On cyclic decompositions of $K_{n+1,n+1} - I$ into a 2-regular graph with at most 2 components*

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Abstract

Let G with n edges be a 2-regular bipartite graph with one or two components. We show that there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$, where I is a 1-factor.

1 Introduction

If m and n are integers with $m \leq n$, we denote $\{m, m+1, \ldots, n\}$ by $[m, n]$. Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. Let K_m have vertex set $\mathbb{Z}_m$ and let G be a subgraph of K_m. By *clicking* G we mean applying the isomorphism $i \mapsto i+1$ to $V(G)$. Likewise, if we let $V(K_{m,m}) = \mathbb{Z}_m \times \mathbb{Z}_2$ with the obvious vertex bipartition, *clicking* a subgraph G of $K_{m,m}$ means to apply the isomorphism $(i, j) \mapsto (i+1, j)$ to $V(G)$.

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Let $V(K_m) = \{0, 1, \ldots, m-1\}$. The *length* of an edge $e = \{i, j\}$ in K_m is $\min\{|i-j|, m-|i-j|\}$. Note that clicking an edge does not change its length.

Now, let $V(K_{m,m}) = \{0, 1, \ldots, m-1\} \times \mathbb{Z}_2$. The *length* of an edge $e = \{(i, 0), (j, 1)\}$ in $K_{m,m}$ is $j - i$ if $j \geq i$ and $m + j - i$, otherwise. As with K_m, we note that clicking an edge in $K_{m,m}$ does not change its length. Also note that $K_{m,m}$ consists of n edges of length i for $i \in [0, m-1]$. Moreover, the edges of length i for $i \in [0, m-1]$ form a 1-factor in $K_{m,m}$.

Let K and G be graphs with G a subgraph of K. A *G-decomposition* of K is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that the edge sets of the graphs G_i form a partition of the edge set of K. The elements of Δ are called *G-blocks*. Such a G-decomposition is said to be *cyclic* if clicking preserves the G-blocks of Δ. A G-decomposition of K is also called a *(K, G)-design*. The study of (K, G)-designs is known as the study of graph designs or simply of G-designs.

A *G-factor* of a graph K is a set of G-blocks whose vertex sets partition the vertex set of K. A *G-factorization* is a G-decomposition where the G-blocks are partitioned into G-factors. A G-factorization is also called a *resolvable G-decomposition*.

The following is a commonly investigated question in graph designs.

Question 1. *Given a graph G with n edges, for which $2n$-regular graphs K does there exist a (K, G)-design?*

Question 1 is difficult to answer in general. However, it is often the case that (K_{2n+1}, G)-designs do exist. Similarly, $(K_{2n+2} - I, G)$-designs where I is a 1-factor often exist. If G is bipartite, then the following is also asked.

Question 2. *Given a bipartite graph G with n edges, for which n-regular bipartite graphs K does there exist a (K, G)-design?*

In this case, $K_{n,n}$ and $K_{n+1,n+1} - I$, where I is a 1-factor, are the common candidates for K.

Let G be a 2-regular bipartite graph with n edges. It is of interest to learn whether or not G decomposes $K_{n,n}$ and $K_{n+1,n+1} - I$. These questions relate to the complete bipartite graph version of the Oberwolfach problem. In [5], Piotrowski showed that if $n \equiv 0 \pmod 4$, then there exists a G-decomposition (actually a G-factorization) of $K_{n/2,n/2}$. Since $K_{n/2,n/2}$ decomposes $K_{n,n}$, the the existence of a G-decomposition of $K_{n,n}$ follows in this case. We note however that these decompositions need not be cyclic. If $n \equiv 2 \pmod 4$, then little is known about G-decompositions of $K_{n,n}$ or of $K_{n+1,n+1} - I$, except in a few cases. In [6], Sotteau found necessary and sufficient conditions for the existence of a C_n-decomposition of $K_{v,w}$. The corresponding problem for C_n-decompositions of $K_{v,v} - I$

was first investigated in [1] and settled completely in [4]. In [2], cyclic G-decompositions of $K_{n+1,n+1} - I$ are investigated for 2-regular bipartite graphs G of order $n \equiv 0 \pmod 4$, and the following is proved.

