AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **56** (2013), Pages 201–217

On the cyclic decomposition of circulant graphs into bipartite graphs

Ryan C. Bunge Saad I. El-Zanati*

Chepina Rumsey Charles Vanden Eynden

4520 Mathematics Department
Illinois State University
Normal, Illinois 61790–4520
U.S.A.

Abstract

It is known that if a bipartite graph G with n edges possesses any of three types of ordered labelings, then the complete graph K_{2nx+1} admits a cyclic G-decomposition for every positive integer x. We introduce variations of the ordered labelings and show that whenever a bipartite graph G admits one of these labelings, then there exists a cyclic G-decomposition of an infinite family of circulant graphs. We also show that all 2-regular bipartite graphs admit one of these variant labelings.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$. (If $a > b$, then $[a, b] = \varnothing$.) If A and B are subsets of the integers and if $\max(A) \leq \min(B)$, we will write $A \leq B$. We define $A < B$, $A \geq B$, and $A > B$ analogously. If $\{a\} \leq B$, we will write $a \leq B$. Similarly, if $B \leq \{b\}$, then we will write $B \leq b$. Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. The *order* and the *size* of a graph G are $|V(G)|$ and $|E(G)|$, respectively. Unless otherwise noted, we will only consider graphs with no isolated vertices.

Let $V(K_m) = \mathbb{Z}_m$ and let G be a subgraph of K_m. By *clicking* G, we mean applying the isomorphism $i \mapsto i+1$ to $V(G)$. Let H and G be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of pairwise edge-disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. A G-decomposition of K_m is also known as a (K_m, G)-*design*. A (K_m, G)-design Δ is *cyclic* if clicking is a permutation of Δ. For recent surveys on G-designs, see [3] and [7].

* Research supported by National Science Foundation Grant No. A0649210

Unless noted otherwise, we will let $V(K_m) = [0, m-1]$. The *label* of an edge $\{i, j\}$ in K_m is $|i-j|$ while the *length* of $\{i, j\}$ is $\min\{|i-j|, m-|i-j|\}$. We shall refer to an edge $\{i, j\}$ whose length is not $|i-j|$ as a *wrap-around* edge. Note that if $\{i, j\}$ has length $|i-j|$ in K_m, then $\{i, j\}$ will have length $|i-j|$ in $K_{m'}$ for all $m' \geq m$. If m is odd, then K_m consists of m edges of length i for $i \in [1, \frac{m-1}{2}]$. If m is even, then K_m consists of m edges of length i for $i \in [1, \frac{m}{2}-1]$ and $\frac{m}{2}$ edges of length $\frac{m}{2}$; moreover, in this case the edges of length $\frac{m}{2}$ constitute a 1-factor in K_m.

Let $L \subseteq \{1, 2, \ldots, \lfloor m/2 \rfloor\}$. The subgraph of K_m induced by all the edges with lengths in L is called a *circulant graph* and is denoted by $\langle L \rangle_m$. Of course, circulant graphs are *Cayley graphs* on cyclic groups. As noted earlier, $\langle \{m/2\} \rangle_m$ is a 1-factor in K_m when m is even. Otherwise, for $1 \leq i < m/2$, it is easy to see that $\langle \{i\} \rangle_m$ consists of δ vertex disjoint cycles $C_{m/\delta}$, where $\delta = \gcd(i, m)$.

Let k and n be positive integers and let G be a graph of size n. It would be of interest to know whether there exists a G-decomposition of the circulant $\langle [k, n+k-1] \rangle_{2n+2k-1}$. When $k = 1$, the circulant $\langle [k, n+k-1] \rangle_{2n+2k-1}$ is the complete graph K_{2n+1}. A popular conjecture of Ringel [19] states that there exists a (K_{2n+1}, G)-design for every tree G of size n. It is very likely that every tree of size n will decompose the circulant $\langle [k, n+k-1] \rangle_{2n+2k-1}$ for every positive integer k. In fact, it would be of interest to know what graphs of size n do not decompose $\langle [k, n+k-1] \rangle_{2n+2k-1}$ for some positive k.

A popular approach to dealing with Ringel's Conjecture is the use of graph labelings. In fact, numerous conjectures in graph labelings are stronger than Ringel's Conjecture (see [14]). For example, Kotzig (see [20]) conjectures that every tree admits what is called a ρ-labeling. This would imply that there is a cyclic (K_{2n+1}, G)-design for every tree G of size n. It can be conjectured similarly that there is a cyclic G-decomposition of $\langle [k, n+k-1] \rangle_{2n+2k-1}$ for every tree G of size n.

1.1 Extensions of Rosa-type Labelings

For any graph G, a one-to-one function $f\colon V(G) \to \mathbb{N}$ is called a *labeling* (or *valuation*) of G. In [20], Rosa introduced a hierarchy of labelings. We generalize Rosa's labelings and add a few items to this hierarchy. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u)\colon u \in V(G)\}$. Define a function $\bar{f}\colon E(G) \to \mathbb{N}$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. If $F \subseteq E(G)$, let $\bar{f}(F) = \{\bar{f}(e)\colon e \in F\}$. Let k be a positive integer and consider the following conditions:

(ℓ1) $f(V(G)) \subseteq [0, 2(n+k-1)]$,

(ℓ2) $f(V(G)) \subseteq [0, n+k-1]$,

(ℓ3) $\bar{f}(E(G)) = \{x_k, x_{k+1}, \ldots, x_{n+k-1}\}$, where for each $i \in [k, n+k-1]$ either $x_i = i$ or $x_i = 2(n+k-1)+1-i = 2(n+k)-1-i$,

(ℓ4) $\bar{f}(E(G)) = [k, n+k-1]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ (with every edge in G having one end vertex in A and the other in B), consider also

($\ell 5$) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,

($\ell 6$) there exists an integer λ (called a *boundary value* of f) such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

($\ell 1$) and ($\ell 3$) is called a ρ_k*-labeling*;

($\ell 1$) and ($\ell 4$) is called a σ_k*-labeling*;

($\ell 2$) and ($\ell 4$) is called a β_k*-labeling*.

A β_k-labeling is necessarily a σ_k-labeling which in turn is a ρ_k-labeling. When $k = 1$, these labelings correspond, respectively, to the β, σ, and ρ-labelings that were introduced by Rosa [20]. We shall refer to the labelings introduced above simply as k*-labelings*.

If G is bipartite and a ρ_k, σ_k, or β_k-labeling of G also satisfies ($\ell 5$), then the labeling is *ordered* and is denoted by ρ_k^+, σ_k^+, or β_k^+, respectively. If in addition ($\ell 6$) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ_k^{++}, σ_k^{++}, or β_k^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an α*-labeling* as introduced in [20]. Because the concept of an α-labeling is well known, we will call a β^{++}-labeling an α-labeling, and we will use the notation α_k in place of β_k^{++}. Moreover, what we are calling a β_k-labeling was previously independently introduced as a k*-graceful labeling* by Slater [21] and by Mahéo and Thuillier [18].

The following lemma shows that if a bipartite graph G admits an ordered k-labeling, then G admits a uniformly-ordered $(k + m)$-labeling for all but a finite number of positive integers m.

