An Introduction to Discrete Mathematics in the Classroom: Latin Squares

Teacher’s Guide

Carol T. Benson, Illinois State University
Kyle P. King, University of Illinois
Jeffrey A. Mudrock, University of Illinois

This project was funded in part by grant # from NSF.
Preface for the Instructor:

Hello and welcome to the wonderful world of Latin Squares! In this module, you will get the opportunity to expose your students to ideas in discrete mathematics that they otherwise might not come across until college. This module encourages students to engage in rich mathematical problem solving which will better prepare them for future mathematics involvement. It is designed for the instructor to guide students through these ideas. The student edition of this module sets up investigations. Your edition of the module has a thorough explanation of the mathematics and sample results that students might find. The text in italics is only in the teacher guide. The non-italicized text is the content of the Students’ Guide. The Appendices and the Glossary are only included in the Teacher’s Guide.

Students within a range of ages can benefit from this module with modification made by the teacher. For example, a Calculus teacher may choose to begin with Section 4, while a fifth grade teacher may choose to only do parts of Section 1. Regardless of how you plan to use the module, please keep in mind that the module is set up to help students find their own results; the teacher’s role is to encourage student discovery through investigations and questioning.

Included in the Teacher’s Guide:

1. Student Guide with explanations and answers
2. Appendix A (Internet Search Results)
3. Appendix B (Playing Card Worksheet)
4. Glossary
5. Works Cited

Preface for the Student:

Hello and welcome to the wonderful world of Latin Squares! In this module, you will have the opportunity to pursue ideas in discrete mathematics that you otherwise might not come across until college! As you will discover, this module is unlike any ordinary textbook; instead, this module contains numerous investigations and problems that help you make connections and discoveries on your own. In this module, your instructor will act as a guide, leading you to develop important ideas and concepts. It is your job as the student to hypothesize, investigate, and discover these rich math concepts on your own. Good luck and have fun!

This project was funded in part by grant # from NSF.
Principles, Standards, and Objectives for Module

Standards and Principles

- **Communication** – students will have a variety of opportunities to communicate during this module: through small group discussion, recording their work, thoughts, conjectures, reasoning in a journal, and whole class discussion.
- **Technology** – students will use the computer to do web searches for terms and applications.
- **Reasoning and proof** – students will use reasoning and logic throughout the module for their investigations, search for patterns, conjectures, and explanations.
- **Problem solving** – the pedagogical focus of this module is problem solving. Activities are designed for students to investigate mathematics content looking for patterns, to make conjectures, and to prove or justify their conjectures through reasoning and proof. Students will monitor and reflect on the process of problem solving through reflective journal entries and class discussion.
- **Connections** – the activities within this module are selected to encourage students to make connections among different content areas within mathematics and between mathematics and real world contexts.
- **Representation** – students are given different relationships in real world and mathematical settings and asked to represent patterns they observe in more than one representation.
- **Equity** – because the content is probably new to all students and the learning takes place through investigation, small group and whole class discussion, this module should be equitable for all students in a high school mathematics class.
- **Algebra** – students will represent mathematical patterns that they find with algebraic symbols.

Objectives

- Students will learn basic terminology and uses of Latin Squares.
- Students will investigate Latin Squares and recognize and describe patterns in their construction.
- Students will investigate real world problems using Latin Squares.
- Students will be introduced to affine geometry, how it relates to Latin Squares, and applications.
- Students will learn how to create Latin Squares that meet given restrictions.
- Students will learn how to create and use mutually orthogonal Latin Squares to solve problems.
- Students will solve a real world problem by using Latin Squares.
- By working through this module, students will learn what it is like to be a mathematician.

Introduction Sudoku Puzzle:

This project was funded in part by grant # from NSF.
Please complete the following Sudoku Puzzle. If you are unfamiliar with the rules, in a Sudoku, each row, column, and small box must contain all the digits from 1-9 exactly once.