Theorem 1. *Let G be a 2-regular bipartite graph with n edges where $n \equiv 0 \pmod 4$. Then there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$, where I is a 1-factor.*

Finding cyclic G-decomposition of $K_{n+1,n+1} - I$ when $n \equiv 2 \pmod 4$ seems to be far more challenging. In this note, we show that if $n \equiv 2 \pmod 4$ and if G consists of at most two cycles, then there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$.

As is often the case when studying cyclic graph decompositions, graph labelings provide a convenient and powerful tool. We discuss one of these labelings next, and we give some notation.

1.1 Bilabelings

For a bipartite graph G with n edges, the simplest way to obtain a G-decomposition of $K_{n,n}$ is to embed G in $K_{n,n}$ so that there is exactly one edge of G of length i for each $i \in [0, n-1]$. Then clicking G a total of $n-1$ times would yield the desired design cyclically. This result is considered folklore and is used regularly by researchers in the area. In [3], such an embedding of G is called a ρ-bilabeling of G.

Suppose G with n edges has vertex bipartition $\big\{A \times \{0\}, B \times \{1\}\big\}$. A *bilabeling* of G is a function $f\colon V(G) \to \mathbb{N}$ such that $f|_{A\times\{0\}}$ and $f|_{B\times\{1\}}$ are injective. Now if $f\colon V(G) \to [0, n-1]$ is a bilabeling of G, we also define $\bar{f}\colon E(G) \to [0, n-1]$ such that if $e = \{(a, 0), (b, 1)\} \in E(G)$, then $\bar{f}(e) = f\big((b, 1)\big) - f\big((a, 0)\big)$ if $f\big((b, 1)\big) \geq f\big((a, 0)\big)$ and $\bar{f}(e) = |E(G)| + f\big((b, 1)\big) - f\big((a, 0)\big)$, otherwise (i.e., $\bar{f}(e)$ is the length of edge e). Then f is a *ρ-bilabeling* of G if $\big\{\bar{f}(e)\colon e \in E(G)\big\} = [0, n-1]$. Thus we have the following.

Theorem 2. *Let G be a bipartite graph of size n. There exists a cyclic G-decomposition of $K_{n,n}$ if and only if G has a ρ-bilabeling.*

It should be noted that not every bipartite graph admits a ρ-bilabeling. The following theorem is stated without proof in [3]. We provide a quick proof here.

Theorem 3. *Let G be a bipartite graph of size n and suppose every vertex of G has even degree. If G admits a ρ-bilabeling then $n \equiv 0 \pmod 4$.*

Proof. Let $\big\{A \times \{0\}, B \times \{1\}\big\}$ be a bipartition of $V(G)$. We note first that n must be even since every vertex has even degree and $|E(G)| = \sum_{a\in A} \deg\big((a, 0)\big)$. Let f be a ρ-bilabeling of G. Then $\sum_{e\in E(G)} \bar{f}(e) =$

$\sum_{i=0}^{n-1} i = n(n-1)/2$. Moreover, this sum must be even since n is even, $\bar{f}(e) \in \{f((b,1)) - f((a,0)), n + f((b,1)) - f((a,0))\}$ for every edge $e = \{(a,0),(b,1)\}$ in G, and every vertex in G has even degree. Thus 2 divides $n(n-1)/2$. Since n is even, the result follows. ∎

A strategy similar to that of the above proof is used to obtain cyclic G-decompositions of $K_{n+1,n+1} - I$. In this case, we select a length $j \in [0,n]$, and we embed G in $K_{n+1,n+1}$ so that there is exactly one edge of G of length i for each $i \in [0,n] \setminus \{j\}$. The set of all edges of length j forms the 1-factor I. Clicking G a total of n times would yield the desired design cyclically.