Lemma 1. *Let G be a bipartite graph with no isolated vertices and vertex bipartition $\{A, B\}$ and let k be a positive integer. Let f be an ordered k-labeling of G with $f(a) < f(b)$ for every $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$. Let $D = \{f(a) - f(b)\colon\ a \in A, b \in B$ with $f(a) > f(b)\}$. If f is a β_k^+, σ_k^+, or ρ_k^+-labeling, then G admits a β_{k+m}^+, σ_{k+m}^+, or ρ_{k+m}^+-labeling, respectively for all $m \in \mathbb{N} \setminus D$. Moreover, if $m > D$, then the $(k + m)$-labeling of G is uniformly-ordered.*

Proof. Suppose G has n edges and vertex bipartition $\{A, B\}$. Let k be a positive integer and let f be a β_k^+, σ_k^+, or ρ_k^+-labeling of G such that $f(a) < f(b)$ for all $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$. Also, let $D = \{f(a) - f(b)\colon\ a \in A, b \in B$ with $f(a) > f(b)\}$ and let $m \in \mathbb{N} \setminus D$. Consider the labeling $f'\colon\ V(G) \to [0, 2n + 2k + 2m - 1]$ defined by $f'(u) = f(u)$ if $u \in A$ and $f'(v) = f(v) + m$ if $v \in B$. Since $m \neq f(a) - f(b)$ for any $a \in A$ and $b \in B$, we have $f'(v) = f(v) + m \neq f(u) = f'(u)$ for any $u \in A$

and $v \in B$. Thus f' is one-to-one. Depending on which type of ordered k-labeling f is, it is simple to verify that f' is the corresponding ordered $(k+m)$-labeling.

Now, suppose m exceeds all elements in D. Then $m > f(a) - f(b)$ for any $a \in A$ and $b \in B$, and we have $f'(v) = f(v) + m > f(u) = f'(u)$ for any $u \in A$ and $v \in B$, i.e. $f'(B) > f'(A)$. Thus f' is uniformly ordered. ∎

If the k-labeling f in Lemma 1 is uniformly-ordered, then D is empty and and the resulting $(k+m)$-labeling is also uniformly-ordered.

Corollary 2. *Let G be a bipartite graph and let k and m be positive integers. If G admits an α_k, σ_k^{++}, or ρ_k^{++}-labeling, then G also admits an α_{k+m}, σ_{k+m}^{++}, or ρ_{k+m}^{++}-labeling, respectively.*

It is well known (see [20]) that if a graph G of size n has all even degrees and if G admits a σ-labeling, then we must have $n \equiv 0$ or $3 \pmod 4$. Moreover, if G is bipartite, then G has an even number of edges, so $n \equiv 0 \pmod 4$. This condition is known as the *parity condition* and has a k-labelings counterpart.

Lemma 3. *Let G be a graph of size n and suppose every vertex of G has even degree. If G admits a σ_k-labeling, then either (a) $n \equiv 0 \pmod 4$, (b) k is even and $n \equiv 1 \pmod 4$, or (c) k is odd and $n \equiv 3 \pmod 4$. Moreover, if G is bipartite, then $n \equiv 0 \pmod 4$.*

Proof. Let f be a σ_k-labeling of G. Then we have the sum of the edge labels in G is $\sum_{\{u,v\}\in E(G)} |f(u) - f(v)|$ which is necessarily even (since every vertex has even degree) and equals $\sum_{i=1}^{n} (k+i-1) = nk + (n-1)n/2$. Thus the conclusions follow. ∎

Lemma 4. *Let G be a bipartite graph of size n and let $d = \gcd(\{\deg(v) \colon v \in V(G)\})$. If G admits a σ_k^+-labeling for some positive integer k, then d divides $n(2k+n-1)/2$.*

Proof. Let G have vertex bipartition $\{A, B\}$, where $A = \{a_1, a_2, \ldots, a_r\}$ and $B = \{b_1, b_2, \ldots, b_s\}$. Let f be a σ_k^+-labeling of G such that $f(a) < f(b)$ for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$. Then the sum of the edge labels in G can be computed with $\sum_{i=1}^{n} (k+i-1) = nk+(n-1)n/2 = n(2k+n-1)/2$ or with $\sum_{j=1}^{s} \deg(b_j)f(b_j) - \sum_{i=1}^{r} \deg(a_i)f(a_i)$. Since $d \mid \sum_{j=1}^{s} \deg(b_j)f(b_j) - \sum_{i=1}^{r} \deg(a_i)f(a_i)$, we have $d \mid n(2k+n-1)/2$. ∎

We note that Lemma 4 has no application if n is odd, since d divides n. However, if n is even and $n \equiv d \pmod{2d}$, then G cannot admit a σ_k^+-labeling. For instance, $K_{5,5} - I$, where I is a 1-factor, does not admit a σ_k^+-labeling for any positive integer k.

We turn our attention briefly to disconnected graphs. It was shown in [13] that the vertex-disjoint union of graphs that admit α-labelings has a σ^+-labeling (called a θ-labeling in [13]). However, for graph decomposition purposes, we need the resulting σ^+-labeling to satisfy additional conditions.

Theorem 5. *Let $G_1, G_2, \ldots, G_t$ be vertex-disjoint bipartite graphs that admit α-labelings and let $H = \bigcup_{i=1}^{t} G_i$. Let H have size n and let $\{A, B\}$ be a bipartition of $V(H)$. Then H admits a σ^+-labeling h that satisfies $h(a) < h(b)$ for every edge $\{a, b\}$ with $a \in A$ and $b \in B$ and satisfies $h(u) - h(v) < n$ for any $u \in A$ and $v \in B$.*

Proof. For $1 \le i \le t$, let bipartite graph G_i have n_i edges (with $n_i \ge 1$), α-labeling g_i with boundary value λ_i, and vertex bipartition $\{A_i, B_i\}$ where $g_i(a) \le \lambda_i < g_i(b)$ for all $a \in A_i$ and $b \in B_i$. Without loss of generality, we can assume that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_t$. Then H is bipartite with vertex bipartition $\{A, B\}$ where $A = \bigcup_{i=1}^t A_i$ and $B = \bigcup_{i=1}^t B_i$. Let $n = |E(H)| = \sum_{i=1}^t n_i$.

If $t = 1$, then g_1 is an α-labeling of H (which necessarily satisfies the conclusion), so assume $t \ge 2$. We define a labeling h on $V(H)$ by

$$h(v) = \begin{cases} g_i(v) + \frac{i-1}{2} + \sum_{j=1}^{\frac{i-1}{2}} \lambda_{2j-1} & \text{for } i \text{ odd, } v \in A_i, \\ g_i(v) + \frac{i-1}{2} + \sum_{j=1}^{\frac{i-1}{2}} \lambda_{2j-1} + \sum_{j=i+1}^{t} n_j & \text{for } i \text{ odd, } v \in B_i, \\ g_i(v) + n + \frac{i}{2} + \sum_{j=1}^{\frac{i-2}{2}} \lambda_{2j} & \text{for } i \text{ even, } v \in A_i, \\ g_i(v) + n + \frac{i}{2} + \sum_{j=1}^{\frac{i-2}{2}} \lambda_{2j} + \sum_{j=i+1}^{t} n_j & \text{for } i \text{ even, } v \in B_i. \end{cases}$$

