= [y, 2y], & B_4 &= [8x+6y+3, 8x+7y+2]. \end{aligned}$$

Using the assumptions that $y \geq x \geq 1$ and $y \geq 2$, we can check that $\{0\} \leq A_3 \leq \{y-2, y\} \leq A_4 \leq \{2y, 4x+2y-1\} \leq A_1 \leq A_2 \leq \{6x+2y-2, 6x+2y-1, 6x+6y+1\} \leq B_2 < B_1 < \{8x+6y+2\} < B_4 < B_3 < \{8x+8y+2, 8x+8y+4\}$. Also note that $V(G_1) \bigcap V(G_2) = \{5x+2y-2\}$;

otherwise, $G_i$ and $G_j$ are vertex-disjoint. Thus, $G_1 + G_2 + (6x + 2y - 2, 6x + 2y - 1, 8x + 6y + 2, 4x + 2y - 1)$ is a cycle of length $4x + 1$, and $G_3 + (y-2, 8x+7y+3, y) + G_4 + (2y, 8x+8y+4, 8x+8y+2, 0)$ is a cycle of length $4y + 1$.

Now let $E_i$ denote the set of edge labels in $G_i$ for $1 \le i \le 4$. By P3, we have

$$\begin{aligned}
E_1^* &= [2x + 4y + 4, 4x + 4y + 1]^* = [2x + 4y + 4, 4x + 4y + 1], \\
E_2^* &= [4y + 3, 2x + 4y + 2]^* = [4y + 3, 2x + 4y + 2], \\
E_3^* &= [8x + 6y + 6, 8x + 8y + 1]^* = [4, 2y - 1], \\
E_4^* &= [8x + 4y + 3, 8x + 6y + 2]^* = [2y + 3, 4y + 2].
\end{aligned}$$

Putting the edge length sets in order, we get:

| cycle | edge lengths | cycle | edge lengths |
|---|---|---|---|
| $C_{4x+1}$ | $1$ | $C_{4y+1}$ | $2y + 2$ |
| $C_{4y+1}$ | $2$ | $C_{4y+1}$ | $[2y + 3, 4y + 2]$ |
| $C_{4y+1}$ | $3$ | $C_{4x+1}$ | $[4y + 3, 2x + 4y + 2]$ |
| $C_{4y+1}$ | $[4, 2y - 1]$ | $C_{4x+1}$ | $2x + 4y + 3$ |
| $C_{4y+1}$ | $2y$ | $C_{4x+1}$ | $[2x + 4y + 4, 4x + 4y + 1]$ |
| $C_{4y+1}$ | $2y + 1$ | $C_{4x+1}$ | $4x + 4y + 2$ |

Thus, the edges of $G'$ have lengths $(\bigcup_{i=1}^{4} E_i^*) \bigcup \{1, 2, 3, 2y, 2y+1, 2y+2, 2x+4y+3, 4x+4y+2\} = [1, 4x+4y+2]$. Therefore, we have a $\rho$-labeling of $G'$. By letting $A = (\bigcup_{i=1}^{4} A_i) \bigcup \{y-2, y, 6x+2y-1\} \subset [0, 6x+2y-1]$ and $B = (\bigcup_{i=1}^{4} B_i) \bigcup \{8x+6y+2, 8x+7y+3, 8x+8y+2, 8x+8y+4\} \subset [6x+6y+1, 8x+8y+4]$, we get $\ell = 2$, $d_1 = 1$, and $d_2 = 4y+2 \ge 10$. Thus our $\rho$-labeling $\alpha$-accommodates graphs of size at least 3.

**Case 3:** $G' = C_{4x+3} \bigcup C_{4y+3}$ where $y \ge x \ge 0$.
Let $C_{4x+3} = G_1 + G_2 + (2x, 2x+2, 6x+4y+8, 0)$ and $C_{4y+3} = G_3 + G_4 + (2x+2y+3, 2x+2y+4, 4x+4y+7, 2x+3)$, where

$$\begin{aligned}
G_1 &= P(0, 4x + 4y + 7, 2x), \\
G_2 &= P(x, 7x + 4y + 9, 2x), \\
G_3 &= P(2x + 3, 4x + 2y + 6, 2y - 2x), \\
G_4 &= P(x + y + 3, x + y + 5, 2x + 2y).
\end{aligned}$$

Note that when $x = 0$, $G_1$ and $G_2$ are empty paths consisting of a single vertex labeled 0, when $x = y$, $G_3$ is an empty path consisting of a single vertex labeled $2x + 3$, and when $y = 0$, $G_4$ is an empty path consisting of a single vertex labeled 3. However, this does not change the proof.

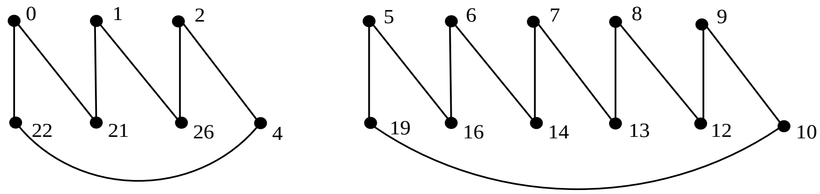


Figure 5: A $\rho$-labeling of $C_7 \bigcup C_{11}$ that $\alpha$-accommodates graphs of size at least 2.

We start by showing that $G_1 + G_2 + (2x, 2x+2, 6x+4y+8, 0)$ is a cycle of length $4x+3$, and $G_3 + G_4 + (2x+2y+3, 2x+2y+4, 4x+4y+7, 2x+3)$ is a cycle of length $4y+3$. Note that by P1, the first vertex of $G_1$ is 0 and the last is $x$, the first vertex of $G_2$ is $x$ and the last is $2x$, the first vertex of $G_3$ is $2x+3$ and the last is $x+y+3$, the first vertex of $G_4$ is $x+y+3$ and the last is $2x+2y+3$. For $1 \leq i \leq 4$, let $A_i$ and $B_i$ denote the sets labeled $A'$ or $B'$ in P2, corresponding to the path $G_i$. Then using P2, we compute

$$\begin{aligned}
A_1 &= [0, x], & B_1 &= [5x+4y+8, 6x+4y+7], \\
A_2 &= [x, 2x], & B_2 &= [8x+4y+10, 9x+4y+9], \\
A_3 &= [2x+3, x+y+3], & B_3 &= [3x+3y+7, 2x+4y+6], \\
A_4 &= [x+y+3, 2x+2y+3], & B_4 &= [2x+2y+6, 3x+3y+5].
\end{aligned}$$

Using the assumptions that $y \geq x \geq 0$, we can check that $\{0\} \leq A_1 \leq A_2 \leq \{2x, 2x+2, 2x+3\} \leq A_3 \leq A_4 \leq \{2x+2y+3, 2x+2y+4\} < B_4 < B_3 < \{4x+4y+7\} < B_1 < \{6x+4y+8\} < B_2 < 8x+8y+12$. Also note that $V(G_1) \bigcap V(G_2) = \{5x+2y-2\}$ and $V(G_3) \bigcap V(G_4) = \{x+y+3\}$; otherwise, $G_i$ and $G_j$ are vertex-disjoint. Thus $G_1 + G_2 + (2x, 2x+2, 6x+4y+8, 0)$ is a cycle of length $4x+3$, and $G_3 + G_4 + (2x+2y+3, 2x+2y+4, 4x+4y+7, 2x+3)$ is a cycle of length $4y+3$.