1.2 Some notation

We denote the directed path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \le i \le k-1$, by $(x_0, x_1, \ldots, x_k)$. The *first vertex* of this path is x_0, the *second vertex* is x_1, and the *last vertex* is x_k. If $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$ are directed paths with $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$.

For the remainder of this manuscript, we consider only subgraphs of a complete bipartite graphs $K_{m,m}$ with vertex set $\{0, 1, \ldots, m-1\} \times \mathbb{Z}_2$ and the obvious vertex bipartition. Furthermore, if m, n, and i are integers with $m \le n$, we denote $\{(m,i),(m+1,i),\ldots,(n,i)\}$ by $[(m,i),(n,i)]$

Let $P(k)$ be the path with k edges and $k+1$ vertices given by $((0,0), (k,1),(1,0),(k-1,1),(2,0),(k-2,1),\ldots,(\lceil k/2\rceil, \lceil k/2\rceil - \lfloor k/2\rfloor))$. Note that the set of vertices of this graph is $A \cup B$, where $A = [(0,0),(\lfloor k/2\rfloor,0)]$, $B = [(\lfloor k/2\rfloor+1,1),(k,1)]$, and every edge joins a vertex of A to one of B. Furthermore, the set of lengths of the edges of $P(k)$ is $[1,k]$.

Now let a and b be nonnegative integers with $a \le b$ and let us add $(a,0)$ to all the vertices of A and $(b,0)$ to all the vertices of B. We denote the resulting graph by $P(a,b,k)$. Note that this graph has the following properties.

- **P1** $P(a,b,k)$ is a path with first vertex $(a,0)$ and second vertex $(b+k,1)$. Its last vertex is $(a+k/2,0)$ if k is even and $(b+(k+1)/2,1)$ if k is odd.
- **P2** Each edge of $P(a,b,k)$ joins a vertex of $A' = [(a,0),(\lfloor k/2\rfloor+a,0)]$ to a vertex of $B' = [(\lfloor k/2\rfloor+1+b,1),(k+b,1)]$.
- **P3** The set of edge lengths of $P(a,b,k)$ is $[b-a+1, b-a+k]$.

Now consider the directed path $Q(k)$ obtained from $P(k)$ replacing each vertex (i,j) with $(k-i,1-j)$. The new graph is the path $((k,1),(0,0),(k-1,1),(1,0),\ldots,(\lfloor k/2\rfloor, \lfloor k/2\rfloor - \lceil k/2\rceil + 1))$. The set of vertices of $Q(k)$ is

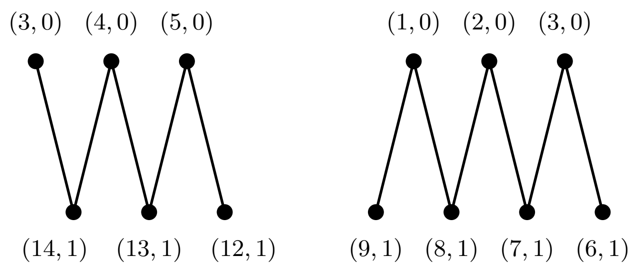
$A \cup B$, where $A = \big[(0,0), (\lceil k/2 \rceil - 1, 0)\big]$ and $B = \big[(\lceil k/2 \rceil, 1), (k, 1)\big]$, and every edge joins a vertex of A to one of B. The set of edge lengths is still $[1, k]$.

We again add $(a, 0)$ to the vertices of A'' and $(b, 0)$ to vertices of B'', where a and b are nonnegative integers with $a \leq b$. We denote the resulting graph by $Q(a, b, k)$. Note that this graph has the following properties.

Q1 $Q(a, b, k)$ is a path with first vertex $(k + b, 1)$. Its last vertex is $(b + k/2, 1)$ if k is even and $(a + (k-1)/2, 0)$ if k is odd.

Q2 Each edge of $Q(a, b, k)$ joins a vertex of $A' = \big[(a, 0), (a + \lceil k/2 \rceil - 1, 0)\big]$ to a vertex of $B = \big[(b + \lceil k/2 \rceil, 1), (b + k, 1)\big]$.