This Sudoku is used to introduce the idea of Latin Squares to the students. As discussed below, a Latin Square is an n x n array such that there is no repeat entry in any of the rows or columns. Please give the class about 15 minutes to complete this task, and then move on to The Problem: Placing Students in Groups.
Introduction

In this module you will learn about Latin Squares and about applications of this area of mathematics. You will quickly find out that Latin Squares are used throughout the world in the form of Sudokus, scheduling, and round robin tournaments. Furthermore Latin Squares are an important area of mathematics (where they are used in graph theory and abstract algebra) and can be enjoyed by all. In this module, you should begin to keep a journal that includes all ideas, descriptions, solutions to problems, attempts at solutions, and questions about the Latin Squares you encounter.

The Culminating Problem: Placing Students in Groups

There are 16 students in a class. The teacher wants them to work on projects throughout the year in groups of 4. How many projects would it take so that each student was in the same group as each other student in the class exactly once? Create a plan to assign students to groups and indicate the groupings for each project.

This project was funded in part by grant # from NSF.
For this problem, students should work individually for a short time. They might be encouraged to share their response to the first question with the class (5 projects) so that all are working on creating 5 groupings. Then students may work individually or in small groups. Allow students 5-10 minutes to work on this, then you might indicate that they will have opportunities to work on this problem later in the module. Please do not provide an answer, but allow students to continue to work on this problem as time permits throughout this module.

Section 1: Introduction to Latin Squares

1.1 Introduction Problem

Three students from your school, Jeff, Kyle, and Desiree, are competing in a round-robin badminton tournament with three students, Seth, Ryan, and Katie, from the rival school. That means each student from your school plays exactly one match with each student from the rival school. Set up a schedule so there are three matches at one time, until all matches are completed.

For this problem, students should work individually for a short time, then they should be encouraged to share with the class that there will be 9 matches, three at a time. Allow 5-10 minutes for students to work individually and in small groups to develop a schedule. Ask students how they might expand and generalize this problem and to describe how they came up with a solution. Conclude with a general form of the problem and a strategy to solve the general problem.

A solution method using Latin Squares is provided below, but it is intended that students merely investigate this problem and come up with their own solutions to generate interest in the use of Latin Squares.
Possible Solution:

<table>
<thead>
<tr>
<th></th>
<th>Seth</th>
<th>Ryan</th>
<th>Katie</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Jeff</strong></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td><strong>Kyle</strong></td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td><strong>Desiree</strong></td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

In the Latin Square above, a number in a cell indicates the round of the tournament in which that match is played. To find who is competing in the match, look to the corresponding names in the labels of the row and column of that cell. For example, Desiree would play against Katie in the second round because the number 2 is found in the row with Desiree’s name and the column with Katie’s name.

There are multiple solutions to this problem. Any valid solution will consist of 9 matches in three rounds, with each member of one team playing one match against each member of the second team. Students should be able to report this.

1.2 Preview

In your journal describe the patterns that you see in the following arrays. What patterns do you see in the rows and columns of each array? What patterns do you see in the diagonals of each array? Are there any patterns that are consistent in all of the following arrays?

a.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

b.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
Latin Squares

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This project was funded in part by grant # from NSF.
Students should note some of the following patterns in their journals:

- All the arrays have no repeats in any row or column (students will find this to be consistent with the definition of Latin square).
- Arrays b, c, and e have consecutive numbers beginning with 1 along the main diagonal (consistent with the definition of idempotent that students will find).
- Arrays a, b, and e are symmetric about the main diagonal (examples of symmetric arrays should be identified as such by students).
- Students may notice that the arrays are of different sizes but always square. This could lead to a discussion of even and odd order arrays.
- These findings should be recorded in the students’ journals. Make sure they are looking for patterns within each individual array and patterns that are consistent among some or all of the above arrays.

Other patterns may be observed and could be discussed. It is anticipated that the definitions of terms and recognition of how they fit the arrays provided should come out in student discussion of patterns they have observed. This activity leads into the student task of looking up the vocabulary terms either on the web if they have access or in the list provided in Appendix A.

1.3 Discussion

a) Discuss the following terms: diagonal, order, even/odd order, and symmetric. What might these terms mean with respect to n x n arrays?

b) One term that has a variety of definitions is idempotent. For the purposes of this module idempotent is defined in a specific manner. Your instructor will show some examples of idempotent Latin Squares. As your instructor does this, try to come up with a definition of idempotent Latin Squares as a class.