Now let $E_i$ denote the set of edge labels in $G_i$ for $1 \leq i \leq 4$. By P3, we have

$$\begin{aligned}
E_1^* &= [4x+4y+8, 6x+4y+7]^* = [2x+4y+6, 4x+4y+5], \\
E_2^* &= [6x+4y+10, 8x+4y+9]^* = [4y+4, 2x+4y+3], \\
E_3^* &= [2x+2y+4, 4y+3]^* = [2x+2y+4, 4y+3], \\
E_4^* &= [3, 2x+2y+2]^* = [3, 2x+2y+2].
\end{aligned}$$

Putting the edge length sets in order, we get:

| cycle | edge labels | cycle | edge labels |
|---|---|---|---|
| $C_{4y+3}$ | $1$ | $C_{4x+3}$ | $[4y+4, 2x+4y+3]$ |
| $C_{4x+3}$ | $2$ | $C_{4y+3}$ | $2x+4y+4$ |
| $C_{4y+3}$ | $[3, 2x+2y+2]$ | $C_{4x+3}$ | $2x+4y+5$ |
| $C_{4y+3}$ | $2x+2y+3$ | $C_{4x+3}$ | $[2x+4y+6, 4x+4y+5]$ |
| $C_{4y+3}$ | $[2x+2y+4, 4y+3]$ | $C_{4x+3}$ | $4x+4y+6$ |

Thus, the edges of $G'$ have lengths $(\bigcup_{i=1}^{4} E_i^*) \bigcup \{1, 2, 2x+2y+3, 2x+4y+4, 2x+4y+5, 4x+4y+6\} = [1, 4x+4y+6]$. Therefore, we have a $\rho$-labeling of $G^{'}$. It is easily checked that this $\rho$-labeling $\alpha$-accommodates graphs of size at least two by taking $A = (\bigcup_{i=1}^{4} A_i) \bigcup \{2x+2, 2x+2y+4\} \subset [0, 2x+2y+4]$ and $B = (\bigcup_{i=1}^{4} B_i) \bigcup \{4x+4y+7, 6x+4y+8\} \subset [2x+2y+6, 9x+4y+9]$ yielding $\ell = 2$, $d_1 = 4y - x + 4 \geq 4$, and $d_2 = 2$.

**Case 4:** $G^{'} = C_{4x+2} \bigcup C_5 \bigcup C_5$ where $x \geq 1$.
Let $C_{4x+2} = G_1 + G_2 + (6x+3, 6x+15, 6x+5, 8x+16, 4x+4)$, the first $C_5 = (0, 8x+17, 3, 8x+19, 8x+18, 0)$, and the second $C_5 = (6x+6, 6x+10, 6x+7, 6x+9, 6x+11, 6x+6)$, where

$$G_1 = P(4x+4, 6x+15, 2x),$$
$$G_2 = P(5x+4, 5x+16, 2x-2).$$

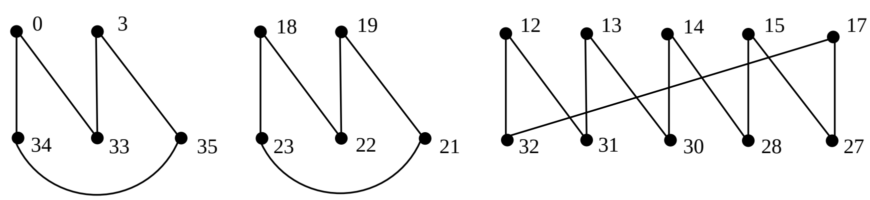


Figure 6: A $\rho$-labeling of $C_5 \bigcup C_5 \bigcup C_{10}$ that $\alpha$-accommodates graphs of size at least 3.

Note that when $x = 1$, $G_2$ is an empty path consisting of a single vertex labeled 9. However, this does not change the proof.

We start by showing that $G_1+G_2+(6x+3, 6x+15, 6x+5, 8x+16, 4x+4)$ is a cycle of length $4x+2$. Note that by P1, the first vertex of $G_1$ is $4x+4$ and the last is $5x+4$, and the first vertex of $G_2$ is $5x+4$ and the last is $6x+3$. For $1 \leq i \leq 2$, let $A_i$ and $B_i$ denote the sets labeled $A'$ or $B'$ in P2, corresponding to the path $G_i$. Then using P2, we compute

$$A_1 = [4x+4, 5x+4], \qquad B_1 = [7x+16, 8x+15],$$
$$A_2 = [5x+4, 6x+3], \qquad B_2 = [6x+16, 7x+14].$$

Using the assumption that $x \geq 1$, we can check that $\{0,3\} < \{4x+4\} \leq A_1 \leq A_2 \leq \{6x+3, 6x+5\} < \{6x+6, 6x+7, 6x+9, 6x+10, 6x+11\} < \{6x+15\} < B_2 < B_1 < \{8x+16\} < \{8x+17, 8x+18, 8x+19\}$. Also note that $V(G_1) \bigcap V(G_2) = \{5x+4\}$; otherwise, $G_i$ and $G_j$ are vertex-disjoint. Thus $G_1 + G_2 + (6x+3, 6x+15, 6x+5, 8x+16, 4x+4)$ is a cycle of length $4x+2$.

Now let $E_i$ denote the set of edge labels in $G_i$ for $1 \leq i \leq 2$. By P3, we have

$$E_1^* = [2x+12, 4x+11]^* = [2x+12, 4x+11],$$
$$E_2^* = [13, 2x+10]^* = [13, 2x+10].$$

Putting the edge length sets in order, we get:

| cycle | edge labels | cycle | edge labels |
|---|---|---|---|
| first $C_5$ | 1 | first $C_5$ | 9 |
| second $C_5$ | 2 | $C_{4x+2}$ | 10 |
| second $C_5$ | 3 | first $C_5$ | 11 |
| second $C_5$ | 4 | $C_{4x+2}$ | 12 |
| second $C_5$ | 5 | $C_{4x+2}$ | $[13, 2x+10]$ |
| second $C_5$ | 6 | $C_{4x+2}$ | $2x+11$ |
| first $C_5$ | 7 | $C_{4x+2}$ | $[2x+12, 4x+11]$ |
| first $C_5$ | 8 | $C_{4x+2}$ | $4x+12$ |

Thus, the edges of $G'$ have lengths $(\bigcup_{i=1}^{2} E_i^*) \bigcup \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 2x+11, 4x+12\} = [1, 4x+12]$. Therefore, we have a $\rho$-labeling of $G^{'}$. It is easily checked that this $\rho$-labeling $\alpha$-accommodates graphs of size at least three by taking $A = (\bigcup_{i=1}^{2} A_i) \bigcup \{0, 3, 6x+5, 6x+6, 6x+7, 6x+9\} \subset [0, 6x+9]$ and $B = (\bigcup_{i=1}^{2} B_i) \bigcup \{6x+10, 6x+11, 6x+15, 8x+16, 8x+17, 8x+18, 8x+19\} \subset [6x+10, 8x+19]$ yielding $\ell = 2$, $d_1 = 6$, and $d_2 = 1$.