Q3 The set of edge lengths of $Q(a, b, k)$ is $[b - a + 1, b - a + k]$.



$P(3, 9, 5)$ $\qquad$ $Q(1, 3, 6)$

Figure 1: Examples of the $P(a, b, k)$ and $Q(a, b, k)$ notation

For ease of notation, we henceforth use i_0 and i_1 to denote the vertices $(i, 0)$ and $(i, 1)$, respectively.

2 Main Results

Lemma 4. *Let G be an even cycle of length n where $n \equiv 2 \pmod 4$ and let I be a 1-factor of $K_{n+1,n+1}$. Then there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$.*

Proof. Let $G = C_{4r+2}$ where $r \in \mathbb{Z}^+$. Let $C_{4r+2} = G_1 + G_2 + ((2r)_0, 0_1, 0_0)$ where

$$G_1 = P(0, 2r+3, 2r-2),$$
$$G_2 = P(r-1, r-1, 2r+2).$$

First, we show that $G_1 + G_2 + ((2r)_0, 0_1, 0_0)$ is a cycle of length $4r + 2$. Note that by **P1**, the first vertex of G_1 is 0_0, and the last is $(r-1)_0$; and the first vertex of G_2 is $(r-1)_0$, and the last is $(2r)_0$. For $1 \le i \le 2$, let A_i and B_i denote the sets labeled A' and B' in **P2** corresponding to the path G_i. Then using **P2**, we compute

$$A_1 = [0_0, (r-1)_0], \qquad B_1 = [(3r+3)_1, (4r+1)_1],$$
$$A_2 = [(r-1)_0, (2r)_0], \qquad B_2 = [(2r+1)_1, (3r1)_1].$$

Note that $V(G_1) \cap V(G_2) = \{(r-1)_0\}$; otherwise, G_1 and G_2 are vertex-disjoint. Therefore, $G_1 + G_2 + ((2r)_0, 0_1, 0_0)$ is a cycle of length $4r+2$.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 2$. By **P3**, we have edge labels

$$E_1 = [2r+4, 4r+1],$$
$$E_2 = [1, 2r+2]$$

yielding edge lengths of the same values. Moreover, the path $((2r)_0, 0_1, 0_0)$ consists of edges with lengths $(-2r)^* = 2r+3$ and 0. Thus, the edge set of G has one edge of each length $i \in [0, 4r+2] \setminus \{4r+2\}$. An example of this labeling is given in Figure 2 with $r = 2$.

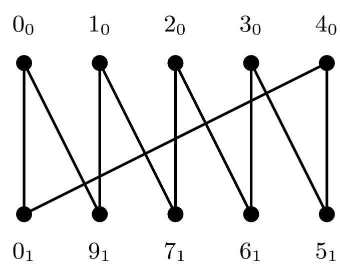


Figure 2: C_{10} with the described labeling

Thus there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$, where I is the 1-factor consisting of all edges of length $4r+2$. ∎

Theorem 5. *Let G be a 2-regular bipartite graph with n edges and at most two components. Then there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$, where I is a 1-factor.*

Proof. If $n \equiv 0 \pmod 4$, then the result follows from Theorem 1. If G is a single cycle of (even) length $n \equiv 2 \pmod 4$, the result is proved in Lemma 4.

Now let $G = C_{4r} \cup C_{4s+2}$ where $r, s \in \mathbb{Z}^+$. We consider four cases.

