On standard Stanton 4-cycle designs

R. C. Bunge[1], S. I. El-Zanati[1]*, A. Hakes[2], J. Jeffries[3],
E. Mastalio[4], J. Torf[1]

[1]Department of Mathematics
Illinois State University
Normal, IL 61790–4520

[2]Department of Mathematics
Purdue University
West Lafayette, IN 47907–2067

[3]Department of Mathematics and Computer Science
University of Central Missouri
Warrensburg, MO 64093

[4]Mathematics Department
University of Northern Iowa
Cedar Falls, IA 50614–0506

Abstract

The standard Stanton 4-cycle is the multigraph with vertex set $\{a, b, c, d\}$ and edge multiset $\big\{\{a, b\}, \{a, d\}, \{a, d\}, \{c, d\}, \{c, d\}, \{c, d\}, \{c, b\}, \{c, b\}, \{c, b\}, \{c, b\}\big\}$. For each integer $v \geq 4$, we find the smallest λ such that there exists a standard Stanton 4-cycle decomposition of ${}^{\lambda}K_v$.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, then $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_m$ the group of integers modulo m. For a finite set S and a positive integer λ, we let ${}^{\lambda}S$

*Research supported by National Science Foundation Grant No. A1063038

denote the multiset that contains every element of S exactly λ times. For example ${}^2[a,b]$ is the multiset $\{a,a,a+1,a+1,\ldots,b-1,b-1,b,b\}$. For a simple graph G and a positive integer λ, we use ${}^\lambda G$ to denote the graph obtained from G by replacing each edge in $E(G)$ with λ parallel edges. Thus ${}^\lambda K_m$ denotes the λ-fold complete mutigraph of order m. Alternatively, we let λG denote the graph consisting of λ vertex-disjoint copies of G. We use $K_{r\times s}$ to denote the complete simple multipartite graph with r parts of size s. We note that a multigraph is not required to contain multiple edges. Thus a simple graph is a multigraph. By the *order* and *size* of a graph G, we mean $|V(G)|$ and $|E(G)|$, respectively.

Let $V({}^\lambda K_m) = \{0,1,\ldots,m-1\}$. The *label* of an edge $\{i,j\}$ in ${}^\lambda K_m$ is defined to be $|i-j|$. The *length* of an edge $\{i,j\}$ in ${}^\lambda K_m$ is defined to be $\min\{|i-j|, m-|i-j|\}$. Note that if m is odd, then ${}^\lambda K_m$ consists of λm edges of length i for $i \in \{1,2,\ldots,\frac{m-1}{2}\}$. If m is even, then ${}^\lambda K_m$ consists of λm edges of length i for $i \in \{1,2,\ldots,\frac{m}{2}-1\}$, and $\lambda m/2$ edges of length $\frac{m}{2}$.

Let $V({}^\lambda K_m) = \mathbb{Z}_m$ and let G be a subgraph of ${}^\lambda K_m$. By *clicking* G, we mean applying the permutation $i \mapsto i+1$ to $V(G)$. Moreover in this case, if $j \in \mathbb{N}$, then $G+j$ is the graph obtained from G by successively clicking G a total of j times. Note that clicking an edge does not change its length. Also note that $G+j$ is isomorphic to G for every $j \in \mathbb{N}$.

Alternatively, we may let $V({}^\lambda K_m) = \mathbb{Z}_{m-1} \cup \{\infty\}$. As expected, clicking a subgraph G of ${}^\lambda K_m$ in this case continues to mean applying the permutation $i \mapsto i+1$ to $V(G)$, with the convention that $\infty + 1 = \infty$. If $i,j \in \mathbb{Z}_{m-1}$, then the label and length of the edge $\{i,j\}$ are defined as if $\{i,j\}$ were an edge in ${}^\lambda K_{m-1}$. The label and length of an edge $\{i,\infty\}$ are both defined to be ∞. Also, $G+j$ is defined as before, and clicking an edge does not change its length.