**Case 5:** $G^{'} = C_{4x+2} \bigcup C_{4y+1} \bigcup C_{4z+1}$ where $x, z \geq 1$, $y \geq 2$, and $y \geq z$. Let $C_{4x+2} = G_1 + G_2 + (6x+2y+1, 6x+6y+4z+5, 6x+2y+3, 8x+6y+4z+6, 4x+2y+2)$, $C_{4y+1} = G_3 + (y-1, 8x+7y+4z+6, y+1) + G_4 + (2y-1, 8x+6y+4z+7, 2y+1, 8x+8y+4z+7, 8x+8y+4z+6, 0)$, and $C_{4z+1} = G_5 + G_6 + (6x+2y+2z+3, 6x+2y+2z+5, 6x+2y+4z+6, 6x+2y+4)$, where

$$G_1 = P(4x+2y+2, 6x+6y+4z+5, 2x),$$
$$G_2 = P(5x+2y+2, 5x+6y+4z+6, 2x-2),$$
$$G_3 = P(0, 8x+6y+4z+7, 2y-2),$$
$$G_4 = P(y+1, 8x+5y+4z+9, 2y-4),$$
$$G_5 = P(6x+2y+4, 6x+2y+2z+5, 2z),$$
$$G_6 = P(6x+2y+z+4, 6x+2y+z+6, 2z-2).$$

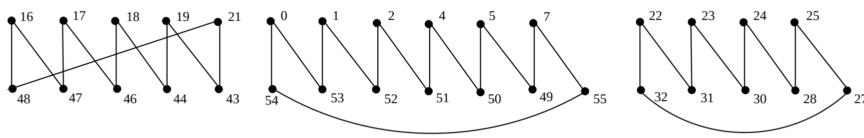


Figure 7: A $\rho$-labeling of $C_{10} \bigcup C_{13} \bigcup C_9$ that $\alpha$-accommodates graphs of size at least 3.

Note that when $x = 1$, $G_2$ is an empty path consisting of a single vertex labeled $2y + 7$, when $y = 2$, $G_4$ is an empty path consisting of a single vertex labeled 3, and when $z = 1$, $G_6$ is an empty path consisting of a single vertex labeled $6x + 2y + 5$. However, this does not change the proof.

We start by showing that $G_1 + G_2 + (6x + 2y + 1, 6x + 6y + 4z + 5, 6x + 2y + 3, 8x + 6y + 4z + 6, 4x + 2y + 2)$ is a cycle of length $4x + 2$, $G_3 + (y - 1, 8x + 7y + 4z + 6, y + 1) + G_4 + (2y - 1, 8x + 6y + 4z + 7, 2y + 1, 8x + 8y + 4z + 7, 8x + 8y + 4z + 6, 0)$ is a cycle of length $4y + 1$, and $G_5 + G_6 + (6x + 2y + 2z + 3, 6x + 2y + 2z + 5, 6x + 2y + 4z + 6, 6x + 2y + 4)$ is a cycle of length $4z+1$. Note that by P1, the first vertex of $G_1$ is $4x+2y+2$ and the last is $5x + 2y + 2$, the first vertex of $G_2$ is $5x + 2y + 2$ and the last is $6x+2y+1$, the first vertex of $G_3$ is 0 and the last is $y-1$, the first vertex of $G_4$ is $y + 1$ and the last is $2y - 1$, the first vertex of $G_5$ is $6x + 2y + 4$ and the last is $6x + 2y + z + 4$, and the first vertex of $G_6$ is $6x + 2y + z + 4$ and the last is $6x + 2y + 2z + 3$. For $1 \leq i \leq 6$, let $A_i$ and $B_i$ denote the sets labeled $A'$ or $B'$ in P2, corresponding to the path $G_i$. Then using P2, we compute

$$A_1 = [4x + 2y + 2, 5x + 2y + 2], \qquad B_1 = [7x + 6y + 4z + 6, 8x + 6y + 4z + 5],$$
$$A_2 = [5x + 2y + 2, 6x + 2y + 1], \qquad B_2 = [6x + 6y + 4z + 6, 7x + 6y + 4z + 4],$$
$$A_3 = [0, y - 1], \qquad B_3 = [8x + 7y + 4z + 7, 8x + 8y + 4z + 5],$$
$$A_4 = [y + 1, 2y - 1], \qquad B_4 = [8x + 6y + 4z + 8, 8x + 7y + 4z + 5],$$
$$A_5 = [6x + 2y + 4, 6x + 2y + z + 4], \qquad B_5 = [6x + 2y + 3z + 6, 6x + 2y + 4z + 5],$$
$$A_6 = [6x + 2y + z + 4, 6x + 2y + 2z + 3], \qquad B_6 = [6x + 2y + 2z + 6, 6x + 2y + 3z + 4].$$

Using the assumptions that $x, z \geq 1$, $y \geq 2$, and $y \geq z$, we can check that $\{0\} \leq A_3 \leq \{y - 1, y + 1\} \leq A_4 \leq \{2y - 1, 2y + 1, 4x + 2y + 2\} \leq A_1 \leq A_2 \leq \{6x + 2y + 1, 6x + 2y + 3, 6x + 2y + 4\} \leq A_5 \leq A_6 \leq \{6x + 2y + 2z +$

$3, 6x+2y+2z+5\} < B_6 < B_5 < \{6x+2y+4z+6, 6x+6y+4z+5\} < B_2 < B_1 < \{8x+6y+4z+6, 8x+6y+4z+7\} < B_4 < \{8x+7y+4z+6\} < B_3 < \{8x+8y+4z+6, 8x+8y+4z+7\}$. Also note that $V(G_1) \bigcap V(G_2) = \{5x+2y+2\}$, and $V(G_5) \bigcap V(G_6) = \{6x+2y+z+4\}$; otherwise, $G_i$ and $G_j$ are vertex-disjoint. Thus the vertices of our graph are distinct. Thus $G_1+G_2+(6x+2y+1, 6x+6y+4z+5, 6x+2y+3, 8x+6y+4z+6, 4x+2y+2)$ is a cycle of length $4x+2$, $G_3+(y-1, 8x+7y+4z+6, y+1)+G_4+(2y-1, 8x+6y+4z+7, 2y+1, 8x+8y+4z+7, 8x+8y+4z+6, 0)$ is a cycle of length $4y+1$, and $G_5+G_6+(6x+2y+2z+3, 6x+2y+2z+5, 6x+2y+4z+6, 6x+2y+4)$ is a cycle of length $4z+1$.