Case 1: $r < s$.
Let $C_{4r} = G_1 + G_2 + ((2r+1)_0, 0_1, 0_0)$ and $C_{4s+2} = G_3 + G_4 + ((2r+2s+3)_0, 2_1, (2r+2)_0)$ where

$$\begin{aligned}
G_1 &= P(0, 2r+4s+3, 2r-1),\\
G_2 &= Q(r+2, r+4s+4, 2r-1),\\
G_3 &= P(2r+2, 4r+2s+4, 2s-2r-1),\\
G_4 &= Q(r+s+3, r+s+3, 2r+2s+1).
\end{aligned}$$

First, we show that $G_1 + G_2 + ((2r+1)_0, 0_1, 0_0)$ is a cycle of length $4r$, and $G_3 + G_4 + ((2r+2s+3)_0, 2_1, (2r+2)_0)$ is a cycle of length $4s+2$. Note that by **P1** and **Q1**, the first vertex of G_1 is 0_0, and the last is $(3r+4s+3)_1$; the first vertex of G_2 is $(3r+4s+3)_1$, and the last is $(2r+1)_0$; the first vertex of G_3 is $(2r+2)_0$, and the last is $(3r+3s+4)_1$; and the first vertex of G_4 is $(3r+3s+4)_1$, and the last is $(2r+2s+3)_0$. For $1 \le i \le 4$, let A_i and B_i denote the sets labeled A' and B' in **P2** and **Q2** corresponding to the path G_i. Then using **P2** and **Q2**, we compute

$$\begin{aligned}
A_1 &= [0_0, (r-1)_0], & B_1 &= [(3r+4s+3)_1, (4r+4s+2)_1],\\
A_2 &= [(r+2)_0, (2r+1)_0], & B_2 &= [(2r+4s+4)_1, (3r+4s+3)_1],\\
A_3 &= [(2r+2)_0, (r+s+1)_0], & B_3 &= [(3r+3s+4)_1, (2r+4s+3)_1],\\
A_4 &= [(r+s+3)_0, (2r+2s+3)_0], & B_4 &= [(2r+2s+4)_1, (3r+3s+4)_1].
\end{aligned}$$

Note that $V(G_1) \cap V(G_2) = \{(3r+4s+3)_1\}$ and $V(G_3) \cap V(G_4) = \{(3r+3s+4)_1\}$; otherwise, G_i and G_j are vertex-disjoint for $i \ne j$. Therefore, $G_1 + G_2 + ((2r+1)_0, 0_1, 0_0)$ is a cycle of length $4r$, and $G_3 + G_4 + ((2r+2s+3)_0, 2_1, (2r+2)_0)$ is a cycle of length $4s+2$.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 4$. By **P3** and **Q3**, we have edge lengths

$$\begin{aligned}
E_1 &= [2r+4s+4, 4r+4s+2],\\
E_2 &= [4s+3, 2r+4s+1],\\
E_3 &= [2r+2s+3, 4s+1],\\
E_4 &= [1, 2r+2s+1].
\end{aligned}$$

Moreover, the path $((2r+1)_0, 0_1, 0_0)$ consists of edges with lengths $4r+4s+3+(-2r-1) = 2r+4s+2$ and 0, and the path $((2r+2s+3)_0, 2_1, (2r+2)_0)$ consists of edges with lengths $4r+4s+3+(-2r-2s-1) = 2r+2s+2$ and $4r+4s+3+(-2r) = 2r+4s+3$. Thus, the edge set of G has one edge of each length $i \in [0, 4r+4s+2] \setminus \{4s+2\}$. An example of this labeling is given in Figure 3 with $r = 1$ and $s = 2$.

Case 2: $r = s$.
Let $C_{4r} = G_1 + G_2 + ((2r+1)_0, 0_1, 0_0)$ and $C_{4s+2} = G_3 + ((4r+3)_0, 2_1, (2r+

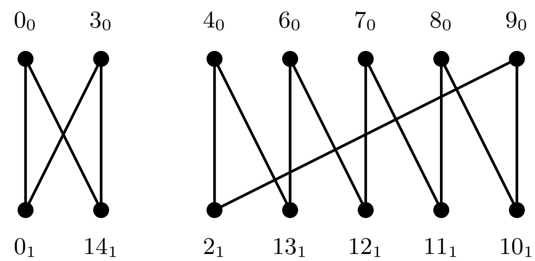


Figure 3: $C_4 \cup C_{10}$ with the described labeling

$2)_0, (6r+3)_1)$ where

$$G_1 = P(0, 6r+3, 2r-1),$$
$$G_2 = Q(r+2, 5r+4, 2r-1),$$
$$G_3 = Q(2r+4, 2r+4, 4r-1).$$

