Labelings of comets plus an edge

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Abstract

The *comet* $S_{m,2}$ is the graph obtained from $K_{1,m}$ by replacing each edge with a path with two edges. It is known that there exists a cyclic $S_{m,2}$-decomposition of K_{4mx+1} for every positive integer x. We let $G = S_{m,2} + e$ and show via graph labelings that there exists a cyclic G-decomposition of $K_{(4m+2)x+1}$ for every positive integer x.

1 Introduction

If a and b are integers with $a \leq b$, we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively.

Let $V(K_k) = \mathbb{Z}_k$ and let G be a subgraph of K_k. By *clicking* G, we mean applying the isomorphism $i \to i+1$ to $V(G)$. Let H and G be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of edgewise disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. If H is K_k, a G-decomposition Γ of H is *cyclic* if clicking is a permutation of Γ. If G is a graph and r is a positive integer, rG denotes the vertex disjoint union of r copies of G.

For any graph G, a one-to-one function $f : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. We will refer to $f(v)$ as the *label* of v. In [7], Rosa introduced a hierarchy of labelings. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{f}(E(G)) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

(ℓ1) $f(V(G)) \subseteq [0, 2n]$,

(ℓ2) $f(V(G)) \subseteq [0, n]$,

(ℓ3) $\bar{f}(E(G)) = \{x_1, x_2, \dots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,

(ℓ4) $\bar{f}(E(G)) = [1, n]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ consider also

(ℓ5) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,

(ℓ6) there exists an integer λ such that $f(a) \le \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

(ℓ1) and (ℓ3) is called a *ρ-labeling*;

(ℓ1) and (ℓ4) is called a *σ-labeling*;

(ℓ2) and (ℓ4) is called a *β-labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. Suppose G is bipartite. If a ρ, σ, or β-labeling of G satisfies condition (ℓ5), then the labeling is *ordered* and is denoted by ρ^+, σ^+ or β^+, respectively. If in addition (ℓ6) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++} or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an *α-labeling* as introduced in [7]. Labelings of the types above are called *Rosa-type* because of Rosa's original article [7] on the topic. (See [4] for a recent comprehensive survey of Rosa-type labelings). A dynamic survey on general graph labelings is maintained by Gallian [5].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [7] and [3], respectively.

Theorem 1. *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G admits a ρ-labeling.*

Theorem 2. *If G is a graph with n edges that admits a ρ^+-labeling, then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

A non-bipartite graph G is said to be *almost-bipartite* if $G-e$ is bipartite for some $e \in E(G)$. Note that if G is almost-bipartite with $e = \{\hat{b}, c\}$, then G is necessarily tripartite and $V(G)$ can be partitioned into three sets A, B and $C = \{c\}$ such that $\hat{b} \in B$ and e is the only edge joining an element of B to c.

Let G be an almost-bipartite graph with n edges with vertex tripartition A, B, C as above. A labeling h of the vertices of G is called a γ*-labeling* of G if the following conditions hold:

($g1$) The function h is a ρ-labeling of G.

($g2$) If $\{a, v\}$ is an edge of G with $a \in A$, then $h(a) < h(v)$.

($g3$) We have $h(c) - h(\hat{b}) = n$.

Several classes of almost-bipartite graphs have been shown to have γ-labelings (see [4]). As seen in [1], γ-labelings yield results similar to ρ^+-labelings.

Theorem 3. *If G is a graph with n edges that admits a γ-labeling, then there exists a cyclic G-deomposition of K_{2nx+1} for all positive integers x.*

2 The graph $S_{m,2} + e$

Consider the complete bipartite graph $K_{1,m}$. This graph is also known as an *m-star*, denoted S_m. If every edge of S_m is replaced with a path of length n, we call this a *comet* and denote it with $S_{m,n}$. Thus, the comet $S_{m,1}$ is an m-star.

In [7], Rosa stated that a tree of diameter 4 that contains the comet $S_{3,2}$ as a subtree would not admit an α-labeling. In fact, $S_{3,2}$ is the smallest tree that does not admit an α-labeling. El-Zanati, Kenig, and Vanden Eynden [2] showed that every comet $S_{m,2}$ has a β^+-labeling (what they called a *near α-labeling*). Hence there exists a cyclic $S_{m,2}$-decomposition of K_{4mx+1} for every positive integer x.