To show that h is a σ^+-labeling, we note that, if t_{o} and t_{e} are respectively the greatest odd integer and greatest even integer less than or equal to t, then we have

$$0 \le h(A_1) < h(A_3) < \cdots < h(A_{t_{\mathrm{o}}}) < h(B_{t_{\mathrm{o}}}) < h(B_{t_{\mathrm{o}}-2}) < \cdots < h(B_1) \le n$$

and

$$n + 1 \le h(A_2) < h(A_4) < \cdots < h(A_{t_{\mathrm{e}}}) < h(B_{t_{\mathrm{e}}}) < h(B_{t_{\mathrm{e}}-2}) < \cdots < h(B_2) \le 2n.$$

Hence h is one-to-one and no edge label will exceed n. Furthermore, $\bar{h}(E(G_i)) = [1, n_i] + \sum_{j=i+1}^t n_j = [1 + \sum_{j=i+1}^t n_j, \sum_{j=i}^t n_j]$, and we have

$$1 \le \bar{h}(E(G_t)) < \bar{h}(E(G_{t-1})) < \cdots < \bar{h}(E(G_1)) \le n.$$

Hence $\bar{h}(E(H)) = [1, n]$, and the conditions for h being a σ^+-labeling are satisfied.

Now, we consider the difference of $\max(h(A))$ and $\min(h(B))$. Note that since g_i is an α-labeling, the boundary value λ_i is unique and $\min G_i(B_i) = \lambda_i + 1$.

Case 1: t is even.
Then we have

$$\begin{aligned} \max(h(A)) &= \max(g_t(A_t)) + n + \tfrac{t}{2} + \textstyle\sum_{j=1}^{\frac{t-2}{2}} \lambda_{2j} \\ &= \lambda_t + n + \tfrac{t}{2} + \textstyle\sum_{j=1}^{\frac{t-2}{2}} \lambda_{2j}, \end{aligned}$$

and, since g_{t-1} is an α-labeling,

$$\begin{aligned} \min(h(B)) &= \min(g_{t-1}(B_{t-1})) + \tfrac{t-2}{2} + \textstyle\sum_{j=1}^{\frac{t-2}{2}} \lambda_{2j-1} + n_t \\ &= (\lambda_{t-1} + 1) + \tfrac{t-2}{2} + \textstyle\sum_{j=1}^{\frac{t-2}{2}} \lambda_{2j-1} + n_t. \end{aligned}$$

Hence

$$\begin{aligned} \max(h(A)) - \min(h(B)) &= \lambda_t + n + \tfrac{t}{2} + \sum_{j=1}^{\frac{t-2}{2}} \lambda_{2j} - \left(\lambda_{t-1} + 1 + \tfrac{t-2}{2} + \sum_{j=1}^{\frac{t-2}{2}} \lambda_{2j-1} + n_t \right) \\ &= n - n_t + \textstyle\sum_{j=1}^{\frac{t}{2}} (\lambda_{2j} - \lambda_{2j-1}). \end{aligned}$$

By the assumption that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t$, we have $\sum_{j=1}^{\frac{t}{2}}(\lambda_{2j} - \lambda_{2j-1}) \leq 0$, and thus $\max(h(A)) - \min(h(B)) \leq n - n_t < n$. Therefore $h(u) - h(v) < n$ for any $u \in A$ and $v \in B$.

Case 2: t is odd.
Then we have

$$\begin{aligned}\max\left(h(A)\right) &= \max\left(g_{t-1}(A_{t-1})\right) + n + \tfrac{t-1}{2} + \textstyle\sum_{j=1}^{\frac{t-3}{2}} \lambda_{2j} \\ &= \lambda_{t-1} + n + \tfrac{t-1}{2} + \textstyle\sum_{j=1}^{\frac{t-3}{2}} \lambda_{2j},\end{aligned}$$

and, since g_t is an α-labeling,

$$\begin{aligned}\min\left(h(B)\right) &= \min\left(g_t(B_t)\right) + \tfrac{t-1}{2} + \textstyle\sum_{j=1}^{\frac{t-1}{2}} \lambda_{2j-1} + 0 \\ &= (\lambda_t + 1) + \tfrac{t-1}{2} + \textstyle\sum_{j=1}^{\frac{t-1}{2}} \lambda_{2j-1}.\end{aligned}$$

Hence

$$\begin{aligned}\max(h(A)) - \min(h(B)) &= \lambda_{t-1} + n + \tfrac{t-1}{2} + \sum_{j=1}^{\frac{t-3}{2}} \lambda_{2j} - \left(\lambda_t + 1 + \tfrac{t-1}{2} + \sum_{j=1}^{\frac{t-1}{2}} \lambda_{2j-1}\right) \\ &= n - (\lambda_t + 1) + \textstyle\sum_{j=1}^{\frac{t-1}{2}}(\lambda_{2j} - \lambda_{2j-1}).\end{aligned}$$

By the assumption that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t$, we have $\sum_{j=1}^{\frac{t-1}{2}}(\lambda_{2j} - \lambda_{2j-1}) \leq 0$, and thus $\max(h(A)) - \min(h(B)) \leq n - (\lambda_t + 1) < n$. Therefore, $h(u) - h(v) < n$ for any $u \in A$ and $v \in B$. ∎

The following was given in [6].

Theorem 6. *Let $G_1, G_2, \ldots, G_t$ be vertex-disjoint bipartite graphs with $n_1, n_2, \ldots, n_t$ edges, respectively. If G_1 admits a ρ^{++}-labeling in which no vertex is labeled $2n_1$ and $G_2, G_3, \ldots, G_t$ admit α-labelings, then the vertex-disjoint union $G_1 \cup G_2 \cup G_3 \cup \cdots \cup G_t$ admits a ρ^{++}-labeling.*

We discovered recently that the proof of Theorem 6 is incomplete as published in [6]. We prove a stronger result here that subsumes the results from Theorem 6. We first give two lemmas.

Lemma 7. *Let G_1 be a bipartite graph with n_1 edges that admits a ρ^{++}-labeling g_1 with boundary value λ_1. Let G_2 be a bipartite graph with n_2 edges that admits an α-labeling g_2 with boundary value λ_2. If $\lambda_1 + 1 \notin g_1(V(G_1))$, then the vertex-disjoint union $G_1 \cup G_2$ admits a ρ^{++}-labeling h such that $2(n_1 + n_2) \notin h(V(G_1 \cup G_2))$.*

Proof. For $i \in \{1, 2\}$, let G_i have vertex bipartition $\{A_i, B_i\}$ such that $g_i(A_i) \leq \lambda_i < g_i(B_i)$. Furthermore, let $\lambda_1 + 1 < g_1(B_1)$. Let H denote the vertex-disjoint union $G_1 \cup G_2$ and let $n = n_1 + n_2$.

We define a labeling h on $V(H)$ by

$$h(v) = \begin{cases} g_1(v) & v \in A_1, \\ g_1(v) + n_2 & v \in B_1, \\ g_2(v) + \lambda_1 + 1 & v \in V(G_2). \end{cases}$$

Clearly, h is one-to-one on each of the sets A_1, B_1, and $V(G_2)$. Since $\lambda_1 + 1 < g_1(B_1)$, we have

$$0 \le h(A_1) < \lambda_1 + 1 \le h(V(A_2)) < h(V(B_2)) \le \lambda_1 + 1 + n_2 < h(B_1) \le 2n_1 + n_2 < 2n. \quad (1)$$

Hence h is one-to-one and $h(V(H)) \subseteq [0, 2n]$.