Let K and G be multigraphs with G a subgraph of K. A *G-decomposition* of K is a collection $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that each edge of K appears in exactly one G_i. The elements of Δ are called *G-blocks*. If there exists a G-decomposition of K, then we say G *divides* K and write $G|K$. A G-decomposition of K is also known as a *(K,G)-design*. A $({}^\lambda K_m, G)$-design is called a *G-design* of *order* m and *index* λ. A $({}^\lambda K_m, G)$-design Δ is said to be *cyclic* if clicking preserves the G-blocks in Δ. If $V({}^\lambda K_m) = \mathbb{Z}_{m-1} \cup \{\infty\}$, then a cyclic $({}^\lambda K_m, G)$-design is also called a *1-rotational* $({}^\lambda K_m, G)$-design. The study of graph decompositions is generally known as the study of graph designs, or G-designs. For recent surveys on G-designs of index 1, see [1] and [2].

Let G be a graph of size n. A primary question in the study of graph designs is, "*For what values of k does there exist a $({}^\lambda K_k, G)$-design*?" The set of all such k is called the *spectrum for G-designs of index λ*. For simple graphs G, the spectrum for G-designs of index 1 has been determined for several classes of graphs including cycles, paths, stars, and complete graphs of order at most 5. For simple graphs G of order at most 5, the spectrum for G-designs of index 1 has been determined for all but 11 values of k (see [1]).

In recent years, there have been some investigations of G-designs of index λ where G is a multigraph with edge multiplicity at least 2. For example, in [4] Carter determined the spectrum for G-designs of index λ for all connected cubic multigraphs G of order at most 6. Sarvate and various co-authors have investigated G-designs of index λ for various multigraphs G of small order (see for example, [5], [7], [8], and [9]).

The concept of a Stanton graph was recently introduced by Chan and Sarvate in [5] as follows: a *Stanton graph of order* k, denoted S_k, is a graph on k vertices where for each $i \in \{1, 2, \ldots, \binom{k}{2}\}$ there is exactly one pair of vertices with i parallel edges between them. The graph S_3 is shown in Figure 1 below. Thus, the simple graph underlying S_k is the complete graph K_k. Clearly, S_k is not unique when $k \geq 4$. One can obviously generalize this interesting concept. Given a simple graph G with q edges $\{e_1, e_2, \ldots, e_q\}$, a Stanton graph SG can be obtained from G by replacing e_i with i parallel edges for $1 \leq i \leq q$. In light of this generalization and because S_k is often used to denote the star with k edges, it would be reasonable to use the notation SK_k in place of S_k. Since K_3 is isomorphic to C_3, the 3-cycle, we use SC_3 in place of SK_3, and we call SC_3 the *Stanton* 3*-cycle.*

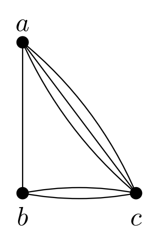

Figure 1: The Stanton graph SK_3 which we also denote by SC_3.

In [5], Chan and Sarvate investigated decompositions of ${}^{\lambda}K_v$ into SC_3. They settled the following problem.

Problem 1.1. *For each integer $v \geq 3$, find the minimum λ such that SC_3 decomposes ${}^{\lambda}K_v$.*

The following theorem summarizes the results of Chan and Sarvate [5].

Theorem 1.1. *Given an integer $v \geq 3$, the minimum λ for which ${}^{\lambda}K_v$ can be decomposed into SC_3 is as follows:*

- $\lambda = 3$ *for* $v \equiv 0, 1, 4, 5, 8,$ *or* $9 \pmod{12}$,

- $\lambda = 4$ *for* $v \equiv 3, 6, 7,$ *or* $10 \pmod{12}$,
- $\lambda = 6$ *for* $v \equiv 2$ *or* $11 \pmod{12}$.

In [6], El-Zanati, Lapchinda, Tangsupphathawat, and Wannasit find necessary and sufficient conditions for the existence of an SC_3-decomposition of ${}^{\lambda}K_v$ for every pair of integers $\lambda \geq 3$ and $v \geq 3$.

In this note, we settle the problem corresponding to Theorem 1.1 for what we call the standard Stanton 4-cycle. There are three non-isomorphic Stanton 4-cycles (see Figure 2). The *standard* Stanton 4-cycle, denoted here by $G_1(a, b, c, d)$ or simply by G_1, is the multigraph with vertex set $\{a, b, c, d\}$ and edge multiset $\big\{\{a, b\}, \{a, d\}, \{a, d\}, \{c, d\}, \{c, d\}, \{c, d\}, \{c, b\}, \{c, b\}, \{c, b\}, \{c, b\}\big\}$. For each integer $v \geq 4$, we find the smallest λ such that there exists a G_1-decomposition of ${}^{\lambda}K_v$.