We consider the graph $G = S_{m,2} + e$ and investigate labelings of G that lead to cyclic G-decompositions of $K_{(4m+2)x+1}$ for every positive integer x. We note that G is either bipartite or almost-bipartite. Using exhaustive computer searches, we discovered that the bipartite graphs $S_{6,2} + e$ and $S_{7,2} + e$ do not admit α-labelings. However, we are able to show that G admits a σ^+-labeling if it is bipartite and a γ-labeling, otherwise.

3 Main Results

For ease of notation, it is helpful to define a new operator on the edge label values and sets. Let G be a graph with n edges. If m is the label of an edge, we call $m^* = \min\{m, 2n+1-m\}$ the *length* of the edge. Also, let $S^* = \{m^* : m \in S\}$ be the corresponding set of edge lengths. Thus if the set of vertex labels of G is a subset of $[0, 2n]$ and the set E of edge labels of G satisfies $E^* = [1, n]$, then conditions ($\ell 1$) and ($\ell 3$) are satisfied, and G has a ρ-labeling.

Theorem 4. *If G is obtained from $S_{m,2}$ by adding an edge between two existing vertices, then there exists a a cyclic G-decomposition of $K_{(4m+2)x+1}$ for every positive integer x.*

Proof. Let $V(S_{m,2}) = \{w\} \cup \{u_i, v_i : 1 \le i \le m\}$ and $E(S_{m,2}) = \{\{w, u_i\}, \{u_i, v_i\} : 1 \le i \le m\}$. Figure 1 shows the graph $S_{6,2}$ as defined here.

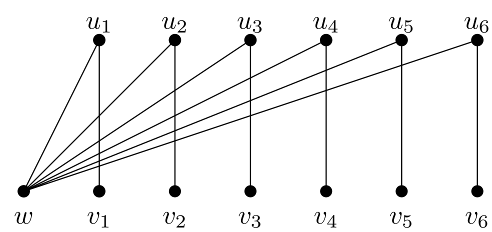

Figure 1: The comet $S_{6,2}$ with the corresponding vertex names

If $m = 1$, then the only edge that can be added is $e = \{w, v_1\}$. This would would give $G = K_3$. A cyclic K_3-decomposition of K_{6x+1} is a cyclic Steiner Triple System of order $6x + 1$, and these are known to exist for all positive integers x, (see [6]). For $m \ge 2$, we will give a σ^+-labeling of G if G is bipartite, and a γ-labeling, otherwise. Since the paths (w, u_i, v_i) are interchangeable for $1 \le i \le m$, it suffices to consider the following four possibilities for the added edge e:

1) $e = \{u_i, v_j\}$ where $i \ne j$,
2) $e = \{v_i, v_j\}$ where $i \ne j$,
3) $e = \{w, v_i\}$,
4) $e = \{u_i, u_j\}$ where $i \ne j$.

Case 1: $e = \{u_1, v_2\}$.
If $m = 2$, let the vertices u_1, u_2, w, v_1, and v_2 be labeled 0, 1, 2, 3, and 5, respectively. It is easily checked that this is an α-labeling of G.

If $m \geq 3$, let $h : V(G) \to [0, 4m+2]$ be defined for $1 \leq i \leq m$ as follows:

$$h(w) = 2m + 1,$$

$$h(u_i) = \begin{cases} 0 & \text{for } i = 1 \\ 2 & \text{for } i = 2 \\ 2m-1 & \text{for } i = 3 \\ 2m-i & \text{for } i \in [4, m], \end{cases}$$

$$h(v_i) = \begin{cases} 4 & \text{for } i = 1 \\ 3 & \text{for } i = 2 \\ 4m-1 & \text{for } i = 3 \\ 4m-2i+2 & \text{for } i \in [4, m]. \end{cases}$$

An example of this labeling can be seen in Figure 2.

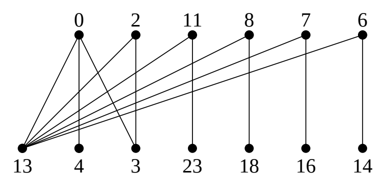

Figure 2: A σ^+-labeling of $S_{6,2} + \{u_1, v_2\}$

First, we show that h is a σ-labeling. Note that

$$\begin{aligned} 0 = h(u_1) &< h(u_2) < h(v_2) < h(v_1) \\ &< h(u_m) < h(u_{m-1}) < \cdots < h(u_4) < h(u_3) < h(w) \\ &< h(v_m) < h(v_m - 1) < \cdots < h(v_4) < h(v_3) = 4m - 1. \quad (1) \end{aligned}$$

Thus, all vertices in $V(G)$ have distinct labels in $[0, 4m+2]$.