Now let $E_i$ denote the set of edge labels in $G_i$ for $1 \le i \le 6$. By P3, we have

$$E_1^* = [2x+4y+4z+4, 4x+4y+4z+3]^* = [2x+4y+4z+4, 4x+4y+4z+3],$$
$$E_2^* = [4y+4z+5, 2x+4y+4z+2]^* = [4y+4z+5, 2x+4y+4z+2],$$
$$E_3^* = [8x+6y+4z+8, 8x+8y+4z+5]^* = [4z+4, 2y+4z+1],$$
$$E_4^* = [8x+4y+4z+9, 8x+6y+4z+4]^* = [2y+4z+5, 4y+4z],$$
$$E_5^* = [2z+2, 4z+1]^* = [2z+2, 4z+1],$$
$$E_6^* = [3, 2z]^* = [3, 2z].$$

Putting the edge length sets in order, we get:

| cycle | edge labels |
|---|---|
| $C_{4y+1}$ | $1$ |
| $C_{4z+1}$ | $2$ |
| $C_{4z+1}$ | $[3, 2z]$ |
| $C_{4z+1}$ | $2z+1$ |
| $C_{4z+1}$ | $[2z+2, 4z+1]$ |
| $C_{4z+1}$ | $4z+2$ |
| $C_{4y+1}$ | $4z+3$ |
| $C_{4y+1}$ | $[4z+4, 2y+4z+1]$ |
| $C_{4y+1}$ | $2y+4z+2$ |
| $C_{4z+1}$ | $2y+4z+3$ |

| cycle | edge labels |
|---|---|
| $C_{4y+1}$ | $2y+4z+4$ |
| $C_{4y+1}$ | $[2y+4z+5, 4y+4z]$ |
| $C_{4y+1}$ | $4y+4z+1$ |
| $C_{4x+2}$ | $4y+4z+2$ |
| $C_{4y+1}$ | $4y+4z+3$ |
| $C_{4x+2}$ | $4y+4z+4$ |
| $C_{4x+2}$ | $[4y+4z+5, 2x+4y+4z+2]$ |
| $C_{4x+2}$ | $2x+4y+4z+3$ |
| $C_{4x+2}$ | $[2x+4y+4z+4, 4x+4y+4z+3]$ |
| $C_{4x+2}$ | $4x+4y+4z+4$ |

Thus, the edges of $G'$ have lengths $(\bigcup_{i=1}^{6} E_i^*) \bigcup \{1, 2, 2z+1, 4z+2, 4z+3, 2y+4z+2, 2y+4z+3, 2y+4z+4, 4y+4z+1, 4y+4z+2, 4y+4z+3, 4y+4z+4, 2x+4y+4z+3, 4x+4y+4z+4\} = [1, 4x+4y+4z+4]$. Therefore, we have a $\rho$-labeling of $G^{'}$. It is easily checked that this $\rho$-labeling $\alpha$-accommodates graphs of size at least 3 by taking $A = (\bigcup_{i=1}^{6} A_i) \bigcup \{2y+1, 6x+2y+3, 6x+2y+2z+5\} \subset [0, 6x+2y+2z+5]$ and $B = (\bigcup_{i=1}^{6} B_i) \bigcup \{6x+2y+4z+6, 6x+6y+4z+5, 6x+6y+4z+6, 8x+6y+4z+7, 8x+7y+4z+6, 8z+8y+4z+6, 8x+8y+4z+7\} \subset [6x+2y+2z+6, 8x+8y+4z+7]$ yielding $\ell = 2$, $d_1 = 4z+2 \ge 6$, and $d_2 = 1$.

**Case 6:** $G^{'} = C_{4x+2} \bigcup C_3 \bigcup C_{4y+3}$ where $x \ge 1$ and $y \ge 0$.
Let $C_{4x+2} = G_1 + G_2 + (2x+1, 4x+4y+8, 1)$, $C_3 = (0, 4x+4y+11, 4x+4y+9, 0)$, and $C_{4y+3} = G_3 + G_4 + (2x+2y+2, 2x+2y+3, 2x+4y+6, 2x+2)$,

where

$$\begin{aligned} G_1 &= P(1, 6x + 4y + 11, 2x + 2), \\ G_2 &= P(x + 2, 5x + 4y + 13, 2x - 2), \\ G_3 &= P(2x + 2, 2x + 2y + 5, 2y), \\ G_4 &= P(2x + y + 2, 2x + y + 4, 2y). \end{aligned}$$

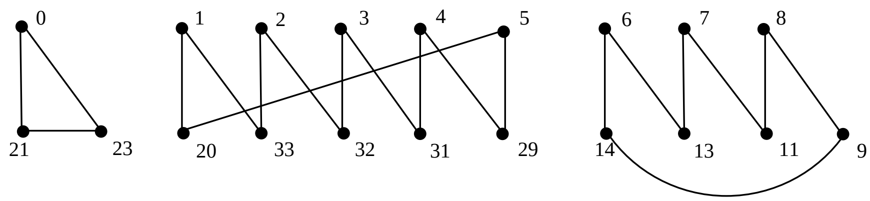


Figure 8: A $\rho$-labeling of $C_3 \bigcup C_{10} \bigcup C_7$ that $\alpha$-accommodates graphs of size at least 2.

Note that when $x = 1$, $G_2$ is an empty path consisting of a single vertex labeled 3, and when $y = 0$, $G_3$ and $G_4$ are empty paths consisting of a single vertex labeled $2x + 2$. However, this does not change the proof.

We start by showing that $G_1 + G_2 + (2x+1, 4x+4y+8, 1)$ is a cycle of length $4x+2$, and $G_3 + G_4 + (2x + 2y + 2, 2x + 2y + 3, 2x + 4y + 6, 2x + 2)$ is a cycle of length $4y + 3$. Note that by P1, the first vertex of $G_1$ is 1 and the last is $x + 2$, the first vertex of $G_2$ is $x + 2$ and the last is $2x + 1$, the first vertex of $G_3$ is $2x + 2$ and the last is $2x + y + 2$, and the first vertex of $G_4$ is $2x + y + 2$ and the last is $2x + 2y + 2$. For $1 \leq i \leq 4$, let $A_i$ and $B_i$ denote the sets labeled $A'$ or $B'$ in P2, corresponding to the path $G_i$. Then using P2, we compute

$$\begin{aligned} A_1 &= [1, x + 2], & B_1 &= [7x + 4y + 13, 8x + 4y + 13], \\ A_2 &= [x + 2, 2x + 1], & B_2 &= [6x + 4y + 13, 7x + 4y + 11], \\ A_3 &= [2x + 2, 2x + y + 2], & B_3 &= [2x + 3y + 6, 2x + 4y + 5], \\ A_4 &= [2x + y + 2, 2x + 2y + 2], & B_4 &= [2x + 2y + 5, 2x + 3y + 4]. \end{aligned}$$

Using the assumptions that $x \geq 1$ $y \geq 0$, we can check that $\{0\} < \{1\} \leq A_1 \leq A_2 \leq \{2x+1, 2x+2\} \leq A_3 \leq A_4 \leq \{2x+2y+2, 2x+2y+3\} < B_4 < B_3 < \{2x+4y+6, 4x+4y+8\} < \{4x+4y+9, 4x+4y+11\} < B_2 < B_1$. Also note that $V(G_1) \bigcap V(G_2) = \{x + 2\}$, and $V(G_3) \bigcap V(G_4) = \{2x + y + 2\}$; otherwise, $G_i$ and $G_j$ are vertex-disjoint. Thus $G_1 + G_2 + (2x + 1, 4x + 4y + 8, 1)$ is a cycle of length $4x + 2$, and $G_3 + G_4 + (2x + 2y + 2, 2x + 2y + 3, 2x + 4y + 6, 2x + 2)$ is a cycle of length $4y + 3$.