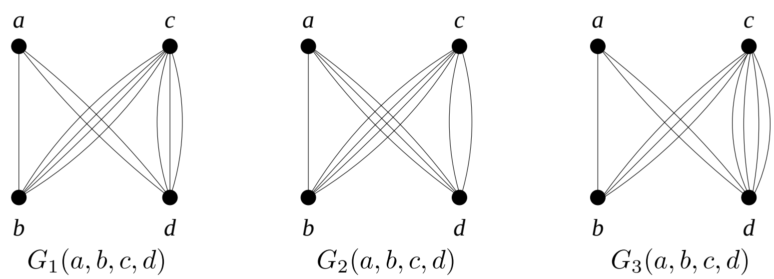

Figure 2: The three non-isomorphic Stanton 4-cycles.

2 Decompositions via Graph Labelings

Suppose G is a multigraph of order p, size n, and maximum edge multiplicity μ. If there exists a G-decomposition of ${}^{\lambda}K_v$, then we must have $v \geq p$, $\lambda \geq \mu$, and $2n | \lambda v(v-1)$. If $\lambda | 2n$, then $v \equiv 0$ or $1 \pmod{\frac{2n}{\lambda}}$ always satisfies the last condition. It is often the case that G-decompositions of ${}^{\lambda}K_v$ when $v \equiv 0$ or $1 \pmod{\frac{2n}{\lambda}}$ can be found via certain types of graph labelings. We define four of these labelings next.

Let n, k, and λ be positive integers such that $n = \lambda k$ or such that λ is even and $n = \lambda k + \frac{\lambda}{2}$. Let G be a multigraph of size n, order at most $\frac{2n}{\lambda} + 1$, and edge multiplicity at most λ. A *λ-fold ρ-labeling* of G is a one-to-one function $f\colon V(G) \to \{0, 1, \ldots, \frac{2n}{\lambda}\}$ such that the multiset

$$\Big\{\min\big\{|f(u) - f(v)|, \tfrac{2n}{\lambda} + 1 - |f(u) - f(v)|\big\} \colon \{u, v\} \in E(G)\Big\}$$
$$= \begin{cases} {}^{\lambda}[1, k] & \text{if } n = \lambda k, \\ {}^{\lambda}[1, k] \cup {}^{\frac{\lambda}{2}}\{k + 1\} & \text{if } n = \lambda k + \frac{\lambda}{2}. \end{cases}$$

Thus a λ-fold ρ-labeling of such a G induces an embedding of G in ${}^{\lambda}K_{\frac{2n}{\lambda}+1}$ so that G has either (1) λ edges of length i for each $i \in [1, k]$ when $n = \lambda k$ or (2) λ edges of length i for each $i \in [1, k]$ and $\frac{\lambda}{2}$ edges of length $k + 1$ when $n = \lambda k + \frac{\lambda}{2}$.

If f is a λ-fold ρ-labeling of a bipartite multigraph G with vertex bipartition $\{A, B\}$ and if for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$ we have $f(a) < f(b)$, then f is called an *ordered λ-fold ρ-labeling*, or λ-fold ρ^{+}-labeling.

The next four theorems are proved in [3].

Theorem 2.1. *Let G be a subgraph of ${}^{\lambda}K_{\frac{2n}{\lambda}+1}$ such that $|E(G)| = n$. There exists a cyclic $({}^{\lambda}K_{\frac{2n}{\lambda}+1}, G)$-design if and only if G admits a λ-fold ρ-labeling.*

Theorem 2.2. *Let G be a bipartite subgraph of ${}^{\lambda}K_{\frac{2n}{\lambda}+1}$ such that $|E(G)| = n$. If G admits a 2-fold ρ^{+}-labeling, then there exists a cyclic $({}^{\lambda}K_{\frac{2n}{\lambda}x+1}, G)$-design for each positive integer x.*