First, we show that $G_1 + G_2 + ((2r+1)_0, 0_1, 0_0)$ is a cycle of length $4r$, and $G_3 + ((4r+3)_0, 2_1, (2r+2)_0, (6r+3)_1)$ is a cycle of length $4s+2$. Note that by **P1** and **Q1**, the first vertex of G_1 is 0_0, and the last is $(7r+3)_1$; the first vertex of G_2 is $(7r+3)_1$, and the last is $(2r+1)_0$; and the first vertex of G_3 is $(6r+3)_1$, and the last is $(4r+3)_0$; For $1 \le i \le 3$, let A_i and B_i denote the sets labeled A' and B' in **P2** and **Q2** corresponding to the path G_i. Then using **P2** and **Q2**, we compute

$$A_1 = [0_0, (r-1)_0], \qquad B_1 = [(7r+3)_1, (8r+2)_1],$$
$$A_2 = [(r+2)_0, (2r+1)_0], \qquad B_2 = [(6r+4)_1, (7r+3)_1],$$
$$A_3 = [(2r+4)_0, (4r+3)_0], \qquad B_3 = [(4r+4)_1, (6r+3)_1].$$

Note that $V(G_1) \cap V(G_2) = \{(7r+3)_1\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1 + G_2 + ((2r+1)_0, 0_1, 0_0)$ is a cycle of length $4r$, and $G_3 + ((4r+3)_0, 2_1, (2r+2)_0, (6r+3)_1)$ is a cycle of length $4s+2$.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 4$. By **P3** and **Q3**, we have edge lengths

$$E_1 = [6r+4, 8r+2],$$
$$E_2 = [4r+3, 6r+1],$$
$$E_3 = [1, 4r-1].$$

Moreover, the path $((2r+1)_0, 0_1, 0_0)$ consists of edges with lengths $4r+4s+3+(-2r-1) = 6r+2$ and 0, and the path $((4r+3)_0, 2_1, (2r+2)_0, (6r+3)_1)$ consists of edges with lengths $4r+4s+3+(-4r-1) = 4r+2$, $4r+4s+$

$3 + (-2r) = 6r + 3$, and $4r + 1$. Thus, the edge set of G has one edge of each length $i \in [0, 8r + 2] \setminus \{4r\}$. An example of this labeling is given in Figure 4 with $r = s = 2$.

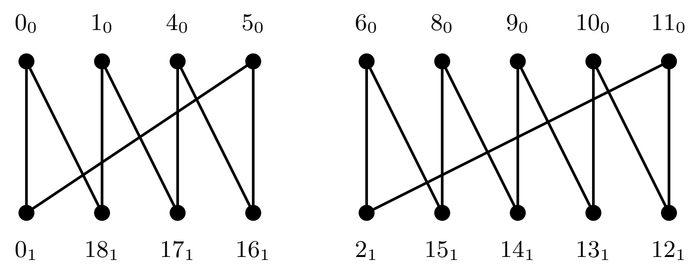


Figure 4: $C_8 \cup C_{10}$ with the described labeling

Case 3: $r = s + 1$.
Let $C_{4s+2} = G_1 + G_2 + ((2s + 2)_0, 1_1, 0_0)$ and $C_{4r} = G_3 + ((2r + 2s + 4)_1, (2r + 2s + 4)_0, 3_1, (2s + 3)_0, (4r + 2s + 2)_1)$ where

$$G_1 = P(0, 4r + 2s + 3, 2s - 1),$$
$$G_2 = Q(s + 2, 4r + s + 2, 2s + 1),$$
$$G_3 = Q(2s + 5, 2s + 6, 4r - 4),$$