Next, we examine the set of edge labels $\bar{h}(E(H))$. For each $\ell \in [1, n_2]$, there exists an edge $e \in E(G_2)$ such that $\bar{g_2}(e) = \ell$. Hence

$$\bar{h}(e) = \bar{g_2}(e) = \ell.$$

Moreover, for each $\ell \in [n_2 + 1, n]$, there exists an edge $e \in E(G_1)$ such that either $\bar{g_1}(e) + n_2 = \ell$ or $(2n_1 + 1 - \bar{g_1}(e)) + n_2 = \ell$. Hence either

$$\bar{h}(e) = \bar{g_1}(e) + n_2 = \ell$$

or

$$2n + 1 - \bar{h}(e) = 2(n_1 + n_2) + 1 - (\bar{g_1}(e) + n_2) = (2n_1 + 1 - \bar{g_1}(e)) + n_2 = \ell.$$

Therefore, $\bar{h}(E(H)) \supseteq \{x_1, x_2, \ldots, x_n\}$, where for each $\ell \in [1, n]$ either $x_\ell = \ell$ or $2n + 1 - x_\ell = \ell$. Since $|\bar{h}(E(H))| = n$, we have $\bar{h}(E(H)) = \{x_1, x_2, \ldots, x_n\}$. Thus h is a ρ-labeling.

Finally, let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Then $\{A, B\}$ is a bipartition of $V(H)$. It is clear from (1) that $h(A) < h(B)$ and thus h is a ρ^{++}-labeling. Moreover, it is clear from (1) that $2n \notin h(V(H))$. ∎

Lemma 8. *Let G_1 be a bipartite graph with n_1 edges that admits a ρ^{++}-labeling g_1 with boundary value λ_1. Let G_2 be a bipartite graph with n_2 edges that admits an α-labeling g_2 with boundary value λ_2. If $2n_1 \notin g_1(V(G_1))$, then the vertex-disjoint union $G_1 \cup G_2$ admits a ρ^{++}-labeling h with boundary value λ such that $\lambda + 1 \notin h(V(G_1 \cup G_2))$.*

Proof. For $i \in \{1, 2\}$, let G_i have vertex bipartition $\{A_i, B_i\}$ such that $g_i(A_i) \le \lambda_i < g_i(B_i)$. Furthermore, let $g_1(B_1) < 2n_1$. Let H denote the vertex-disjoint union $G_1 \cup G_2$ and let $n = n_1 + n_2$.

We define a labeling h on $V(H)$ by

$$h(v) = \begin{cases} g_1(v) + \lambda_2 + 1 & v \in A_1, \\ g_1(v) + \lambda_2 + 1 + n_2 & v \in B_1, \\ g_2(v) & v \in A_2, \\ g_2(v) + 2n_1 + n_2 & v \in B_2. \end{cases}$$

Clearly, h is one-to-one on each of the sets A_1, B_1, A_2, and B_2. Since $g_1(B_1) < 2n_1$, we have

$$0 \le h(A_2) < \lambda_2 + 1 \le h(V(A_1)) \le \lambda_1 + \lambda_2 + 1 < h(V(B_1)) < 2n_1 + \lambda_2 + 1 + n_2 \le h(B_2) \le 2n. \quad (2)$$

Hence h is one-to-one and $h(V(H)) \subseteq [0, 2n]$.

Next, we examine the set of edge labels $\bar{h}(E(H))$. For each $\ell \in [1, n_2]$, there exists an edge $e \in E(G_2)$ such that $\bar{g_2}(e) = n_2 + 1 - \ell$. Hence

$$2n + 1 - \bar{h}(e) = 2(n_1 + n_2) + 1 - (\bar{g_2}(e) + 2n_1 + n_2) = n_2 + 1 - \bar{g_2}(e) = \ell.$$

Moreover, for each $\ell \in [n_2 + 1, n]$, there exists an edge $e \in E(G_1)$ such that either $\bar{g_1}(e) + n_2 = \ell$ or $(2n_1 + 1 - \bar{g_1}(e)) + n_2 = \ell$. Hence either

$$\bar{h}(e) = \bar{g_1}(e) + n_2 = \ell$$

or

$$2n + 1 - \bar{h}(e) = 2(n_1 + n_2) + 1 - (\bar{g_1}(e) + n_2) = (2n_1 + 1 - \bar{g_1}(e)) + n_2 = \ell.$$

Therefore, $\bar{h}(E(H)) \supseteq \{x_1, x_2, \ldots, x_n\}$, where for each $\ell \in [1, n]$ either $x_\ell = \ell$ or $2n + 1 - x_\ell = \ell$. Since $|\bar{h}(E(H))| = n$, we have $\bar{h}(E(H)) = \{x_1, x_2, \ldots, x_n\}$. Thus h is a ρ-labeling.

Finally, let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Then $\{A, B\}$ is a bipartition of $V(H)$. It is clear from (2) that $h(A) \le \lambda_1 + \lambda_2 + 1 < h(B)$ and thus h is a ρ^{++}-labeling with boundary value $\lambda_1 + \lambda_2 + 1$. Moreover, since $\min(h(B)) = \min(h(B_1)) \ge \lambda_1 + \lambda_2 + 2 + n_2$, we have $\lambda_1 + \lambda_2 + 2 \notin h(V(H))$. ∎

By combining the results from Lemmas 7 and 8, we obtain the following theorem which subsumes Theorem 6.

Theorem 9. *Let G be a bipartite graph with n edges that admits ρ^{++}-labeling g with boundary value λ. Let H_1, H_2, $\ldots$, H_k be bipartite graphs that admit α-labelings. If $\{\lambda + 1, 2n\} \not\subseteq g(V(G))$, then the vertex-disjoint union $G \cup H_1 \cup H_2 \cup \cdots \cup H_k$ admits a ρ^{++}-labeling.*

Labelings that are used in graph decompositions are called *Rosa-type* because of Rosa's original article [20] on the topic. For a survey of Rosa-type labelings and their graph decomposition applications, see [14]. A comprehensive dynamic survey on general graph labelings is maintained by Gallian [16].

Rosa-type labelings are critical to the study of cyclic graph decompositions as seen in the following results from Rosa [20].

Theorem 10. *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

Theorem 11. *Let G be a graph with n edges that has a σ-labeling. Then there exists a cyclic G-decomposition of $K_{2n+2} - I$, where I is a 1-factor in K_{2n+2}.*

Theorem 12. *Let G be a bipartite graph with n edges that has an α-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

From a graph decompositions perspective, Theorem 12 offers a great advantage over the other two theorems. However, many bipartite graphs, including infinite classes of trees, fail to admit α-labelings. In [13] it is shown that ρ^+-labelings yield similar results to Theorem 12.

Theorem 13. *Let G be a bipartite graph with n edges that has a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

Unlike with α-labelings, it is not currently known if there is a bipartite graph (without too many isolated vertices) that fails to admit a ρ^+-labeling. In fact, El-Zanati and Vanden Eynden conjecture that every bipartite graph admits a ρ^{++}-labeling (see [14]).

