# A Unit on Graph Theory Prepared by the ISU REU 2012

Here we put the introduction and instruction to teachers and table of contents of the module.

# Lesson Introduction Day 1

During this unit you will work on discovering more about mathematics than you ever thought was possible! Throughout this journey you will discover the amazing branch of mathematics known as **Discrete Mathematics**. Within this branch of mathematics you will be learning about ***Graph Theory***. On your journey to expand your mathematical prowess you will first train your brain to think in terms of the basic terminology and notation used within graph theory. You will then demonstrate an understanding of the terminology and notation by creating graphs as instructed. Once you have shown your growth as a mathematician into this new domain, you will then begin a journey into investigation of unsolved mathematical problems in an emerging area of graph theory studying graphs and looking for what happens.

## TERMINOLOGY and NOTATION Part 1

### Defining the Basics

**Vertex**

A **vertex** is an individual point. **Note:** The plural of "vertex" is ***vertices***.

*Mathematical Analogy: A vertex of a graph is like a vertex of a polygon.*

**Order**

The **order** of graph $G$ is the number of *vertices* found in graph $G$.

**Edge**

An **edge** connects two vertices.

*Mathematical Analogy: An edge of a graph is like an edge of a polygon.*

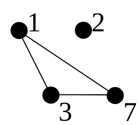
**Size**

The **size** of graph $G$ is the number of *edges* found in graph $G$.

**Graph**

**Formal Mathematical Definition:** A **graph** $G$ is an ordered pair $(V(G), E(G))$ where $V(G)$ is a non-empty finite set of vertices $\{v_1, v_2, v_3, ..., v_n\}$ and $E(G)$ is a set of 2-element subsets of $V(G)$ (Ex: $\{\{v_1, v_3\}, \{v_3, v_2\}, \{v_2, v_n\}\}$) that define the edges.

**Example:** $G = \Big(\{1, 2, 3, 7\}, \{\{1, 3\}, \{1, 7\}, \{3, 7\}\}\Big)$ where 1, 2, 3, and 7 are the vertices and $\{1, 3\}$ denotes the edge connecting vertices 1 and 3, $\{1, 7\}$ denotes the edge connecting 1 and 7, and $\{3, 7\}$ denotes the edge connecting 3 and 7. Below is a *drawing* of graph $G$.




***Note:*** The ***order*** of $G$ is 4 while the ***size*** of $G$ is 3.

***Note:*** The location of the *vertices* is arbitrary and can be changed at any time.

**Define *graph* in your own words and then draw another design for the above graph $G$ changing the location of the vertices.**

Once you like the definition you've formulated for the term ***graph*** and have a new design, ask your teacher to check your work. If you are fully comfortable with all the words presented so far and your teacher approves of your work, show your mathematical prowess and answer the questions found on the next page.

**Now It's Your Turn #1: Working with the Basic Vocabulary**

1. **Moving from mathematical notation to a visual representation.**

   A. Draw the graph $G = \Big(\{2,3,5,7,11\}, \{\{2,5\},\{3,7\},\{2,11\},\{5,11\},\{3,11\}\}\Big)$.

   B. What is the order of graph $G$? What is the size of graph $G$?

   **Order:** ______________ **Size:** ______________

   C. Compare your graph $G$ with a few classmates. Are your drawings identical? If there is a difference in the drawings, does that imply that someone drew graph $G$ incorrectly? Why or why not?

2. **Moving from a visual representation to mathematical notation.**

   A. Draw a graph $H$ with 7 vertices and 8 edges making sure to label your vertices.

   B. Provide the mathematical notation of your graph $H$ in the form $(V(H), E(H))$.

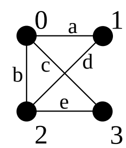
   C. Find a partner and swap the formal definitions you created in part 2B.. Draw your partner's graph $H$ using his/her formal definition and then compare your drawing with his/hers. Is your drawing identical to his/hers? If they are not 100% identical could they still represent the same graph? Why or why not?

**Defining Basic Structure Vocabulary**

**Incident and Endvertex**

**Formal Mathematical Definition:** If $e$ is an edge in a graph $G$ where $e = \{u, v\}$, then we say vertex $u$ is **incident** with edge $e$. We also say edge $e$ is **incident** with $u$ and $v$. Moreover, $u$ and $v$ are called the ***endvertices*** of $e$.