It is not difficult to see how Theorem 2.2 works. Let $\{A, B\}$ be a bipartition of $V(G)$ and let f be a λ-fold ρ^{+}-labeling of G such that $f(u) < f(v)$ for every edge $\{u, v\} \in E(G)$ with $u \in A$ and $v \in B$. Let $A = \{u_1, u_2, \ldots, u_r\}$ and $B = \{v_1, v_2, \ldots, v_s\}$. Let x be a positive integer. For $1 \leq i \leq x$, let H_i be a copy of G with vertex bipartition (A, B_i) where $B_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,s}\}$ and $v_{i,j}$ corresponds to v_j in B. Let $G(x) = H_1 \cup H_2 \cup \cdots \cup H_x$. Thus $G(x)$ is bipartite with bipartition $\{A, B_1 \cup B_2 \cup \cdots \cup B_x\}$. Define a labeling f' of $G(x)$ as follows: $f'(u_j) = f(u_j)$ for each $u_j \in A$ for $1 \leq j \leq r$ and $f'(v_{i,j}) = f(v_j) + (i-1)\frac{2n}{\lambda}$ for $1 \leq i \leq x$ and $1 \leq j \leq s$. It is easy to see that f' is a λ-fold ρ-labeling of $G(x)$, and thus Theorem 2.1 applies.

Now, let G of size n be a subgraph of ${}^{\lambda}K_{\frac{2n}{\lambda}}$. Let w be a vertex in $V(G)$ of degree λ and let y and z be the neighbors of w (y and z need not be distinct). A *1-rotational λ-fold labeling* of G is a one-to-one function $f\colon V(G) \to \mathbb{Z}_{\frac{2n}{\lambda}-1} \cup \{\infty\}$ such that f restricted to $G - w$ is a λ-fold ρ-labeling, $f(w) = \infty$, $f(y) = 0$, and $f(z) \in \{0, 1\}$. If in addition G is bipartite and f restricted to $G - w$ is a λ-fold ρ^{+}-labeling, then f is *ordered.*

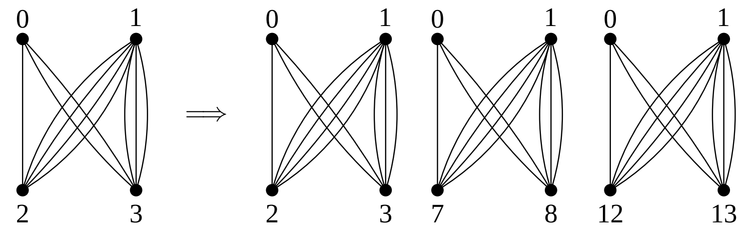

Figure 3: A 4-fold ρ^+-labeling of G_1 and three G_1-blocks that can be used as starters for a cyclic G_1-decomposition of ${}^4K_{16}$.

Theorem 2.3. *Let G be a subgraph of ${}^\lambda K_{\frac{2n}{\lambda}}$ such that $|E(G)| = n$. There exists a 1-rotational G-decomposition of ${}^\lambda K_{\frac{2n}{\lambda}}$ if and only if G admits a 1-rotational λ-fold labeling.*

Theorem 2.4. *Let Gbe a bipartite subgraph of ${}^\lambda K_{\frac{2n}{\lambda}}$ such that $|E(G)| = n$. If G admits an ordered 1-rotational λ-fold labeling, then there exists a 1-rotational G-decomposition of ${}^\lambda K_{\frac{2n}{\lambda}x}$ for every positive integer x.*

Again we illustrate how Theorem 2.4 works. Let $\{A, B\}$ be a bipartition of $V(G)$ and let $w \in B$ and $y, z \in A$ be as in the definition of an ordered 1-rotational λ-fold labeling. Let f be such a labeling of G. Let $B = \{w, v_1, v_2, \ldots, v_s\}$. Let x be a positive integer. For $1 \le i \le x$, let H_i be a copy of G with bipartition (A, B_i) where $B_i = \{w_i, v_{i,1}, v_{i,2}, \ldots, v_{i,s}\}$ and w_i corresponds to w and $v_{i,j}$ corresponds to v_j in B. Let $G(x) = H_1 \cup H_2 \cup \cdots \cup H_x$. Thus $G(x)$ is bipartite with bipartition $\{A, B_1 \cup B_2 \cup \cdots \cup B_x\}$. Define a labeling f' of $G(x)$ as follows: $f'(a) = f(a)$ for each $a \in A$ and $f'(b) = f(b)$ for each $b \in B_1$. For $2 \le i \le x$ and $1 \le j \le s$, let $f'(w_i) = (i-1)n$ and $f'(v_{i,j}) = f(v_j) + (i-1)n$. Then f' is a 1-rotational λ-fold labeling of $G(x)$, and thus Theorem 2.3 applies.