In this manuscript, we show how to use k-labelings to get extensions of the above theorems to cyclic G-decompositions of the corresponding circulant graphs. We also investigate which bipartite 2-regular graphs admit the various k-labelings.

2 Main Results

We note that a ρ_k-labeling of G of size n induces an embedding of G in $K_{2n+2k-1}$ (with $V(K_{2n+2k-1}) = [0, 2n+2k-2]$) so that for each ℓ, such that $k \le \ell \le n+k+\ell$, there an edge of the length ℓ in $E(G)$. Moreover, $\langle [k, n+k-1] \rangle_{2n+2k-1} = K_{2n+2k-1} - \langle [1, k-1] \rangle_{2n+2k-1}$. Thus we have the following result corresponding to Theorem 10.

Theorem 14. *Let G be a graph with n edges and let k be a positive integer. There exists a cyclic G-decomposition of $\langle [k, n+k-1] \rangle_{2n+2k-1}$ if and only if G has a ρ_k-labeling.*

Similarly, a σ_k-labeling of G can be viewed as inducing an embedding of G in K_{2n+2k} (with $V(K_{2n+2k}) = [0, 2n+2k-1]$) so that there is one edge in $E(G)$ with each label ℓ for $k \le \ell \le n+k-1$. (Recall that σ-labelings do not allow wrap-around edges.) Moreover, $\langle [k, n+k-1] \rangle_{2n+2k} = K_{2n+2k} - \langle [1, k-1] \cup \{k+n\} \rangle_{2n+2k}$. Thus we have the following result corresponding to Theorem 11.

Theorem 15. *Let G be a graph with n edges and let k be a positive integer. If G admits a σ_k-labeling, then there exists a cyclic G-decomposition of $\langle [k, n+k] \rangle_{2n+2k} - I$, where I is a 1-factor.*

Because σ_k-labelings do not allow wrap-around edges, Theorem 14 can be broadened greatly in terms of decompositions of circulant graphs.

Theorem 16. *Let G be a graph with n edges and let $k \ge 1$ be an integer. If G admits a σ_k-labeling, then there exists a cyclic G-decomposition of $\langle [k, n+k-1] \rangle_{2n+2k-1+t}$ for each nonnegative integer t.*

As would be expected, Theorem 13 has a k-labelings counterpart.

Theorem 17. *Let G be a bipartite graph with n edges and let k be a positive integer. If G admits a ρ_k^+-labeling, then there exists a cyclic G-decomposition of $\langle[k, nx+k-1]\rangle_{2nx+2k-1}$ for each positive integer x.*

Proof. Let $\{A, B\}$ be a bipartition of $V(G)$. Let h be a ρ_k^+-labeling of G, so that $h(u) < h(v)$ for every $\{u, v\} \in E(G)$ with $u \in A$ and $v \in B$. We define a multigraph G' with

$$V(G') = \{h(a)\colon\ a \in A\} \cup \{h(b) + 2n(i-1)\colon\ b \in B, i \in [1, x]\} \subseteq [0, 2(nx+k-1)]$$

and

$$E(G') = \big\{\{h(a), h(b) + 2n(i-1)\}\colon\ \{a, b\} \in E(G), a \in A, b \in B, i \in [1, x]\big\}.$$

We will show that the lengths in $K_{2nx+2k-1}$ of the nx edges of G' are exactly the nx integers in $[k, nx+k-1]$, and so G' is actually a graph. Then if we define $h'(v) = v$ for $v \in V(G')$, then h' is a ρ_k-labeling of G' and there is a cyclic G'-decomposition of $\langle[k, nx+k-1]\rangle_{2nx+2k-1}$ by Theorem 14. But for fixed $i \in [1, x]$ the corresponding edges of G' induce a graph isomorphic to G, so G' has a G-decomposition and the theorem follows.

Let $\ell \in [k, nx+k-1]$. We will show that some edge of G' has label ℓ or $2nx+2k-1-\ell$. First, we show that there exist integers q and r, with $0 \le q < x$ and $k \le r < n+k$, such that either $\ell = 2nq + r$ or $2nx+2k-1-\ell = 2nq+r$. By the division algorithm there exist integers q_i and r_i, for $i \in \{1, 2\}$, such that

$$\ell - k = 2nq_1 + r_1, \quad \text{where } 0 \le r_1 < 2n,$$

and

$$2nx + k - 1 - \ell = 2nq_2 + r_2, \quad \text{where } 0 \le r_2 < 2n.$$

Note that since $\ell \in [k, nx+k-1]$, $q_1 \ge 0$ and $q_2 \ge 0$. Also,

$$q_1 = \tfrac{\ell-k-r_1}{2n} \le \tfrac{nx+k-1-k-r_1}{2n} = \tfrac{nx-1-r_1}{2n} < \tfrac{nx}{2n} < x,$$

while also

$$q_2 = \tfrac{2nx+k-1-\ell-r_2}{2n} \le \tfrac{2nx+k-1-k-r_2}{2n} = \tfrac{2nx-1-r_2}{2n} < x.$$

We claim that $r_i < n$ for either $i = 1$ or $i = 2$. For if not, then $r_1 + r_2 \ge 2n$. Now

$$q_1 + q_2 = \tfrac{\ell-k-r_1}{2n} + \tfrac{2nx+k-1-\ell-r_2}{2n} = x - \tfrac{r_1+r_2+1}{2n},$$

and so $2n$ divides $r_1 + r_2 + 1 > 2n$. Thus, $r_1 + r_2 + 1 \ge 4n$, but this contradicts the fact that neither r_1 nor r_2 exceeds $2n-1$. Therefore, $r_I < n$ for some $I \in \{1, 2\}$. Set $q = q_I$ and $r = r_I + k$. Then $k \le r < n + k$, and we noted already that $0 \le q < x$. If $I = 1$, then

$$\ell = 2nq_1 + r_1 + k = 2nq + r$$

while if $I = 2$, then

$$2nx + 2k - 1 - \ell = 2nx + k - 1 - \ell + k = 2nq_2 + r_2 + k = 2nq + r.$$

Since h is a ρ_k^+-labeling of G, there exists an edge $\{a, b\}$, where $a \in A$ and $b \in B$, with label either r or $2n + 2k - 1 - r$. In what follows, if $b \in B$, we denote $h(b) + 2n(i - 1)$ by b_i.

Case 1: The label of $\{a, b\}$ is r.
Since $h(b) - h(a) = r$, we have

$$\begin{aligned} h'(b_{q+1}) - h'(a) &= h(b) + 2nq - h(a) \\ &= 2nq + r. \end{aligned}$$

Thus, $h'(b_{q+1}) - h'(a) = \ell$ if $\ell = 2nq + r$, and $h'(b_{q+1}) - h'(a) = 2nx + 2k - 1 - \ell$ if $2nx + 2k - 1 - \ell = 2nq + r$.