First, we show that $G_1 + G_2 + ((2s + 2)_0, 1_1, 0_0)$ is a cycle of length $4s+2$, and $G_3 + ((2r + 2s + 4)_1, (2r + 2s + 4)_0, 3_1, (2s + 3)_0, (4r + 2s + 2)_1)$ is a cycle of length $4r$. Note that by **P1** and **Q1**, the first vertex of G_1 is 0_0, and the last is $(4r+3s+3)_1$; the first vertex of G_2 is $(4r+3s+3)_1$, and the last is $(2s+2)_0$; and the first vertex of G_3 is $(4r+2s+2)_1$, and the last is $(2r + 2s + 4)_1$. For $1 \leq i \leq 4$, let A_i and B_i denote the sets labeled A' and B' in **P2** and **Q2** corresponding to the path G_i. Then using **P2** and **Q2**, we compute

$$A_1 = [0_0, (s - 1)_0], \qquad B_1 = [(4r + 3s + 3)_1, (4r + 4s + 2)_1],$$
$$A_2 = [(s + 2)_0, (2s + 2)_0], \qquad B_2 = [(4r + 2s + 3)_1, (4r + 3s + 3)_1],$$
$$A_3 = [(2s + 5)_0, (2r + 2s + 2)_0], \quad B_3 = [(2r + 2s + 4)_1, (4r + 2s + 2)_1].$$

Note that $V(G_1) \cap V(G_2) = \{(4r + 3s + 3)_1\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1+G_2+((2s+2)_0, 1_1, 0_0)$ is a cycle of length $4s+2$, and $G_3+((2r+2s+4)_1, (2r+2s+4)_0, 3_1, (2s+3)_0, (4r+2s+2)_1)$ is a cycle of length $4r$.

Next, let E_i denote the set of edge labels in G_i for $1 \leq i \leq 3$. By **P3** and **Q3**, we have edge lengths

$$E_1 = [4r + 2s + 4, 4r + 4s + 2],$$
$$E_2 = [4r + 1, 4r + 2s + 1],$$
$$E_3 = [2, 4r - 3].$$

Moreover, the path $((2s + 2)_0, 1_1, 0_0)$ consists of edges with lengths $4r + 4s + 3 + (-2s - 1) = 4r + 2s + 2$ and 1, and the path $((2r + 2s + 4)_1, (2r + 2s + 4)_0, 3_1, (2s + 3)_0, (4r + 2s + 2)_1)$ consists of edges with lengths 0, $4r + 4s + 3 + (-2r - 2s - 1) = 2r + 2s + 2 = 4r$, $4r + 4s + 3 + (-2s) = 4r + 2s + 3$, and $4r - 1$. Thus, the edge set of G has one edge of each length $i \in [0, 4r + 4s + 2] \setminus \{4r - 2\}$. An example of this labeling is given in Figure 5 with $r = 2$ and $s = 1$.

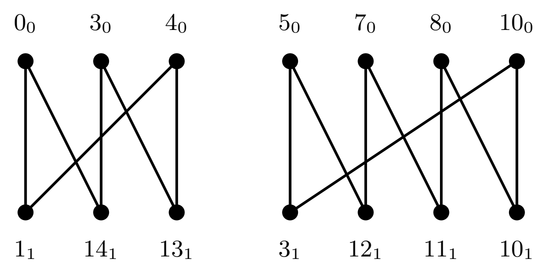


Figure 5: $C_6 \cup C_8$ with the described labeling

Case 4: $r > s + 1$.
Let $C_{4s+2} = G_1 + G_2 + ((2s + 2)_0, 1_1, 0_0)$ and $C_{4r} = G_3 + G_4 + ((2s + 2r + 4)_1, (2s + 2r + 4)_0, 3_1, (2s + 3)_0)$ where

$$G_1 = P(0, 4r + 2s + 3, 2s - 1),$$
$$G_2 = Q(s + 2, 4r + s + 2, 2s + 1),$$
$$G_3 = P(2s + 3, 2r + 4s + 5, 2r - 2s - 3),$$
$$G_4 = Q(r + s + 3, r + s + 4, 2s + 2r).$$