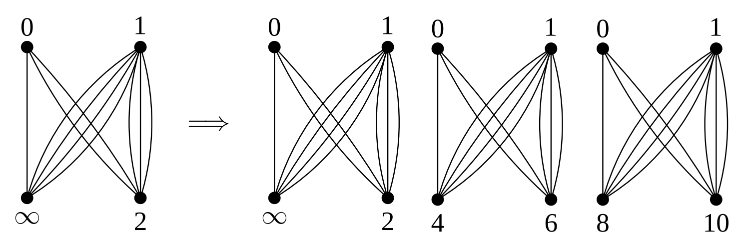

Figure 4: A 1-rotational 5-fold labeling of G_1 and three G_1-blocks that can be used as starters for a 1-rotational G_1-decomposition of ${}^5K_{12}$.

3 Some Small Examples

In this section we present G_1-decompositions of various graphs that are needed for the constructions used in Section 4. A G_1-decomposition of a graph H with vertex set V may be written as an ordered pair (V, B) where B is a collection of copies of G_1 that partitions the edge set of H.

Example 3.1. Let $V = \mathbb{Z}_{10}$ and let $B = \{G_1(1,8,0,4),\ G_1(7,5,0,1),\ G_1(8,3,0,2),\ G_1(1,2,3,7),\ G_1(4,5,3,1),\ G_1(7,6,3,8),\ G_1(1,0,6,5),\ G_1(4,8,6,7),\ G_1(5,9,6,2),\ G_1(7,3,9,8),\ G_1(7,2,9,4),\ G_1(2,0,9,5),\ G_1(4,9,1,2),\ G_1(2,6,1,8),\ G_1(1,3,4,5),\ G_1(5,6,4,8),\ G_1(4,0,7,2),\ G_1(8,9,7,5)\}$. Then (V, B) is a G_1-decomposition of ${}^4K_{10}$.

Example 3.2. Let $V = V({}^4K_{5,5}) = \mathbb{Z}_5 \times \mathbb{Z}_2$ with the obvious bipartition and let $B = \{G_1((i,0),(1+i,1),(2+i,0),(3+i,1)) : i \in \mathbb{Z}_5\} \cup \{G_1((2+i,0),(2+i,1),(i,0),(i,1)) : i \in \mathbb{Z}_5\}$. Then (V, B) is a G_1-decomposition of ${}^4K_{5,5}$.

Example 3.3. Let $V = V({}^4K_{5,9}) = \{u_0, ..., u_4\} \cup \{v_0, ..., v_8\}$ with the obvious bipartition and let $B = \{G_1(u_0, v_0, u_1, v_6),\ G_1(u_2, v_7, u_1, v_1),\ G_1(u_4, v_4, u_1, v_5),\ G_1(u_0, v_3, u_3, v_2),\ G_1(u_2, v_6, u_3, v_8),\ G_1(u_4, v_1, u_3, v_0),\ G_1(u_4, v_8, u_0, v_7),\ G_1(u_0, v_5, u_2, v_4),\ G_1(u_2, v_2, u_4, v_3),\ G_1(v_6, u_1, v_3, u_0),\ G_1(v_1, u_1, v_2, u_2),\ G_1(v_5, u_1, v_8, u_4),\ G_1(v_2, u_3, v_5, u_0),\ G_1(v_8, u_3, v_7, u_2),\ G_1(v_0, u_3, v_4, u_4),\ G_1(v_7, u_0, v_1, u_4),\ G_1(v_4, u_2, v_0, u_0),\ G_1(v_3, u_4, v_6, u_2)\}$. Then (V, B) is a G_1-decomposition of ${}^4K_{5,9}$.