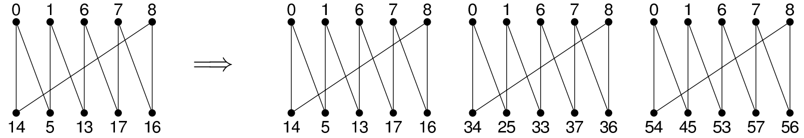
Case 2: The label of $\{a, b\}$ is $2n + 2k - 1 - r$.
Since $h(b) - h(a) = 2n + 2k - 1 - r$, we have

$$\begin{aligned} h'(b_{x-q}) - h'(a) &= h(b) + 2n(x - q - 1) - h(a) \\ &= h(b) - h(a) + 2nx - 2nq - 2n \\ &= 2n + 2k - 1 - r + 2nx - 2nq - 2n \\ &= 2nx + 2k - 1 - (2nq + r). \end{aligned}$$

Thus, $h'(b_{x-q}) - h'(a) = 2nx + 2k - 1 - \ell$ if $\ell = 2nq + r$, and $h'(b_{x-q}) - h'(a) = \ell$ if $2nx + 2k - 1 - \ell = 2nq + r$.

Since G' has size nx and each of the nx edge lengths $k, k + 1, \ldots, nx + k - 1$ is the length of an edge, h' is a ρ_k-labeling of G', and we have a cyclic G-decomposition of $\langle[k, nx + k - 1]\rangle_{2nx+2k-1}$. ∎

In Figure 1, we show an example of a ρ_4^+-labeling of C_{10} and the three starters for a cyclic C_{10}-decomposition of $\langle[4, 33]\rangle_{67}$.



(a) A ρ_4^+-labeling of C_{10}.

(b) The three starters for a cyclic C_{10}-decomposition of $\langle[4, 33]\rangle_{67}$.

Figure 1: An ordered k-labeling yielding decompositions of more than one circulant graph.

If the ordered labeling in Theorem 17 is a σ_k^+-labeling with a slight restriction, then a more general result can be obtained.

Theorem 18. *Let G be a bipartite graph with n edges and let k be a positive integer. Let $\{A, B\}$ be a bipartition of $V(G)$ and let h be a σ_k^+-labeling of G with the property that $h(u) < h(v)$ for every $\{u, v\} \in E(G)$ with $u \in A$ and $v \in B$. Suppose moreover that $h(a) - h(b) \neq n$ for any $a \in A$ and $b \in B$. Then for all integers $x \geq 1$ and $t \geq 0$, there exists a cyclic G-decomposition of both $\langle[k, nx + k - 1]\rangle_{2nx+2k-1+t}$ and $\langle[k, nx + k]\rangle_{2nx+2k} - I$, where I is a 1-factor.*

Proof. We define a multigraph G' with

$$V(G') = \{h(a)\colon\ a \in A\} \cup \{h(b) + n(i-1)\colon\ b \in B, i \in [1, x]\} \subseteq [0, 2(nx + k - 1)]$$

and

$$E(G') = \big\{\{h(a), h(b) + n(i-1)\}\colon\ \{a, b\} \in E(G), a \in A, b \in B, i \in [1, x]\big\}.$$

Because $h(a) - h(b) \neq n$ for all $a \in A$ and $b \in B$, the two sets whose union comprises $V(G')$ are disjoint. We will show that the labels of the nx edges of G' are exactly the nx integers in $[k, nx + k - 1]$, and so G' is actually a graph. Thus if we define $h'(v) = v$ for $v \in V(G')$, then h' is a σ_k-labeling of G' and by Theorem 16, there is a cyclic G'-decomposition of $\langle[k, nx + k - 1]\rangle_{2nx+2k-1+t}$. Also, by Theorem 15, there is a cyclic G'-decomposition of $\langle[k, nx + k]\rangle_{2nx+2k} - I$, where I is the 1-factor induced by the edges of length $nx + k$. But for fixed $i \in [1, x]$ the corresponding edges of G' induce a graph isomorphic to G, so G' has a G-decomposition and the theorem follows.

Let $\ell \in [k, nx + k - 1]$. We will show that some edge of G' has label ℓ. By the division algorithm, there exist integers q and r, with $0 \leq q < x$ and $0 \leq r < n$, such that $\ell - k = nq + r$. Thus, $\ell = nq + k + r$. Since h is a σ_k^+-labeling of G, there exists an edge $\{a, b\}$, where $a \in A$ and $b \in B$, with label $k + r$. Denote the vertex $h(b) + nq$ by b_{q+1}.

Since $h(b) - h(a) = k + r$, we have

$$\begin{aligned} h'(b_{q+1}) - h'(a) &= h(b) + nq - h(a) \\ &= nq + k + r. \end{aligned}$$

Thus, $h'(b_{q+1}) - h'(a) = \ell$.

Since G' has size nx and each of the nx edge lengths $k, k + 1, \ldots, nx + k - 1$ is the label of an edge, h' is a σ_k-labeling of G', and the result follows. ∎

3 Decompositions of circulant graphs into 2-regular bipartite graphs

Let G be a bipartite graph with n edges and let k be a positive integer. From the perspective of decomposing circulant graphs, and in light of Theorems 17 and 18, the most desirable k-labelings of G would be uniformly-ordered k-labelings. Moreover, σ_k^{++}-labelings would be preferable to ρ_k^{++}-labelings. El-Zanati and Vanden Eynden (see [14]) conjecture that every bipartite graph admits a ρ^{++}-labeling (and thus, by Lemma 1, a ρ_k^{++}-labeling for each $k \geq 1$). Lemmas 3 and 4 rule out the existence of certain variations of σ_k-labelings.

It is known that bipartite 2-regular graphs that satisfy the parity condition (i.e., have size a multiple of 4) and have at most 3 components admit α-labelings, except for the graph $3C_4$ (see [15]). In fact, $3C_4$ is currently the only known example of a 2-regular bipartite graph that satisfies the parity condition and fails to have an

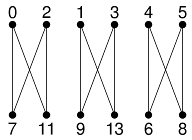


Figure 2: A σ^{++}-labeling of $3C_4$.

α-labeling. A σ^{++}-labeling of $3C_4$ is given in Figure 2. We conjecture that every bipartite 2-regular graph that satisfies the parity condition admits a σ^{++}-labeling. It is also likely that, with the sole exception of $3C_4$, all such graphs admit α-labelings. In [1], it is shown that rC_4 admits an α-labeling for all positive integers $r \neq 3$.

Because both C_{4m} (see [20]) and $C_{4m_1+2} \cup C_{4m_2+2}$ (see [2]) admit α-labelings, we have the following consequence of Theorem 5.

Theorem 19. *Let G be a 2-regular bipartite graph of size $n \equiv 0 \pmod 4$ and let $\{A, B\}$ be a bipartition of $V(G)$. Then G admits a σ^+-labeling f that satisfies $f(a) < f(b)$ for every edge $\{a, b\}$ with $a \in A$ and $b \in B$ and satisfies $f(u) - f(v) < n$ for all $u \in A$ and $v \in B$.*

Moreover, in light of Lemma 1, we have the following.

Corollary 20. *Let G be a 2-regular bipartite graph of size $n \equiv 0 \pmod 4$. Then G admits a σ_k^{++}-labeling for every integer $k \geq n$.*

Because C_{4m+2} admits a ρ^{++}-labeling that does not use $8m + 4$ as a vertex label (see [13]), we can use Theorem 9 to show that bipartite 2-regular graphs that do not satisfy the parity conditions admit ρ^{++}-labelings and hence ρ_k^{++}-labelings for every positive integer k. Moreover, since an α-labeling is a ρ^{++}-labeling, the following holds for all 2-regular bipartite graphs.