We now compute the resulting edge labels. For $1 \leq i \leq m$:

$$\bar{h}(\{w, u_i\}) = \begin{cases} 2m+1 & \text{for } i = 1 \\ 2m-1 & \text{for } i = 2 \\ 2 & \text{for } i = 3 \\ i+1 & \text{for } i \in [4, m], \end{cases}$$

$$\bar{h}(\{u_i, v_i\}) = \begin{cases} 4 & \text{for } i = 1 \\ 1 & \text{for } i = 2 \\ 2m & \text{for } i = 3 \\ 2m-i+2 & \text{for } i \in [4, m], \end{cases}$$

$$\bar{h}(e = \{u_1, v_2\}) = 3.$$

Thus, the set of all edge labels in G is

$$\bar{h}(E(G)) = \{2, 2m-1, 2m+1\} \cup [5, m+1] \cup \{1, 4, 2m\} \cup [m+2, 2m-2] \cup \{3\}.$$

Since we have each edge label in $[1, 2m+1]$ and $h(V(G)) \subseteq [0, 4m+2]$, the defined labeling satisfies condtions ($\ell 1$) and ($\ell 4$), and h is a σ-labeling.

Now, let $A = \{u_i : 1 \leq i \leq m\}$ and $B = \{w\} \cup \{v_i : 1 \leq i \leq m\}$. Then $\{A, B\}$ is a bipartition of $V(G)$. Condition ($\ell 6$) is clear from inequality (1) because $h(A \setminus \{u_1, u_2\}) < h(B \setminus \{v_1, v_2\})$ and $h(\{u_1, u_2\}) < h(\{v_1, v_2\})$. Therefore, we have a σ^+-labeling of G, which is necessarily a ρ^+-labeling, and thus Theorem 2 applies.

Case 2: $e = \{v_1, v_2\}$
Let $h : V(G) \to [0, 4m+2]$ be defined for $1 \leq i \leq m$ as follows:

$$h(w) = m + 1,$$
$$h(u_i) = i - 1,$$
$$h(v_i) = \begin{cases} 4m+2 & \text{for } i = 1 \\ 2m+1 & \text{for } i = 2 \\ 2m+2i & \text{for } i \in [3, m]. \end{cases}$$

An example of this labeling can be seen in Figure 3.

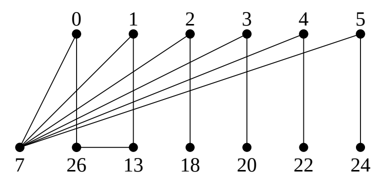

Figure 3: A γ-labeling of $S_{6,2} + \{v_1, v_2\}$

First, we show that h is a ρ-labeling. Note that

$$\begin{aligned} 0 = h(u_1) < h(u_2) < \ldots < h(u_m) < h(w) < h(v_2) \\ < h(v_3) < h(v_4) < \cdots < h(v_m) < h(v_1) = 4m + 2. \end{aligned} \quad (2)$$

Thus, all vertices in $V(G)$ have distinct labels in $[0, 4m+2]$.

We now compute the resulting edge labels. For $1 \leq i \leq m$:

$$\bar{h}(\{w, u_i\}) = m - i + 2,$$
$$\bar{h}(\{u_i, v_i\}) = \begin{cases} 4m+2 & \text{for } i = 1 \\ 2m & \text{for } i = 2 \\ 2m+i+1 & \text{for } i \in [3, m], \end{cases}$$
$$\bar{h}(e = \{v_1, v_2\}) = 2m + 1.$$

Since the edge label $\bar{h}(\{u_i, v_i\})$ exceeds $2m+1$ when $i \neq 2$, the corresponding edge lengths are $(\bar{h}(\{u_1, v_1\}))^* = 1$ and $(\bar{h}(\{u_i, v_i\}))^* = 2m - i + 2$ for $i \in [3, m]$. Thus, the set of all edge labels in G is

$$\bar{h}(E(G)) = [2, m+1] \cup \{4m+2, 2m\} \cup [2m+4, 3m+1] \cup \{2m+1\}$$

yielding the following set of edge lengths:

$$(\bar{h}(E(G)))^* = [2, m+1] \cup \{1, 2m\} \cup [m+2, 2m-1] \cup \{2m+1\}.$$

Since we have each edge length in $[1, 2m+1]$ and $h(V(G)) \subseteq [0, 4m+2]$, h is a ρ-labeling of G, and condition $(g1)$ for a γ-labeling is satisfied.