Example 3.4. Let H be the 4-fold complete graph of order 15 with vertex set $H_1 \cup H_2 \cup \{\infty\}$, with $|H_1| = 9$ and $|H_2| = 5$. Let B_1 be a G_1-decomposition of ${}^4K_{10}$ with vertex set $H_1 \cup \{\infty\}$ and let B_2 be a G_1-decomposition of ${}^4K_6$ with vertex set $H_2 \cup \{\infty\}$ (which is shown to exist in Theorem 4.1). Let B_3 be a G_1-decomposition of ${}^4K_{5,9}$ with vertex set $H_1 \cup H_2$ with the obvious bipartition. Then (V, B), where $V = V(H)$ and $B = B_1 \cup B_2 \cup B_3$ is a G_1-decomposition of ${}^4K_{15}$.

Example 3.5. Let $V = V(K_{2,2}) = \{u_0, u_1\} \cup \{v_0, v_1\}$ with the obvious bipartition and let $B = \{G_1(u_0, v_0, u_1, v_1), G_1(u_1, v_0, u_0, v_1)\}$. Then (V, B) is a G_1-decomposition of ${}^5K_{2,2}$.

Example 3.6. Let $V = V(K_{2,3}) = \{u_0, u_1\} \cup \{v_0, v_1, v_2\}$ with the obvious bipartition and let $B = \{G_1(v_0, u_0, v_1, u_1),\ G_1(v_1, u_0, v_2, u_1),\ G_1(v_2, u_0, v_0, u_1)\}$. Then (V, B) is a G_1-decomposition of ${}^5K_{2,3}$.

Example 3.7. Let $V = \mathbb{Z}_5 \cup \{\infty\}$ and let $B = \{G_1(i, 2+i, \infty, 1+i) : i \in \mathbb{Z}_5\} \cup \{G_1(\infty, 1+i, i, 2+i) : i \in \mathbb{Z}_5\} \cup \{G_1(i, 4+i, 1+i, 2+i) : i \in \mathbb{Z}_5\}$. Then (V, B) is a G_1-decomposition of ${}^{10}K_6$.

Example 3.8. Let $V = \mathbb{Z}_7$ and let $B = \{G_1(1+i, 5+i, i, 3+i) \colon i \in \mathbb{Z}_7\} \cup \{G_1(i, 2+i, 1+i, 3+i) \colon i \in \mathbb{Z}_7\} \cup \{G_1(2+i, 3+i, i, 1+i) \colon i \in \mathbb{Z}_7\}$. Then (V, B) is a G_1-decomposition of ${}^{10}K_7$.

4 Main Results

In this section, we give several general results on G_1-decompositions of ${}^{\lambda}K_v$ for $\lambda \in \{4, 5, 10\}$. These results enable us to find, for each $v \geq 4$, the smallest λ such that there exists a G_1-decomposition of ${}^{\lambda}K_v$.

Theorem 4.1. *There exists a G_1-decomposition of ${}^4K_{5x+1}$ for every positive integer x.*

Proof. By Theorem 2.2, it suffices to show that G_1 admits a 4-fold ρ^+-labeling. We note that $G_1(0, 2, 1, 3)$ induces such a labeling. □

Theorem 4.2. *There exists a G_1-decomposition of ${}^4K_{10x}$ for every positive integer x.*

Proof. Let ${}^4K_{10x} = x({}^4K_{10}) \cup {}^4K_{x \times 10}$. It is easy to see that $K_{5,5}|K_{x\times 10}$ and thus ${}^4K_{5,5}|{}^4K_{x\times 10}$. Since $G_1|{}^4K_{5,5}$ and $G_1|{}^4K_{10}$, we have $G_1|{}^4K_{10x}$. □

Theorem 4.3. *There exists a G_1-decomposition of ${}^4K_{10x+5}$ for every positive integer x.*

Proof. By Example 3.4, we have $G_1|{}^4K_{15}$. For $x \geq 2$, let ${}^4K_{10x+5} = {}^4K_{10(x-1)} \cup {}^4K_{15} \cup {}^4K_{10(x-1),15}$. By Theorem 4.2, we have $G_1|{}^4K_{10(x-1)}$. Since $K_{5,5}|K_{10(x-1),15}$ and $G_1|{}^4K_{5,5}$, we have $G_1|{}^4K_{10x+5}$. □