**Example:** Looking at the graph below, notice that vertex 0 is incident with edges $a$, $b$, and $c$, while vertices 1 and 2 are the endvertices of edge $d$.




**Define *incident* in your own words:**

**Adjacent**

Two edges are **adjacent** if they are *incident* with the same vertex in graph $G$. Similarly, two vertices are **adjacent** if they are *incident* with the same edge in graph $G$.

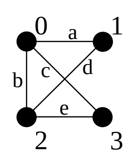
**Degree**

The **degree** of a vertex in graph $G$ is the number of edges *incident* with the vertex.

**Path**

If $u$ and $v$ are vertices in $G$ then a **path** from $u$ to $v$ is an alternating sequence of vertices and edges with starting vertex $u$ and ending vertex $v$. **Note:** No edges may be repeated in any path.

**Example:** In the graph shown below there is a path from vertex 0 to vertex 3 which is comprised of edges $a$, $d$, and $e$ in that order.




**Simple Path**

A *path* is a **simple path** if no vertex is repeated.

**Cycle**

A *path* is a **cycle** if it begins and ends at the same vertex and no other vertices are repeated.

**Star**

A **star** is a graph where all edges share a common vertex.

**Example:** Draw the **star** $S = (\{0, 1, 2, 3, 4, 5\}, \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{0, 5\}\})$.

Once you like the definition you've formulated for the term ***incident*** and have drawn the ***star***, have your work checked by your teacher. If you are fully comfortable with all the words presented so far and your teacher approves of your definition, show your mathematical prowess once again and answer the questions found on the next page.

**Now It's Your Turn #2: Working with Basic Structure Vocabulary**

For the questions in this section you will first need to draw the graph described below as all the questions require use of this graph. Be sure to have your graph checked before moving on!

$M = (\{1,2,3,4,5,6,7\}, \{\{1,3\},\{1,2\},\{1,6\},\{1,5\},\{2,3\},\{2,5\},\{4,6\},\{4,7\},\{5,6\},\{5,7\},\{6,7\}\})$

3. What is the order of graph $M$? What is the size of graph $M$?

   **Order:** __________ **Size:** __________

4. What is the degree of each vertex?

   **1:** ____ **2:** ____ **3:** ____ **4:** ____ **5:** ____ **6:** ____ **7:** ____

5. What edges are incident with vertex 5? (Name the edges in $\{u,v\}$ form.)

6. What edges are adjacent to edge $\{3,1\}$? (Name the edges in $\{u,v\}$ form.)

7. **Working with paths:** Find paths within graph $M$ by listing the vertices of the path in order ***(ex: 1-2-5-7-6)***. Once you have a path indicate whether or not it is simple. Try to be creative with each new path.

   A. 5 edges __________________ Simple? *Yes/No* ________

   B. 6 edges __________________ Simple? *Yes/No* ________

   C. 7 edges __________________ Simple? *Yes/No* ________

   D. 8 edges __________________ Simple? *Yes/No* ________

8. **Working with cycles:** Find cycles within graph $M$ by listing the vertices of the cycle ***(ex: 1-2-5-7-6-1)***. Try to be creative with each new cycle.

   A. 3 edges ______________________________

   B. 4 edges ______________________________

   C. 5 edges ______________________________

   D. 7 edges ______________________________

9. **Working with stars:** Find stars within graph $M$ by listing the edges with the common vertex coming first ***(ex: {0,1}, {0,2}, {0,3}, {0,4} is a star with four edges)***. Try to be creative with each new star.

   A. 2 edges ______________________________

   B. 3 edges ______________________________

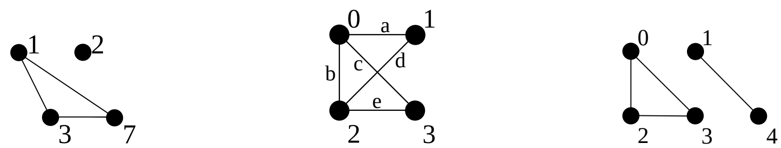
   C. 4 edges ______________________________

**Exploring the Definitions of the Types of Graphs**

**Connected and Disconnected Graphs**

A graph $G$ is **connected** if there is a *path* between every pair of vertices.
A graph $G$ is **disconnected** if there is a pair of vertices for which you cannot find a *path*.