Now, let $A = \{u_i : 1 \leq i \leq m\}$, $B = \{\hat{b} = v_2\} \cup \{w\} \cup \{v_i : 3 \leq i \leq m\}$, and $C = \{c = v_1\}$. Then $\{A, B, C\}$ is a tripartition of $V(G)$. Condition $(g2)$ for a γ-labeling is clear from inequality (2) because $h(A) < h(B \cup C)$. Finally, $h(c) - h(\hat{b}) = (4m+2) - (2m+1) = 2m+1$, the number of edges of G. Thus condition $(g3)$ holds, and h is a γ-labeling of G.

Case 3: $e = \{w, v_1\}$
Let $h : V(G) \to [0, 4m+2]$ be defined for $1 \leq i \leq m$ as follows:

$$h(w) = m,$$
$$h(u_i) = \begin{cases} 0 & \text{for } i = 1 \\ m - i + 1 & \text{for } i \in [2, m], \end{cases}$$
$$h(v_i) = \begin{cases} 3m - 2i + 3 & \text{for } i \in [1, m-1] \\ m + 2 & \text{for } i = m. \end{cases}$$

An example of this labeling can be seen in Figure 4.

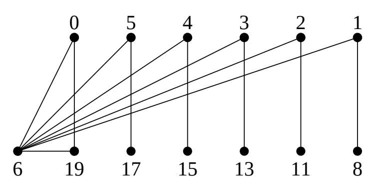

Figure 4: A γ-labeling of $S_{6,2} + \{w, v_2\}$

First, we show that h is a ρ-labeling. Note that

$$\begin{aligned} 0 = h(u_1) < h(u_m) < (u_{m-1}) < \cdots < h(u_2) < h(w) < h(v_m) \\ < h(v_{m-1}) < h(v_{m-1}) < \cdots < h(v_2) < h(v_1) = 3m+1. \quad (3) \end{aligned}$$

Thus, all vertices in $V(G)$ have disjoint labels in $[0, 4m+2]$.

We now compute the resulting edge labels. For $1 \leq i \leq m$:

$$\bar{h}(\{w, u_i\}) = \begin{cases} m & \text{for } i = 1 \\ i - 1 & \text{for } i \in [2, m], \end{cases}$$

$$\bar{h}(\{u_i, v_i\}) = \begin{cases} 3m + 1 & \text{for } i = 1 \\ 2m - i + 2 & \text{for } i \in [2, m-1] \\ m + 1 & \text{for } i = m, \end{cases}$$

$$\bar{h}(e = \{v_1, v_2\}) = 2m + 1.$$

Since the edge label $\bar{h}(\{u_1, v_1\})$ exceeds $2m + 1$, the corresponding edge length is $(\bar{h}(\{u_1, v_1\}))^* = m + 2$. Thus, the set of all edge labels in G is

$$\bar{h}(E(G)) = \{m\} \cup [1, m-1] \cup \{3m+1\} \cup [m+3, 2m] \cup \{m+1\} \cup \{2m+1\}$$

yielding the following set of edge lengths:

$$(\bar{h}(E(G)))^* = \{m\} \cup [1, m-1] \cup \{m+2\} \cup [m+3, 2m] \cup \{m+1\} \cup \{2m+1\}.$$

Since we have each edge length in $[1, 2m+1]$ and $h(V(G)) \subseteq [0, 4m+2]$, h is a ρ-labeling of G, and condition $(g1)$ for a γ-labeling is satisfied.

Now, let $A = \{u_i : 1 \leq i \leq m\}$, $B = \{\hat{b} = w\} \cup \{v_i : 2 \leq i \leq m\}$, and $C = \{c = v_1\}$. Then $\{A, B, C\}$ is a tripartition of $V(G)$. Condition $(g2)$ for a γ-labeling is clear from inequality (3) because $h(A) < h(B \cup C)$. Finally, $h(c) - h(\hat{b}) = (3m + 1) - (m) = 2m + 1$, the number of edges of G. Thus condition $(g3)$ holds, and h is a γ-labeling of G.