Theorem 4.4. *There exists a G_1-decomposition of ${}^5K_{4x}$ for every positive integer x.*

Proof. By Theorem 2.4, it suffices to show that G_1 admits an ordered 1-rotational 5-fold labeling. It is easy to verify that $G_1(0, \infty, 1, 2)$ induces such a labeling. □

Theorem 4.5. *There exists a G_1-decomposition of ${}^5K_{8x+1}$ for every positive integer x.*

Proof. Let G_1' be the graph with 20 edges induced by the following union: $G_1(0, 4, 2, 3) \cup G_1(3, 5, 1, 4)$. It is easy to verify that the given labeling of G_1' is a 5-fold ρ^+-labeling. Thus by Theroem 2.2, there exists a cyclic G_1'-decomposition of ${}^5K_{8x+1}$ for every positive integer x. Since $G_1|G_1'$, the result follows. □

Theorem 4.6. *There exists a G_1-decomposition of ${}^5K_{8x+5}$ for every positive integer x.*

Proof. For $x = 1$, let G_1' be the graph with 30 edges induced by the following: $G_1(0,5,1,3) \cup G_1(1,8,0,3) \cup G_1(2,6,0,1)$. It is easily checked that the given labeling of G_1' is a 5-fold ρ-labeling, and thus there exists a cyclic G_1'-decomposition of ${}^5K_{13}$. Since, $G_1 | G_1'$, we get $G_1 | {}^5K_{13}$.

For $x \geq 2$, let H be the 5-fold complete graph with vertex set $H_1 \cup H_2 \cup \{\infty\}$, with $|H_1| = 8(x-1)$ and $|H_2| = 12$. Let B_1 be a G_1-decomposition of ${}^5K_{8(x-1)+1}$ with vertex set $H_1 \cup \{\infty\}$ and let B_2 be a G_1-decomposition of ${}^5K_{13}$ with vertex set $H_2 \cup \{\infty\}$. Let B_3 be a G_1-decomposition of ${}^5K_{8(x-1),12}$ with vertex set $H_1 \cup H_2$ with the obvious bipartition. Then (V, B), where $V = V(H)$ and $B = B_1 \cup B_2 \cup B_3$ is a G_1-decomposition of ${}^5K_{8x+5}$. □

Theorem 4.7. *There exists a G_1-decomposition of ${}^{10}K_{2x}$ for every integer $x \geq 2$.*

Proof. By Theorem 4.4, we have $G_1 | {}^5K_{4x}$ for every positive integer x. Since ${}^5K_{4x} | {}^{10}K_{4x}$, it suffices to show that $G_1 | {}^{10}K_{4x+2}$. By Example 3.7, we have $G_1 | {}^{10}K_6$. Let $x \geq 2$ and let ${}^{10}K_{4x+2} = {}^{10}K_{4(x-1)} \cup {}^{10}K_6 \cup {}^{10}K_{4(x-1),6}$. Since G_1 divides ${}^{10}K_{4(x-1)}$, ${}^{10}K_6$, and ${}^{10}K_{4(x-1),6}$, the result follows. □

Theorem 4.8. *There exists a G_1-decomposition of ${}^{10}K_{2x+1}$ for every integer $x \geq 2$.*

Proof. Let $V({}^{10}K_5) = \mathbb{Z}_5$ and let G_1' be the subgraph induced by the following; $G_1(1,4,0,2) \cup G_1(0,4,2,3)$. It is easy to check that the given labeling of G_1' is a 10-fold ρ^+-labeling and thus there exists a cyclic G_1'-decomposition of ${}^{10}K_{4x+1}$ for all positive integers x. Since $G_1 | G_1'$, we have $G_1 | {}^{10}K_{4x+1}$.