Now let $E_i$ denote the set of edge labels in $G_i$ for $1 \leq i \leq 4$. By P3, we have

$$E_1^* = [6x+4y+11, 8x+4y+12]^* = [4y+5, 2x+4y+6],$$
$$E_2^* = [4x+4y+12, 6x+4y+9]^* = [2x+4y+8, 4x+4y+5],$$
$$E_3^* = [2y+4, 4y+3]^* = [2y+4, 4y+3],$$
$$E_4^* = [3, 2y+2]^* = [3, 2y+2].$$

Putting the edge length sets in order, we get:

| cycle | edge labels |
|---|---|
| $C_{4y+3}$ | 1 |
| $C_3$ | 2 |
| $C_{4y+3}$ | $[3, 2y+2]$ |
| $C_{4y+3}$ | $2y+3$ |
| $C_{4y+3}$ | $[2y+4, 4y+3]$ |
| $C_{4y+3}$ | $4y+4$ |

| cycle | edge labels |
|---|---|
| $C_{4x+2}$ | $[4y+5, 2x+4y+6]$ |
| $C_{4x+2}$ | $2x+4y+7$ |
| $C_{4x+2}$ | $[2x+4y+8, 4x+4y+5]$ |
| $C_3$ | $4x+4y+6$ |
| $C_{4x+2}$ | $4x+4y+7$ |
| $C_3$ | $4x+4y+8$ |

Thus, the edges of $G'$ have lengths $(\bigcup_{i=1}^4 E_i^*) \bigcup \{1, 2, 2x+3, 4y+4, 2x+4y+7, 4x+4y+6, 4x+4y+7, 4x+4y+8\} = [1, 4x+4y+8]$. Therefore, we have a $\rho$-labeling of $G^{'}$. It is easily checked that this $\rho$-labeling $\alpha$-accommodates graphs of size at least 2 by taking $A = (\bigcup_{i=1}^4 A_i) \bigcup \{0, 2x+2y+3\} = [0, 2x+2y+3]$ and $B = (\bigcup_{i=1}^4 B_i) \bigcup \{2x+4y+6, 4x+4y+8, 4x+4y+9, 4x+4y+11\} \subset [2x+2y+5, 8x+4y+13]$ yielding $\ell = 2$, $d_1 = 4y+4 \geq 4$, and $d_2 = 2$.

**Case 7:** $G^{'} = C_{4x+2} \bigcup C_{4y+3} \bigcup C_{4z+3}$ where $x, y, z \geq 1$ and $y \geq z$.
Let $C_{4x+2} = G_1 + G_2 + (6x+2y+2, 6x+6y+4z+10, 6x+2y+4, 8x+6y+4z+11, 4x+2y+3)$, $C_{4y+3} = G_3 + (y-1, 8x+7y+4z+12, y+1) + G_4 + (2y, 8x+6y+4z+12, 2y+2, 8x+8y+4z+14, 8x+8y+4z+12, 0)$, and $C_{4z+3} = G_5 + G_6 + (6x+2y+2z+5, 6x+2y+2z+6, 6x+2y+4z+9, 6x+2y+5)$, where

$$G_1 = P(4x+2y+3, 6x+6y+4z+10, 2x),$$
$$G_2 = P(5x+2y+3, 5x+6y+4z+11, 2x-2),$$
$$G_3 = P(0, 8x+6y+4z+13, 2y-2),$$
$$G_4 = P(y+1, 8x+5y+4z+13, 2y-2),$$
$$G_5 = P(6x+2y+5, 6x+2y+2z+8, 2z),$$
$$G_6 = P(6x+2y+z+5, 6x+2y+z+7, 2z).$$

Note that when $x = 1$, $G_2$ is an empty path consisting of a single vertex labeled $2y+8$, and when $y = 1$, $G_3$ and $G_4$ are empty paths consisting of

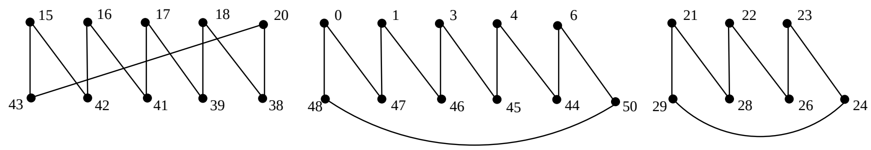


Figure 9: A $\rho$-labeling of $C_{10} \bigcup C_{11} \bigcup C_7$ that $\alpha$-accommodates graphs of size at least 2.

single vertices labeled 0 and 2, respectively. However, this does not change the proof.

We start by showing that $G_1 + G_2 + (6x+2y+2, 6x+6y+4z+10, 6x+2y+4, 8x+6y+4z+11, 4x+2y+3)$ is a cycle of length $4x+2$, $G_3 + (y-1, 8x+7y+4z+12, y+1) + G_4 + (2y, 8x+6y+4z+12, 2y+2, 8x+8y+4z+14, 8x+8y+4z+12, 0)$ is a cycle of length $4y+3$, and $G_5+G_6+(6x+2y+2z+5, 6x+2y+2z+6, 6x+2y+4z+9, 6x+2y+5)$ is a cycle of length $4z+3$. Note that by P1, the first vertex of $G_1$ is $4x+2y+3$ and the last is $5x+2y+3$, the first vertex of $G_2$ is $5x+2y+3$ and the last is $6x+2y+2$, the first vertex of $G_3$ is 0 and the last is $y-1$, the first vertex of $G_4$ is $y+1$ and the last is $2y$, the first vertex of $G_5$ is $6x+2y+5$ and the last is $6x+2y+z+5$, and the first vertex of $G_6$ is $6x+2y+z+5$ and the last is $6x+2y+2z+5$. For $1 \le i \le 6$, let $A_i$ and $B_i$ denote the sets labeled $A'$ or $B'$ in P2, corresponding to the path $G_i$. Then using P2, we compute

$$\begin{aligned}
A_1 &= [4x+2y+3, 5x+2y+3], & B_1 &= [7x+6y+4z+11, 8x+6y+4z+10],\\
A_2 &= [5x+2y+3, 6x+2y+2], & B_2 &= [6x+6y+4z+11, 7x+6y+4z+9],\\
A_3 &= [0, y-1], & B_3 &= [8x+7y+4z+13, 8x+8y+4z+11],\\
A_4 &= [y+1, 2y], & B_4 &= [8x+6y+4z+13, 8x+7y+4z+11],\\
A_5 &= [6x+2y+5, 6x+2y+z+5], & B_5 &= [6x+2y+3z+9, 6x+2y+4z+8],\\
A_6 &= [6x+2y+z+5, 6x+2y+2z+5], & B_6 &= [6x+2y+2z+8, 6x+2y+3z+7].
\end{aligned}$$