Theorem 21. *Every 2-regular bipartite graph admits a ρ_k^{++}-labeling for every positive integer k.*

In light of the above theorems and Theorems 17 and 18, the following hold for 2-regular bipartite graphs.

Corollary 22. *Let G be a 2-regular bipartite graph of size $n \equiv 0 \pmod 4$ and let $k \geq n$ be an integer. Then for all integers $x \geq 1$ and $t \geq 0$, there exists a cyclic G-decomposition of both $\langle [k, nx + k - 1] \rangle_{2nx+2k-1+t}$ and $\langle [k, nx + k] \rangle_{2nx+2k} - I$, where I is a 1-factor.*

Corollary 23. *Let G be a 2-regular bipartite graph of size n. Then there exists a cyclic G-decomposition of $\langle [k, nx + k - 1] \rangle_{2nx+2k-1}$ for all positive integers k and x.*

Little is known about cyclic decompositions of circulant graphs into non-bipartite 2-regular graphs. By using k-labelings of odd cycles, the following is shown in [12].

Corollary 24. *Let $n \geq 3$ be odd and $k \in [1, n]$ with $(n, k) \notin \{(3, 3), (5, 3)\}$. Then there exists a cyclic C_n-decomposition of $\langle [k, nx + k - 1] \rangle_{2nx+2k-1}$ for every positive integer x.*

3.1 Decompositions of K_m into Hamilton cycles and 2-regular bipartite graphs

Several authors have considered the problem of decomposing the complete graph into Hamilton cycles and n-cycles. For example, Bryant and Maenhaut [9] show that for all odd positive integers m, the complete graph K_m can be decomposed into h Hamilton cycles and t triangles (i.e., C_3's) if and only if $hm + 3t = m(m-1)/2$. More recently, Jordon [17] settled the corresponding problem for Hamilton cycles and 5-cycles. Corollaries 22 and 23 can be used to obtain some decompositions of complete graphs into Hamilton cycles and 2-regular graphs.

As noted earlier, $\langle\{m/2\}\rangle_m$ is a 1-factor in K_m when m is even. Otherwise, for $1 \le i < m/2$, it is easy to see that $\langle\{i\}\rangle_m$ consists of δ vertex-disjoint $C_{m/\delta}$'s, where $\delta = \gcd(i, m)$. Thus, $\langle\{i\}\rangle_m$ is a Hamilton cycle if and only if i and m are relatively prime. A special case of a celebrated result by Bermond, Favron, and Mahéo [5] tells us when $\langle\{i, j\}\rangle_m$ can be decomposed into two Hamilton cycles.

Lemma 25. (Bermond et al. [5]) *For positive integers i, j, and m with $i < j < m/2$, the graph $\langle\{i, j\}\rangle_m$ can be decomposed into two Hamilton cycles if and only if* $\gcd(i, j, m) = 1$.

We will make use of the following corollary to the above lemma.

Corollary 26. *Let t and m be positive integers with $t < m/2$ and let $L = [1, t]$. Then $\langle L\rangle_m$ can be decomposed into t Hamilton cycles.*

Proof. If t is even, let $Q_i = \{2i-1, 2i\}$ for $1 \le i \le t/2$. Then $Q = \{Q_i \colon 1 \le i \le t/2\}$ is a partition of L. Since the elements of each Q_i are relatively prime, the circulant graph $\langle Q_i\rangle_m$ can be decomposed into two Hamilton cycles for $1 \le i \le t/2$. If t is odd, let $Q_1 = \{1\}$ and for $2 \le i \le (t+1)/2$, let $Q_i = \{2(i-1), 2i-1\}$. Again, $Q = \{Q_i \colon 1 \le i \le (t+1)/2\}$ is a partition of L. Now $\langle Q_1\rangle_m$ is a Hamilton cycle and for $2 \le i \le (t+1)/2$, each $\langle Q_i\rangle_m$ can be decomposed into two Hamilton cycles. Thus the result holds. ∎

We will also make use of the following result of Dean [10, 11].

Lemma 27. *For integers r, s, t, and n with $r < s < t < n/2$,* $\gcd(r, s, t, n) = 1$, *and either n is odd or* $\gcd(x, n) = 1$ *for some $x \in \{r, s, t\}$, the graph $\langle\{r, s, t\}\rangle_n$ can be decomposed into three Hamilton cycles.*

First we state a basic lemma about decompositions using ρ_k-labelings.

Lemma 28. *Let G be a graph of size n that admits a ρ_k-labeling for some positive integer k. Then there exists a $2(k-1)$-regular spanning subgraph H of $K_{2(n+k)-1}$ that can be decomposed into $k-1$ Hamilton cycles such that $K_{2(n+k)-1} - H$ has a cyclic G-decomposition.*

Proof. Let H be the spanning subgraph of $K_{2(n+k)-1}$ induced by edge-lengths $[1, k-1]$. By Corollary 26, we can decompose H into $k-1$ Hamilton cycles. By Theorem 14, G decomposes $K_{2(n+k)-1} - H$ cyclically. ∎

If the graph has a σ_k-labeling, then more can be done.

Lemma 29. *Let G be a graph of size n that admits a σ_k-labeling for some positive integer k and let t be a nonnegative integer. Then there exists a cyclic G-decomposition of $K_{2(n+k+t)-1} - H$, where H is a $2(t+k-1)$-regular spanning subgraph that can be decomposed into $t+k-1$ Hamilton cycles. Moreover, there exists a cyclic G-decomposition of $K_{2(n+k+t)} - H'$, where H' is a $2(t+k-1)+1$-regular spanning subgraph that can be decomposed into a 1-factor and $t+k-1$ Hamilton cycles.*

Proof. Let H be the spanning subgraph of $K_{2(n+k+t)-1}$ induced by edge-lengths $[1, k-1] \cup [n+k, n+k+t-1]$. By Corollary 26, we can decompose $\langle[1, k-1]\rangle_{2(n+k+t)-1}$ into $k-1$ Hamilton cycles. If t is even, then $Q = \{\{n+k+2i-2, n+k+2i-1\}\colon 1 \le i \le t/2\}$ is a partition of $[n+k, n+k+t-1]$. If t is odd, then $Q = \{\{n+k+2i-2, n+k+2i-1\}\colon 1 \le i \le (t-1)/2\} \cup \{n+k+t-1\}$ is a partition of $[n+k, n+k+t-1]$. In either case, the elements of Q that are pairs of consecutive integers induce graphs that can be decomposed into Hamilton cycles by Lemma 25. Moreover, when t is odd, the graph $\langle\{n+k+t-1\}\rangle_{2(n+k+t)-1}$ is a Hamilton cycle since the $\gcd(n+k+t-1, 2(n+k+t)-1) = 1$. By Theorem 16, G decomposes $K_{2(n+k+t)-1} - H$ cyclically.