What remains to be shown is that $G_1 | {}^{10}K_{4x+3}$. By Example 3.8, we have $G_1 | {}^{10}K_7$. For $x \geq 2$, let ${}^{10}K_{4x+3} = {}^{10}K_{4(x-1)} \cup {}^{10}K_7 \cup {}^{10}K_{4(x-1),7}$. It is easy to see that $K_{4(x-1),7}$ can be decomposed into a combination of $K_{2,2}$'s and $K_{2,3}$'s. Thus, ${}^{10}K_{4(x-1),7}$ can be decomposed into a combination of ${}^{10}K_{2,2}$'s and ${}^{10}K_{2,3}$'s, and hence $G_1 | {}^{10}K_{4(x-1),7}$. Since G_1 divides ${}^{10}K_{4(x-1)}$, ${}^{10}K_7$, and ${}^{10}K_{4(x-1),7}$, the result follows. □

In light of the results in the previous eight theorems, we can now give the theorem that corresponds to the main result from Chan and Sarvate [5].

Theorem 4.9. *Given an integer $v \geq 4$, the minimum λ for which ${}^\lambda K_v$ can be decomposed into the standard Stanton 4-cycle is as follows:*

- $\lambda = 4$ *for* $v \equiv 0, 1, 5, 6, 10, 11, 15, 16 \pmod{20}$,
- $\lambda = 5$ *for* $v \equiv 4, 8, 9, 12, 13, 17 \pmod{20}$,
- $\lambda = 10$ *for* $v \equiv 2, 3, 7, 14, 18, 19 \pmod{20}$.

Proof. If there exists a G_1-decomposition of ${}^{\lambda}K_v$, then we must have $v \geq 4$, $\lambda \geq 4$, and that $\lambda v(v-1)$ is divisible by 20. Thus, if $v \equiv 0 \text{ or } 1 \pmod 5$, then $\lambda = 4$ is the least possible λ that satisfies the above conditions for the existence of a G_1-decomposition of ${}^{\lambda}K_v$. Similarly, if $v \equiv 0 \text{ or } 1 \pmod 4$ but $v \not\equiv 0 \text{ or } 1 \pmod 5$, then $\lambda = 5$ is the least possible λ for the existence of the decomposition. Finally, if $v \not\equiv 0 \text{ or } 1 \pmod 4$ and $v \not\equiv 0 \text{ or } 1 \pmod 5$, then $\lambda = 10$ is necessary. This establishes the necessity of the conditions in the theorem. The sufficiency of these conditions follows from the previous eight theorems in this section. □

5 Acknowledgement and Final Note

This research is partially supported by grant number A1063038 from the Division of Mathematical Sciences at the National Science Foundation. This work was done while the third through sixth authors were participants in *REU Site: Mathematics Research Experience for Pre-service and for In-service Teachers* at Illinois State University. In a forthcoming article, the same set of authors settle the problem corresponding to Theorem 4.9 for the two other Stanton 4-cycles.

References

[1] P. Adams, D. Bryant, and M. Buchanan, A survey on the existence of G-designs, *J. Combin. Des.* **16** (2008), 373–410.

[2] D. Bryant and S. El-Zanati, "Graph decompositions," *in* Handbook of Combinatorial Designs, C. J. Colbourn and J. H. Dinitz (Editors), 2nd ed., Chapman & Hall/CRC, Boca Raton, 2007, pp. 477–485.

[3] R. C. Bunge, S. I. El-Zanati, C. Vanden Eynden, and W. Wannasit, On λ-fold Rosa-type labelings, preprint.

[4] J. E. Carter, Designs on cubic multigraphs, Ph.D. Thesis, Department of Mathematics and Statistics, McMaster University, Canada, 1989.

[5] H. Chan and D. G. Sarvate, Stanton graph decompositions, *Bull. Inst. Combin. Appl.* **64** (2012), 21–29.

[6] S. I. El-Zanati, W. Lapchinda, P. Tangsupphathawat, and W. Wannasit, The spectrum for the Stanton 3-cycle, *Bull. Inst. Combin. Appl.*, to appear.

[7] S. P. Hurd and D. G. Sarvate, Graph decompositions of $K(v,\lambda)$ into modified triangles using Langford and Skolem Sequences, *Ars Combin.*, to appear.

[8] S. Malick and D. G. Sarvate, Decomposition of λK_v into multigraphs with four vertices and five edges, preprint.

[9] D. G. Sarvate and L. Zhang, Decomposition of $3K_{8t}$ into H_2 graphs, *Bull. Inst. Combin. Appl.*, to appear.