Using the assumptions that $x, y, z \ge 1$ and $y \ge z$, we can check that $\{0\} \le A_3 \le \{y-1, y+1\} \le A_4 \le \{2y, 2y+2, 4x+2y+3\} \le A_1 \le A_2 \le \{6x+2y+2, 6x+2y+4, 6x+2y+5\} \le A_5 \le A_6 \le \{6x+2y+2z+5, 6x+2y+2z+6\} < B_6 < B_5 < \{6x+2y+4z+9, 6x+6y+4z+10\} < B_2 < B_1 < \{8x+6y+4z+11, 8x+6y+4z+12\} < B_4 < \{8x+7y+4z+12\} < B_3 < \{8x+8y+4z+12, 8x+8y+4z+14\}$. Also note that $V(G_1) \bigcap V(G_2) = \{5x+2y+3\}$, and $V(G_5) \bigcap V(G_6) = \{6x+2y+z+5\}$; otherwise, $G_i$ and $G_j$ are vertex-disjoint. Thus $G_1+G_2+(6x+2y+2, 6x+6y+4z+10, 6x+2y+4, 8x+6y+4z+11, 4x+2y+3)$ is a cycle of length $4x+2$, $G_3+(y-1, 8x+7y+4z+12, y+1)+G_4+(2y, 8x+6y+4z+12, 2y+2, 8x+8y+4z+14, 8x+8y+4z+12, 0)$ is a cycle of length $4y+3$, and $G_5+G_6+(6x+2y+2z+5, 6x+2y+2z+6, 6x+2y+4z+9, 6x+2y+5)$ is a cycle of length $4z+3$.

Now let $E_i$ denote the set of edge labels in $G_i$ for $1 \le i \le 6$. By P3, we have

$E_1^* = [2x+4y+4z+8, 4x+4y+4z+7]^* = [2x+4y+4z+8, 4x+4y+4z+7]$,
$E_2^* = [4y+4z+9, 2x+4y+4z+6]^* = [4y+4z+9, 2x+4y+4z+6]$,
$E_3^* = [8x+6y+4z+14, 8x+8y+4z+11]^* = [4z+6, 2y+4z+3]$,
$E_4^* = [8x+4y+4z+13, 8x+6y+4z+10]^* = [2y+4z+7, 4y+4z+4]$,
$E_5^* = [2z+4, 4z+3]^* = [2z+4, 4z+3]$,
$E_6^* = [3, 2z+2]^* = [3, 2z+2]$.

Putting the edge length sets in order, we get:

| cycle | edge labels |
|---|---|
| $C_{4z+3}$ | 1 |
| $C_{4y+3}$ | 2 |
| $C_{4z+3}$ | $[3, 2z+2]$ |
| $C_{4z+3}$ | $2z+3$ |
| $C_{4z+3}$ | $[2z+4, 4z+3]$ |
| $C_{4z+4}$ | $4z+4$ |
| $C_{4y+3}$ | $4z+5$ |
| $C_{4y+3}$ | $[4z+6, 2y+4z+3]$ |
| $C_{4y+3}$ | $2y+4z+4$ |
| $C_{4y+3}$ | $2y+4z+5$ |

| cycle | edge labels |
|---|---|
| $C_{4y+3}$ | $2y+4z+6$ |
| $C_{4y+3}$ | $[2y+4z+7, 4y+4z+4]$ |
| $C_{4y+3}$ | $4y+4z+5$ |
| $C_{4x+2}$ | $4y+4z+6$ |
| $C_{4y+3}$ | $4y+4z+7$ |
| $C_{4x+2}$ | $4y+4z+8$ |
| $C_{4x+2}$ | $[4y+4z+9, 2x+4y+4z+6]$ |
| $C_{4x+2}$ | $2x+4y+4z+7$ |
| $C_{4x+2}$ | $[2x+4y+4z+8, 4x+4y+4z+7]$ |
| $C_{4x+2}$ | $4x+4y+4z+8$ |

Thus, the edges of $G'$ have lengths $(\bigcup_{i=1}^{6} E_i^*) \bigcup \{1, 2, 2z+3, 4z+4, 4z+5, 2y+4z+4, 2y+4z+5, 2y+4z+6, 4y+4z+5, 4y+4z+6, 4y+4z+7, 4y+4z+8, 2x+4y+4z+7, 4x+4y+4z+8\} = [1, 4x+4y+4z+8]$. Therefore, we have a $\rho$-labeling of $G^{'}$. It is easily checked that this $\rho$-labeling $\alpha$-accommodates graphs of size at least 2 by taking $A = (\bigcup_{i=1}^{6} A_i) \bigcup \{2y+2, 6x+2y+4, 6x+2y+2z+6\} \subset [0, 6x+2y+2z+6]$ and $B = (\bigcup_{i=1}^{6} B_i) \bigcup \{6x+2y+4z+9, 6x+6y+4z+10, 8x+6y+4z+11, 8x+6y+4z+12, 8x+7y+4z+12, 8x+8y+4z+12, 8x+8y+4z+14\} \subset [6x+2y+2z+8, 8x+8y+4z+14]$ yielding $\ell = 2$, $d_1 = 4z+3 \geq 7$, and $d_2 = 2$.

**Case 8:** $G^{'} = C_{4x+2} \bigcup C_3 \bigcup C_{4y+1}$ where $x, y \geq 1$.
Let $C_{4x+2} = G_1 + G_2 + (6x+1, 8x+4y+8, 4x+1)$, $C_3 = (0, 8x+4y+10, 8x+4y+9, 0)$, and $C_{4y+1} = G_3 + G_4 + (6x+2y+1, 6x+2y+3, 6x+4y+4, 6x+2)$, where

$$G_1 = P(4x+1, 6x+4y+8, 2x-2),$$
$$G_2 = P(5x, 5x+4y+4, 2x+2),$$
$$G_3 = P(6x+2, 6x+2y+3, 2y),$$
$$G_4 = P(6x+y+2, 6x+y+4, 2y-2).$$

Note that when $x = 1$, $G_1$ is an empty path consisting of a single vertex labeled 5, and when $y = 1$, $G_4$ is an empty path consisting of a single vertex labeled $6x+3$. However, this does not change the proof.