First, we show that $G_1 + G_2 + ((2s + 2)_0, 1_1, 0_0)$ is a cycle of length $4s + 2$, and $G_3 + G_4 + ((2s + 2r + 4)_1, (2s + 2r + 4)_0, 3_1, (2s + 3)_0)$ is a cycle of length $4r$. Note that by **P1** and **Q1**, the first vertex of G_1 is 0_0, and the last is $(4r + 3s + 3)_1$; the first vertex of G_2 is $(4r + 3s + 3)_1$, and the last is $(2s + 2)_0$; the first vertex of G_3 is $(2s + 3)_0$, and the last is $(3r + 3s + 4)_1$; and the first vertex of G_4 is $(3r + 3s + 4)_1$, and the last is $(2r + 2s + 4)_1$. For $1 \le i \le 4$, let A_i and B_i denote the sets labeled A' and B' in **P2** and

Q2 corresponding to the path G_i. Then using **P2** and **Q2**, we compute

$$\begin{aligned} A_1 &= [0_0, (s-1)_0], & B_1 &= [(4r+3s+3)_1, (4r+4s+2)_1], \\ A_2 &= [(s+2)_0, (2s+2)_0], & B_2 &= [(4r+2s+3)_1, (4r+3s+3)_1], \\ A_3 &= [(2s+3)_0, (r+s+1)_0], & B_3 &= [(3r+3s+4)_1, (4r+2s+2)_1], \\ A_4 &= [(r+s+3)_0, (2s+2r+2)_0], & B_4 &= [(2r+2s+4)_1, (3r+3s+4)_1]. \end{aligned}$$

Note that $V(G_1) \cap V(G_2) = \{(4r+3s+3)_1\}$ and $V(G_3) \cap V(G_4) = \{(3r+3s+4)_1\}$; otherwise, G_i and G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1 + G_2 + ((2s+2)_0, 1_1, 0_0)$ is a cycle of length $4s+2$, and $G_3 + G_4 + ((2s+2r+4)_1, (2s+2r+4)_0, 3_1, (2s+3)_0)$ is a cycle of length $4r$.

Next, let E_i denote the set of edge labels in G_i for $1 \leq i \leq 4$. By **P3** and **Q3**, we have edge lengths

$$\begin{aligned} E_1 &= [4r+2s+4, 4r+4s+2], \\ E_2 &= [4r+1, 4r+2s+1], \\ E_3 &= [2s+2r+3, 4r-1], \\ E_4 &= [2, 2s+2r+1]. \end{aligned}$$

Moreover, the path $((2s+2)_0, 1_1, 0_0)$ consists of edges with lengths $4r+4s+3+(-2s-1) = 4r+2s+2$ and 1, and the path $((2s+2r+4)_1, (2s+2r+4)_0, 3_1, (2s+3)_0)$ consists of edges with lengths 0, $4r+4s+3+(-2r-2s-1) = 2r+2s+2$, and $4r+4s+3+(-2s) = 4r+2s+3$. Thus, the edge set of G has one edge of each length $i \in [0, 4r+4s+2] \setminus \{4r\}$. An example of this labeling is given in Figure 6 with $r = 3$ and $s = 1$.

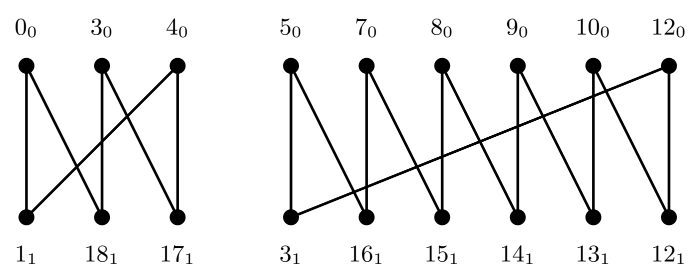


Figure 6: $C_6 \cup C_{12}$ with the described labeling

Thus in each of the four cases, there exists a cyclic G-decomposition of $K_{n+1,n+1} - I$, where I is a 1-factor. ∎

3 Acknowledgement and Final Note

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