**Example:** Looking at the graphs below determine which are and are not connected.




10. In your own words how would you describe to someone connected versus disconnected graph?

**Complete Graph**

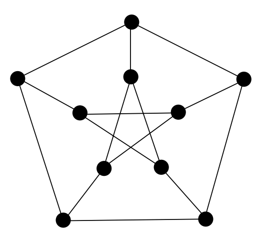
The graph of *order* $n$ where every vertex is adjacent to every other vertex is called the **complete graph** on $n$ vertices and is denoted by $K_n$.

**Example:** Shown below are graphs $K_4$ and $K_5$.


**Regular Graph**

A graph is **regular** if every vertex has the same degree. If every vertex has degree $k$, we call the graph $k$-*regular*.

**Example:** Shown below is a graph that is 3-regular, also known as *cubic*.

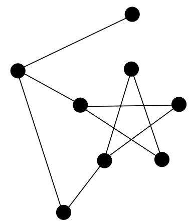


11. Explain the similarities and differences of complete and regular graphs.

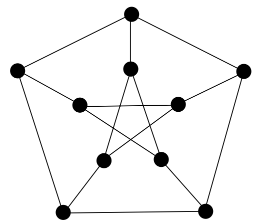
**Subgraph**

Let $G$ and $H$ be graphs. We say that $H$ is a **subgraph** of $G$ if the set of vertices in $H$ is a subset of the set of vertices in $G$ and the set of edges in $H$ is a subset of the edges in $G$. *For $H$ to be a **subgraph** of $G$ then $H$ can fit perfectly within $G$.*

**Example:** The graph on the left is a subgraph of the graph on the right.



**Graph** $H$

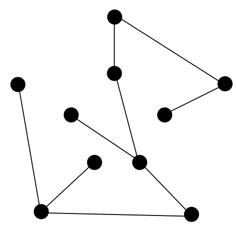


**Graph** $G$

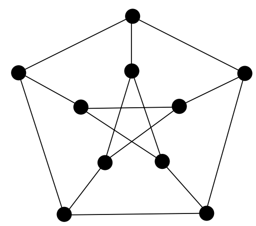
**Spanning Subgraph**

A *subgraph* $H$ of graph $G$ is called **spanning** if $H$ has the same set of vertices as $G$. If $H$ is a *$k$-regular spanning subgraph*, then we say $H$ is a $k$**-factor** of $G$.

**Example:** The graph on the left is a **spanning subgraph** of the graph on the right.



**Graph** $H$



**Graph** $G$

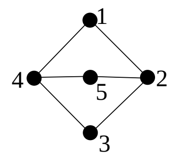
12. Compare and contrast subgraphs and spanning subgraphs.

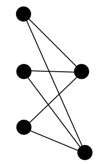
## Defining Properties of Graphs

**Isomorphic**

Two graphs $G$ and $H$ are said to be **isomorphic** if there is a way to manipulate the location of the vertices of $G$ such that edge adjacency is preserved and graph $G$ looks like graph $H$.

**Example:** The two graphs below are isomorphic to each other.



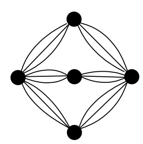



**Your Turn:** Show that the two graphs are **isomorphic** to each other by labeling each vertex in the graph on the right with the same label as the corresponding vertex in the graph on the left.

**Edge Multiplicity and $\lambda$-fold**

If more than one edge is allowed to connect a pair of vertices, the **edge multiplicity** of a graph, commonly denoted by the Greek letter $\lambda$ (lambda), is the highest number of edges between any two vertices. A graph that has the same number of edges between all pairs of adjacent vertices is said to be a **$\lambda$-fold** version of the graph with only 1 edge connecting each pair of adjacent vertices. If a graph is *complete* and **$\lambda$-fold** we symbolize this as ${}^{\lambda}K_n$.

**Example:** Shown below is the 3-fold version of the graph shown above.



**Now try and draw ${}^{2}K_3$.**

**Vertex Coloring**

A **vertex coloring** of a graph is an assignment of colors to each vertex.