Case 4: $e = \{u_1, u_2\}$
Let $h : V(G) \to [0, 4m + 2]$ be defined for $1 \leq i \leq m$ as follows:

$$h(w) = 0,$$

$$h(u_i) = \begin{cases} 2 & \text{for } i = 1 \\ 2m + 2i - 1 & \text{for } i \in [2, m], \end{cases}$$

$$h(v_i) = \begin{cases} 1 & \text{for } i = 1 \\ 4i - 4 & \text{for } i \in [2, m]. \end{cases}$$

An example of this labeling can be seen in Figure 5.

First, we show that h is a ρ-labeling. Note that the labels of the vertices in $\{u_i : 2 \leq i \leq m\}$ are all odd and contained in $[2m + 3, 4m - 1]$ and that the labels of the vertices in $\{v_i : 2 \leq i \leq m\}$ are all even and contained in $[4, 4m - 4]$. Thus, all vertices in $V(G)$ have disjoint labels in $[0, 4m + 2]$.

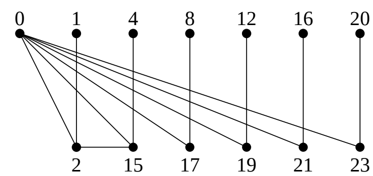

Figure 5: A γ-labeling of $S_{6,2} + \{u_1, u_2\}$

Furthermore, for m even

$$\begin{aligned} 0 = h(w) < h(v_1) < h(u_1) < h(v_2) < h(v_3) < \cdots < h(v_{\lceil m/2 \rceil+1}) < h(u_2) \\ < h(v_{\lceil m/2 \rceil+2}) < h(u_3) < h(u_4) \\ < h(v_{\lceil m/2 \rceil+3}) < h(u_5) < h(u_6) < \cdots \\ < h(v_m) < h(u_{m-1}) < h(u_m) = 4m-1 \quad (4) \end{aligned}$$

and for m odd

$$\begin{aligned} 0 = h(w) < h(v_1) < h(u_1) < h(v_2) < h(v_3) < \cdots < h(v_{\lceil m/2 \rceil+1}) \\ < h(u_2) < h(u_3) < h(v_{\lceil m/2 \rceil+2}) \\ < h(u_4) < h(u_5) < h(v_{\lceil m/2 \rceil+3}) < \cdots \\ < h(u_{m-3}) < h(u_{m-2}) < h(v_m) \\ < h(u_{m-1}) < h(u_m) = 4m-1. \quad (5) \end{aligned}$$

We now compute the resulting edge labels. For $1 \le i \le m$:

$$\bar{h}(\{w, u_i\}) = \begin{cases} 2 & \text{for } i = 1 \\ 2m + 2i - 1 & \text{for } i \in [2, m], \end{cases}$$

$$\bar{h}(\{u_i, v_i\}) = \begin{cases} 1 & \text{for } i = 1 \\ 2m - 2i + 3 & \text{for } i \in [2, m], \end{cases}$$

$$\bar{h}(e = \{u_1, u_2\}) = 2m + 1.$$

Since the edge label $\bar{h}(\{w, u_i\})$ exceeds $2m + 1$ when $i \in [2, m]$, the corresponding edge lengths are $(\bar{h}(\{w, u_i\}))^* = 2m - 2i + 4$ for $i \in [2, m]$. Thus, the set of all edge labels in G is

$$\bar{h}(E(G)) = \{2\} \cup \{2m+3, 2m+5, \ldots, 4m-1\} \cup \{1\} \cup \{3, 5, \ldots, 2m-1\} \cup \{2m+1\}$$

yielding the following set of edge lengths:

$$(\bar{h}(E(G)))^* = \{2\} \cup \{4, 6, \ldots, 2m\} \cup \{1\} \cup \{3, 5, \ldots, 2m-1\} \cup \{2m+1\}.$$

Since we have each edge length in $[1, 2m+1]$ and $h(V(G)) \subseteq [0, 4m-1]$, condition $(g1)$ for a γ-labeling is satisfied.

Now, let $A = \{w\} \cup \{v_i : 1 \leq i \leq m\}$, $B = \{\hat{b} = u_1\}\{u_i : 3 \leq i \leq m\}$, and $C = \{c = u_2\}$. Then $\{A, B, C\}$ is a tripartition of $V(G)$. Condition $(g2)$ is clear from inequalities (4) and (5). Finally, $h(c) - h(\hat{b}) = (2m+3) - (2) = 2m+1$, the number of edges of G. Thus condition $(g3)$ holds, and h is a γ-labeling of G. ■

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