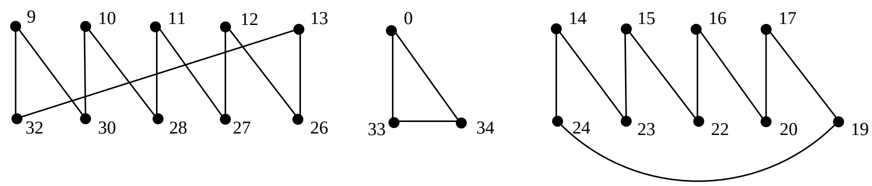


Figure 10: A $\rho$-labeling of $C_{10} \bigcup C_3 \bigcup C_9$ that $\alpha$-accommodates graphs of size at least 3.

We start by showing that $G_1 + G_2 + (6x+1, 8x+4y+8, 4x+1)$ is a cycle of length $4x+2$, and $G_3 + G_4 + (6x+2y+1, 6x+2y+3, 6x+4y+4, 6x+2)$ is a cycle of length $4y+1$. Note that by P1, the first vertex of $G_1$ is $4x+1$ and the last is $5x$, the first vertex of $G_2$ is $5x$ and the last is $6x+1$, the first vertex of $G_3$ is $6x+2$ and the last is $6x+y+2$, and the first vertex of $G_4$ is $6x+y+2$ and the last is $6x+2y+1$. For $1 \le i \le 4$, let $A_i$ and $B_i$ denote the sets labeled $A'$ or $B'$ in P2, corresponding to the path $G_i$. Then using P2, we compute

$$\begin{aligned} A_1 &= [4x+1, 5x], & B_1 &= [7x+4y+8, 8x+4y+6], \\ A_2 &= [5x, 6x+1], & B_2 &= [6x+4y+6, 7x+4y+6], \\ A_3 &= [6x+2, 6x+y+2], & B_3 &= [6x+3y+4, 6x+4y+3], \\ A_4 &= [6x+y+2, 6x+2y+1], & B_4 &= [6x+2y+4, 6x+3y+2]. \end{aligned}$$

Using the assumptions that $x, y \ge 1$, we can check that $\{0\} < \{4x+1\} \le A_1 \le A_2 \le \{6x+1, 6x+2\} \le A_3 \le A_4 \le \{6x+2y+1, 6x+2y+3\} < B_4 < B_3 < \{6x+4y+4\} < B_2 < B_1 < \{8x+4y+8\} < \{8x+4y+9, 8x+4y+10\}$. Also note that $V(G_1) \bigcap V(G_2) = \{5x\}$, and $V(G_3) \bigcap V(G_4) = \{6x+y+2\}$; otherwise, $G_i$ and $G_j$ are vertex-disjoint. Thus $G_1 + G_2 + (6x+1, 8x+4y+8, 4x+1)$ is a cycle of length $4x+2$, and $G_3 + G_4 + (6x+2y+1, 6x+2y+3, 6x+4y+4, 6x+2)$ is a cycle of length $4y+1$.

Now let $E_i$ denote the set of edge labels in $G_i$ for $1 \le i \le 4$. By P3, we have

$$\begin{aligned} E_1^* &= [2x+4y+8, 4x+4y+5]^* &&= [2x+4y+8, 4x+4y+5], \\ E_2^* &= [4y+5, 2x+4y+6]^* &&= [4y+5, 2x+4y+6], \\ E_3^* &= [2y+2, 4y+1]^* &&= [2y+2, 4y+1], \\ E_4^* &= [3, 2y]^* &&= [3, 2y]. \end{aligned}$$

Putting the edge length sets in order, we get:

| cycle | edge labels |
|---|---|
| $C_3$ | $1$ |
| $C_{4y+1}$ | $2$ |
| $C_{4y+1}$ | $[3, 2y]$ |
| $C_{4y+1}$ | $2y+1$ |
| $C_{4y+1}$ | $[2y+2, 4y+1]$ |
| $C_{4y+1}$ | $4y+2$ |

| cycle | edge labels |
|---|---|
| $C_3$ | $4y+3$ |
| $C_3$ | $4y+4$ |
| $C_{4x+2}$ | $[4y+5, 2x+4y+6]$ |
| $C_{4x+2}$ | $2x+4y+7$ |
| $C_{4x+2}$ | $[2x+4y+8, 4x+4y+5]$ |
| $C_{4x+2}$ | $4x+4y+6$ |

Thus, the edges of $G'$ have lengths $(\bigcup_{i=1}^{4} E_i^*) \bigcup \{1, 2, 2y+1, 4y+2, 4y+3, 4y+4, 2x+4y+7, 4x+4y+6\} = [1, 4x+4y+6]$. Therefore, we have a $\rho$-labeling of $G^{'}$. It is easily checked that this $\rho$-labeling $\alpha$-accommodates graphs of size at least 3 by taking $A = (\bigcup_{i=1}^{4} A_i) \bigcup \{0, 6x+2y+3\} \subset [0, 6x+2y+3]$ and $B = (\bigcup_{i=1}^{4} B_i) \bigcup \{6x+4y+4, 8x+4y+8, 8x+4y+9, 8x+4y+10\} \subset [6x+2y+4, 8x+4y+10]$ yielding $\ell = 2$, $d_1 = 4y+3 \geq 7$, and $d_2 = 1$.

**Case 9:** $G^{'} = C_{4x+2} \bigcup C_{4y+3} \bigcup C_{4z+1}$ where $x, y, z \geq 1$.
Let $C_{4x+2} = G_1 + G_2 + (6x+2y+2, 8x+6y+4z+9, 4x+2y+2)$, $C_{4y+3} = G_3 + (y-1, 8x+7y+4z+10, y+1) + G_4 + (2y+1, 8x+8y+4z+11, 8x+8y+4z+10, 0)$, and $C_{4z+1} = G_5 + G_6 + (6x+2y+2z+2, 6x+2y+2z+4, 6x+2y+4z+5, 6x+2y+3)$, where

$$G_1 = P(4x+2y+2, 6x+6y+4z+9, 2x-2),$$
$$G_2 = P(5x+2y+1, 5x+6y+4z+5, 2x+2),$$
$$G_3 = P(0, 8x+6y+4z+11, 2y-2),$$
$$G_4 = P(y+1, 8x+5y+4z+9, 2y),$$
$$G_5 = P(6x+2y+3, 6x+2y+2z+4, 2z),$$
$$G_6 = P(6x+2y+z+3, 6x+2y+z+5, 2z-2).$$

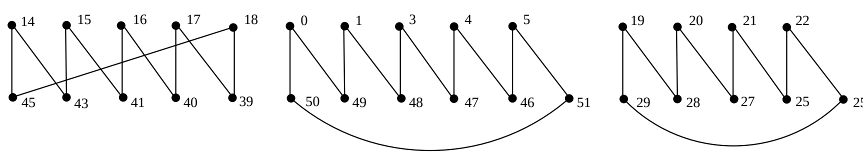


Figure 11: A $\rho$-labeling of $C_{10} \bigcup C_{11} \bigcup C_9$ that $\alpha$-accommodates graphs of size at least 3.