**Proper Coloring**

A *vertex coloring* is a **proper coloring** if no two adjacent vertices receive the same color.

**Chromatic Number**

The **chromatic number** of graph $G$, denoted by $\chi(G)$, is the minimum number of colors needed to *properly color* the vertices.

**Bipartite**

A graph $G$ is **bipartite** if the vertices can be *colored properly* with two colors, thus $\chi(G) = 2$.

**Complete Bipartite**

A graph $G$ is a **complete bipartite** graph if each vertex of a given color in a proper coloring is connected to every vertex of the other color. A **complete bipartite** graph is denoted by $K_{m,n}$ where there are $m$ vertices of one color and $n$ vertices of the other color yielding $m \cdot n$ edges.

Once again, after you drawn the ${}^{2}K_3$, have your work checked by your teacher. If you are fully comfortable with all the words presented so far and your teacher approves of your work, show your mathematical prowess and answer the questions found on the next page.

**Now It's Your Turn #3: Working with properties of graphs**

13. **Investigating Bipartite Graphs**

A. Draw the graphs as described with their respective mathematical notation.

i. $Z = \Big(\{A, B, C\}, \big\{\{A, B\}, \{A, C\}\big\}\Big)$.

ii. $J = \Big(\{2, 0, 3, 1, 4\}, \big\{\{0, 2\}, \{1, 4\}, \{3, 1\}, \{0, 4\}, \{0, 3\}\big\}\Big)$.

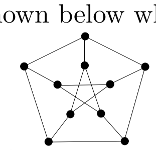
iii. $N = \Big(\{A, B, C, D\}, \big\{\{A, B\}, \{B, C\}, \{C, D\}, \{C, A\}, \{A, D\}, \{B, D\}\big\}\Big)$.

B. Which graphs are bipartite? Which graphs are not bipartite?

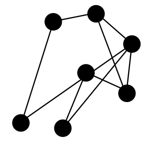
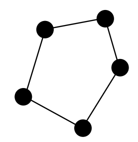
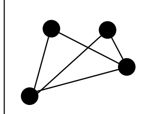
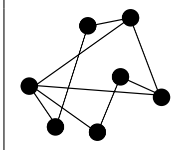
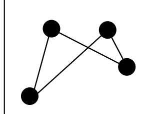
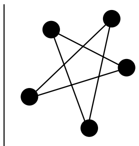
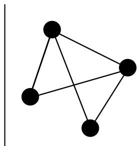
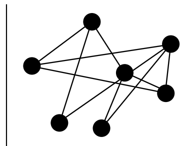
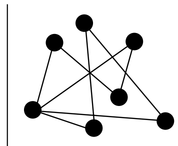
**Bipartite:** ____________ **Not Bipartite:** ____________

C. For the graph(s) that is(are) not bipartite, what edge(s) can you remove to make the graph(s) bipartite?

14. Given graph P shown below what is $\chi(P)$? Why?



15. Match the isomorphic graphs!

| A. | B. | C. | D. | E. | F. | G. | H. | I. |
|---|---|---|---|---|---|---|---|---|
|  |  |  |  |  |  |  |  |  |

16. For the graphs that do not have an isomorphic match, create an isomorphic match.

## Putting it all together!

### Time to Demonstrate Your New Mathematical Knowledge

Now it is time to see what you have learned from the vocabulary just reviewed. Working as instructed, draw the images and then have them checked for accuracy.

17. Describe everything you know about ${}^4K_6$ graph using proper terminology.

18. Every subgraph is a spanning subgraph.
    A. True
    B. False, because: ____________________

19. Draw a 4-regular graph with order 5.

20. Draw two graphs with order 5 of size 7 that are isomorphic with all vertices labeled.

21. Draw the graph for ${}^3K_4$ with all vertices labeled.

22. Draw $K_7$ and then create three subgraphs that can be put together to recreate $K_7$.

23. Draw two different bipartite graphs of order 9.

24. Draw $K_{5,6}$.

25. ***Reflection***: Reflect on today's introduction to graph theory. What was fun and what wasn't? What did you find easy and what was difficult? What do you think is cool and what would you like to investigate? What do you think about graph theory?