A similar argument can be used to obtain the decomposition of $K_{2(n+k+t)} - H'$, where H' is the graph induced by edge-lengths $[1, k-1] \cup [n+k, n+k+t]$. As before, $\langle[1, k-1]\rangle_{2(n+k+t)}$ can be decomposed into $k-1$ Hamilton cycles. If t is even, then $[n+k, n+k+t-1]$ can be partitioned into pairs of consecutive integers. By Lemma 25, the subgraphs induced by these pairs can be decomposed into Hamilton cycles. The subgraph $\langle\{n+k+t\}\rangle_{2(n+k+t)}$ is the 1-factor. If $t > 1$ is odd, then $[n+k, n+k+t-4]$ can be partitioned into pairs of consecutive integers and thus $\langle[n+k, n+k+t-4]\rangle_{2(n+k+t)}$ can be decomposed into Hamilton cycles (if $t = 3$, then the circulant is empty and the decomposition is trivial). Moreover, $\langle[n+k+t-3, n+k+t-1]\rangle_{2(n+k+t)}$ can be decomposed into 3 Hamilton cycles by Lemma 27. If $t = 1$ and $n+k$ is odd, then $\langle\{n+k\}\rangle_{2(n+k+t)}$ is a Hamilton cycle and thus $\langle\{n+k, n+k+1\}\rangle_{2(n+k+t)}$ can be decomposed into a Hamilton cycle and a 1-factor. Here, the subgraph $\langle\{n+k+t\}\rangle_{2(n+k+t)}$ is the 1-factor. Finally, if $t = 1$ and $n+k$ is even, then $\langle\{n+k, n+k+t\}\rangle_{2(n+k+t)}$ is isomorphic to $C_{n+k+t} \times K_2$ and can thus be decomposed into a Hamilton cycle and a 1-factor. ∎

If the graph G in the previous two lemmas is bipartite and admits a uniformly-ordered k-labeling, then we have the following.

Lemma 30. *Let G be a graph of size n that admits a ρ_k^{++}-labeling for some positive integer k. Then there exists a cyclic G-decomposition of $K_{2(nx+k)-1} - H$, where H is a $2(k-1)$-regular spanning subgraph that can be decomposed into $k-1$ Hamilton cycles.*

Lemma 31. *Let G be a graph of size n that admits a σ_k^{++}-labeling for some positive integer k and let t be a nonnegative integer. Then there exists a cyclic G-decomposition of $K_{2(nx+k+t)-1} - H$, where H is a $2(t+k-1)$-regular spanning subgraph that can be decomposed into $t+k-1$ Hamilton cycles. Moreover, there exists* *a cyclic G-decomposition of $K_{2(nx+k+t)} - H'$, where H' is a $(2(t+k-1)+1)$-regular spanning subgraph that can be decomposed into a 1-factor and $t+k-1$ Hamilton cycles.*

Since every 2-regular bipartite graph admits a ρ_k^{++}-labeling for every positive integer k, we have the following.

Corollary 32. *Let G be a 2-regular bipartite graph of size n and let k be a positive integer. There exists a cyclic G-decomposition of $K_{2(nx+k)-1} - H$, where H is a $2(k-1)$-regular spanning subgraph that can be decomposed into $k-1$ Hamilton cycles.*

In conclusion, we remark that Corollary 32 contributes towards a solution of a conjecture of Alspach [4] that there exists a decomposition of K_n (n odd) into cycles of lengths $m_1, m_2, \ldots, m_t$ whenever $3 \leq m_i \leq n$ for $1 \leq i \leq t$ and $m_1 + m_2 + \cdots + m_t = n(n-1)/2$. Alspach's Conjecture was settled recently by Bryant, Horsley, and Pettersson [8].

4 Acknowledgement

This research is partially supported by grant number A0649210 from the Division of Mathematical Sciences at the National Science Foundation. Part of this work was done while the first author was a teacher participant in *REU Site: Mathematics Research Experience for Pre-service and for In-service Teachers* at Illinois State University.

References

[1] J. Abrham and A. Kotzig, All 2-regular graphs consisting of 4-cycles are graceful, *Discrete Math.* **135** (1994), 1–14.

[2] J. Abrham and A. Kotzig, Graceful valuations of 2-regular graphs with two components, *Discrete Math.* **150** (1996), 3–15.

[3] P. Adams, D. Bryant, and M. Buchanan, A survey on the existence of G-designs, *J. Combin. Des.* **16** (2008), 373–410.

[4] B. Alspach, Research problems, Problem 3, *Discrete Math.* **36** (1981), 333.

[5] J. Bermond, O. Favaron, and M. Mahéo, Hamilton decomposition of Cayley graphs of degree 4, *J. Combin. Theory, Ser. B* **46**, (1989), 142–153.

[6] A. Blinco and S. I. El-Zanati, A note on the cyclic decomposition of complete graphs into bipartite graphs, *Bull. Inst. Combin. Appl.* **40** (2004), 77–82.

[7] D. Bryant and S. El-Zanati, "Graph decompositions," *in* Handbook of Combinatorial Designs, C. J. Colbourn and J. H. Dinitz (Editors), 2nd ed., Chapman & Hall/CRC, Boca Raton, 2007, pp. 477–485.

[8] D. Bryant, D. Horsley, and W. Pettersson, Cycle decompositions V: Complete graphs into cycles of arbitrary lengths, preprint.

[9] D. Bryant and B. Maenhaut, Decompositions of complete graphs into triangles and Hamilton cycles, *J. Combin. Des.* **12** (2004), 221–232.

[10] M. Dean, On Hamilton cycle decomposition of 6-regular circulant graphs, *Graphs Combin.* **22** (2006), 331–340.

[11] M. Dean, Hamilton cycle decomposition of 6-regular circulants of odd order, *J. Combin. Des.* **15** (2007), 91–97.

[12] S. I. El-Zanati, K. King, and J. Mudrock, On the cyclic decomposition of circulant graphs into almost-bipartite graphs, *Australas. J. Combin.* **49** (2011), 61–76.

[13] S. I. El-Zanati, C. Vanden Eynden, and N. Punnim, On the cyclic decomposition of complete graphs into bipartite graphs, *Australas. J. Combin.* **24** (2001), 209–219.

[14] S. I. El-Zanati and C. Vanden Eynden, On Rosa-type labelings and cyclic graph decompositions, *Mathematica Slovaca* **59** (2009), 1–18.

[15] K. Eshghi, The existence and construction of α-valuations of 2-regular graphs with 3 components, Ph.D. Thesis, Industrial Engineering Dept., University of Toronto, 1997.

[16] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* **18** (2011), #DS6.

[17] H. Jordon, Alspach's problem: The case of Hamilton cycles and 5-cycles, *Electron. J. Combin.* **18** (2011), Paper 82, 17 pp.

[18] M. Maheo and H. Thuillier, On d-graceful graphs, *Ars Combin.* **13** (1982), 181–192.

[19] G. Ringel, Problem 25, in *Theory of Graphs and Its Applications* (Proc. Symposium, Smolenice, 1963), ed. M. Fielder, Publishing House of the Czechoslovak Academy of Sciences, Prague, 1964, p. 162.

[20] A. Rosa, On certain valuations of the vertices of a graph, in *Theory of Graphs* (Internat. Sympos., Rome, 1966), ed. P. Rosenstiehl, Dunod, Paris; Gordon and Breach, New York, 1967, pp. 349–355.

[21] P. Slater, On k-graceful graphs, *Congr. Numer.* **36** (1982), 53–57.

(Received 2 Oct 2012; revised 3 Mar 2013)