Note that when $x = 1$, $G_1$ is an empty path consisting of a single vertex labeled $2y+6$, when $y = 1$, $G_3$ is an empty path consisting of a single vertex labeled 0, and when $z = 1$, $G_6$ is an empty path consisting of a single vertex labeled $6x + 2y + 4$. However, this does not change the proof.

We start by showing that $G_1+G_2+(6x+2y+2, 8x+6y+4z+9, 4x+2y+2)$ is a cycle of length $4x+2$, $G_3+(y-1, 8x+7y+4z+10, y+1)+G_4+(2y+1, 8x+8y+4z+11, 8x+8y+4z+10, 0)$ is a cycle of length $4y+3$, and $G_5+G_6+(6x+2y+2z+2, 6x+2y+2z+4, 6x+2y+4z+5, 6x+2y+3)$ is a cycle of length $4z+1$. Note that by P1, the first vertex of $G_1$ is $4x+2y+2$ and the last is $5x+2y+1$, the first vertex of $G_2$ is $5x+2y+1$ and the last is $6x+2y+2$, the first vertex of $G_3$ is 0 and the last is $y-1$, the first vertex of $G_4$ is $y+1$ and the last is $2y+1$, the first vertex of $G_5$ is $6x+2y+3$ and the last is $6x+2y+z+3$, and the first vertex of $G_6$ is $6x+2y+z+3$ and the last is $6x+2y+2z+2$. For $1 \leq i \leq 6$, let $A_i$ and $B_i$ denote the sets labeled $A'$ or $B'$ in P2, corresponding to the path $G_i$. Then using P2, we compute

$$
\begin{aligned}
A_1 &= [4x+2y+2, 5x+2y+1], & B_1 &= [7x+6y+4z+9, 8x+6y+4z+7],\\
A_2 &= [5x+2y+1, 6x+2y+2], & B_2 &= [6x+6y+4z+7, 7x+6y+4z+7],\\
A_3 &= [0, y-1], & B_3 &= [8x+7y+4z+11, 8x+8y+4z+9],\\
A_4 &= [y+1, 2y+1], & B_4 &= [8x+6y+4z+10, 8x+7y+4z+9],\\
A_5 &= [6x+2y+3, 6x+2y+z+3], & B_5 &= [6x+2y+3z+5, 6x+2y+4z+4],\\
A_6 &= [6x+2y+z+3, 6x+2y+2z+2], & B_6 &= [6x+2y+2z+5, 6x+2y+3z+3].
\end{aligned}
$$

Using the assumption that $x, y, z \geq 1$, we can check that $\{0\} \leq A_3 \leq \{y-1, y+1\} \leq A_4 \leq \{2y+1, 4x+2y+2\} \leq A_1 \leq A_2 \leq \{6x+2y+2, 6x+2y+3\} \leq A_5 \leq A_6 \leq \{6x+2y+2z+2, 6x+2y+2z+4\} < B_6 < B_5 < \{6x+2y+4z+5\} < B_2 < B_1 < \{8x+6y+4z+9\} < B_4 < \{8x+7y+4z+10\} < B_3 < \{8x+8y+4z+10, 8x+8y+4z+11\}$. Also note that $V(G_1) \bigcap V(G_2) = \{5x+2y+1\}$, and $V(G_5) \bigcap V(G_6) = \{6x+2y+z+3\}$. Thus $G_1+G_2+(6x+2y+2, 8x+6y+4z+9, 4x+2y+2)$ is a cycle of length $4x+2$, $G_3+(y-1, 8x+7y+4z+10, y+1)+G_4+(2y+1, 8x+8y+4z+11, 8x+8y+4z+10, 0)$ is a cycle of length $4y+3$, and $G_5+G_6+(6x+2y+2z+2, 6x+2y+2z+4, 6x+2y+4z+5, 6x+2y+3)$ is a cycle of length $4z+1$.

Now let $E_i$ denote the set of edge labels in $G_i$ for $1 \leq i \leq 6$. By P3, we have

$$
\begin{aligned}
E_1^* &= [2x+4y+4z+8, 4x+4y+4z+5]^* = [2x+4y+4z+8, 4x+4y+4z+5],\\
E_2^* &= [4y+4z+5, 2x+4y+4z+6]^* = [4y+4z+5, 2x+4y+4z+6],\\
E_3^* &= [8x+6y+4z+12, 8x+8y+4z+9]^* = [4z+4, 2y+4z+1],\\
E_4^* &= [8x+4y+4z+9, 8x+6y+4z+8]^* = [2y+4z+5, 4y+4z+4],\\
E_5^* &= [2z+2, 4z+1]^* = [2z+2, 4z+1],\\
E_6^* &= [3, 2z]^* = [3, 2z].
\end{aligned}
$$

Putting the edge length sets in order, we get:

| cycle | edge labels |
|---|---|
| $C_{4y+3}$ | $1$ |
| $C_{4z+1}$ | $2$ |
| $C_{4z+1}$ | $[3, 2z]$ |
| $C_{4z+1}$ | $2z+1$ |
| $C_{4z+1}$ | $[2z+2, 4z+1]$ |
| $C_{4z+1}$ | $4z+2$ |
| $C_{4y+3}$ | $4z+3$ |
| $C_{4y+3}$ | $[4z+4, 2y+4z+1]$ |

| cycle | edge labels |
|---|---|
| $C_{4y+3}$ | $2y+4z+2$ |
| $C_{4y+3}$ | $2y+4z+3$ |
| $C_{4y+3}$ | $2y+4z+4$ |
| $C_{4y+3}$ | $[2y+4z+5, 4y+4z+4]$ |
| $C_{4x+2}$ | $[4y+4z+5, 2x+4y+4z+6]$ |
| $C_{4x+2}$ | $2x+4y+4z+7$ |
| $C_{4x+2}$ | $[2x+4y+4z+8, 4x+4y+4z+5]$ |
| $C_{4x+2}$ | $4x+4y+4z+6$ |

Thus, the edges of $G'$ have lengths $(\bigcup_{i=1}^{6} E_i^*) \bigcup \{1, 2, 2z+1, 4z+2, 4z+3, 2y+4z+2, 2y+4z+3, 2y+4z+4, 2x+4y+4z+7, 4x+4y+4z+6\} = [1, 4x+4y+4z+6]$. Therefore, we have a $\rho$-labeling of $G^{'}$. It is easily checked that this $\rho$-labeling $\alpha$-accommodates graphs of size at least 3 by taking $A = (\bigcup_{i=1}^{6} A_i) \bigcup \{6x+2y+2z+4\} \subset [0, 6x+2y+2z+4]$ and $B = (\bigcup_{i=1}^{6} B_i) \bigcup \{6x+2y+4z+5, 8x+6y+4z+9, 8x+7y+4z+10, 8x+8y+4z+10, 8x+8y+4z+11\} \subset [6x+2y+2z+5, 8x+8y+4z+11]$ yielding $\ell = 2$, $d_1 = 4z+2 \geq 6$, and $d_2 = 1$. □

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