If students are able to sufficiently define any of the above terms, then it will not be necessary for them to find the definitions of those terms in the following Exercise.

In the preview, Latin Squares b, c and e are examples of idempotent Latin Squares and can be used to help the class define what it means to have an idempotent Latin Square. For an exact definition refer to Appendix A. The class should come up with something along the lines of the following definition of idempotent Latin Squares: The main diagonal contains consecutive numbers 1 through n, beginning with 1 in the upper-left corner.
1.4 Exercise

Below are some important terms for our module. Please search for definitions and examples of these terms using the internet or, if you do not have access to the internet, your teacher will provide some results from such a search so that you may form your definitions from a combination of sources. Note that these terms may have other uses outside of this module. You may wish to use the phrase “graph theory” to find an appropriate definition for our work. In your journal, record in your own words what you find and reflect on how this information relates to what we have been doing in class:

- Latin Square
- diagonal
- symmetric
- order
- even/odd order

Key Components of Each Definition Should Include:

- Latin Square: an n x n array using a set of n distinct symbols such that the entries of any row or column are distinct

- diagonal: The main diagonal, or the one that runs from the upper-left corner to the lower-right corner

- symmetric: Given an entry in the nth row, mth column, the same entry is found in the mth row, nth column. Arrays a, b, and e from 1.2 Preview are examples of symmetric arrays.

- order: Given an n x n array, the order of that array is n

- even/odd order: Given an n x n array, the order is odd if n is odd; similarly, the order is even if n is even

Note: All the terms except Latin Square and idempotent can be applied to any n x n arrays – not just Latin Squares. The term idempotent as defined above only applies to Latin Squares.

If the students are having difficulty finding the above definitions, they may need to use the term “Latin Squares” with each of the terms above.
1.5 Exercise

Now that you have found the definitions of the basic terms listed above, describe the following arrays using the terms that you just defined. Write down all the terms that apply to each array in your journal and explain why each term does apply. Also record any other interesting observations about the arrays below.

a.  

\begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{array}

b.  

\begin{array}{ccc}
1 & 3 & 2 \\
3 & 2 & 1 \\
2 & 1 & 3 \\
\end{array}

c.  

\begin{array}{cccc}
1 & 3 & 4 & 2 \\
4 & 2 & 1 & 3 \\
2 & 4 & 3 & 1 \\
3 & 1 & 2 & 4 \\
\end{array}

d.  

\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
4 & 1 & 2 & 3 \\
2 & 4 & 1 & 2 \\
\end{array}
This project was funded in part by grant # from NSF.
Through discussion, share and modify your descriptions of the above arrays with other classmates and your instructor.

Please find below suggested responses to the exercises above. Please note that a discussion may take place for array g regarding whether the array should be considered idempotent if letters are used instead of numbers. The custom is that, if the rows and columns would be labeled with the letters in alphabetic order left to right and top to bottom, then the Latin Square would be considered idempotent. However, if the rows and columns would be labeled with numbers, then the Latin Square would not be considered idempotent. The class has the option to determine whether the labels would be letters or numbers since no labels were listed. By convention of this module, unless otherwise specified, rows are labeled top to bottom and columns labeled left to right with consecutive numbers beginning with 1.
Answers:

<table>
<thead>
<tr>
<th></th>
<th>Latin Square</th>
<th>Idempotent</th>
<th>Diagonal</th>
<th>Symmetric</th>
<th>Order</th>
<th>Even/odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>Yes</td>
<td>No</td>
<td>1, 3, 2</td>
<td>Yes</td>
<td>3</td>
<td>Odd</td>
</tr>
<tr>
<td>b.</td>
<td>Yes</td>
<td>Yes</td>
<td>1, 2, 3</td>
<td>Yes</td>
<td>3</td>
<td>Odd</td>
</tr>
<tr>
<td>c.</td>
<td>Yes</td>
<td>Yes</td>
<td>1, 2, 3, 4</td>
<td>No</td>
<td>4</td>
<td>Even</td>
</tr>
<tr>
<td>d.</td>
<td>No</td>
<td>NA</td>
<td>1, 3, 2, 2</td>
<td>No</td>
<td>4</td>
<td>Even</td>
</tr>
<tr>
<td>e.</td>
<td>Yes</td>
<td>Yes</td>
<td>1, 2, 3, 4, 5</td>
<td>Yes</td>
<td>5</td>
<td>Odd</td>
</tr>
<tr>
<td>f.</td>
<td>Yes</td>
<td>No</td>
<td>4, 7, 12, 5, 9</td>
<td>Yes</td>
<td>5</td>
<td>Odd</td>
</tr>
<tr>
<td>g.</td>
<td>Yes</td>
<td>No unless labels are assumed to be a, b, c</td>
<td>a, b, c</td>
<td>Yes</td>
<td>3</td>
<td>Odd</td>
</tr>
<tr>
<td>h.</td>
<td>Yes</td>
<td>No</td>
<td>3, 3, 3</td>
<td>No</td>
<td>3</td>
<td>Odd</td>
</tr>
<tr>
<td>i.</td>
<td>Yes</td>
<td>No</td>
<td>1, 2, 1, 2</td>
<td>Yes</td>
<td>4</td>
<td>Even</td>
</tr>
<tr>
<td>j.</td>
<td>Yes</td>
<td>No</td>
<td>1, 2, 1, 2</td>
<td>No</td>
<td>4</td>
<td>Even</td>
</tr>
</tbody>
</table>

Section 2: The “Clicking” and “Diagonal” Methods

2.1 Problem

Alvin wants to set up Christmas lights on his roof the same as last year, but he has forgotten which string of lights goes where. He wants one string of lights in each marked region, and wants each row and column to have exactly one light of each color. Each string of lights consists of seven bulbs, one of each color. Note that the order of the colors on each string may vary.

Alvin remembers where one of each color should go, but you must help him place each string so that his rooftop has one light of each color in every row and column.
This is an application of concepts learned in Section 1. It is an exercise to assess the students’ understanding of Latin Squares. One solution is the following:

If you do not have a color copy of this module, the coloring is listed below: G for green, P for pink, Y for yellow, O for orange, T for teal, R for red and B for blue.
If you do not have a color copy of this module, the coloring is listed below: G for green, P for pink, Y for yellow, O for orange, T for teal, R for red and B for blue.

2.2 Critical Thinking
Now, develop a strategy to create your own Latin Squares. Attempt to create different strategies for finding idempotent and/or symmetric Latin Squares. In your journal, write a full explanation
of how someone might use your strategy. Write it so that a classmate or teacher could easily understand.

During this process, students should discover both the “clicking” and “diagonal” methods, as demonstrated below.

“Clicking” Method: This method will always produce a symmetric Latin Square for any order. Given a first row: \(a_1 \ a_2 \ ... \ a_n\), the second row is formed by moving each entry in the first row one space to the left, with wraparound. Thus, row two would be: \(a_2 \ ... \ a_n \ a_1\), and one would continue the pattern for the remaining rows, each time moving the entries one place to the left. The array below is an example of a symmetric Latin Square constructed using this strategy.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

The “Diagonal” Method: This strategy will produce idempotent symmetric Latin Squares for arrays of odd order. Given an array of order \(n\) and beginning at the upper left cell, place consecutive numbers 1 through \(n\) along the main diagonal. Now look at the diagonals that run from upper right to lower left (using the wrap-around concept (shown below) and copy the number from the main diagonal into the other cells along each upper right to lower left diagonal. For clarification, please see the sequence below.
Step 1:

```
 1  2  3  4  5
2   3   4
3   4
5
```

Step 2:

```
 1
 2
 3
2  3
2
3
3
4
3
4
5
```
Step 3: The concept of wraparound diagonals is illustrated in the diagram below. To fill in any remaining diagonals, imagine shifting the leftmost column to the right side of the rightmost column and, at the same time, shifting the rightmost column to the left side of the leftmost column. Then, following the arrows within the array, fill in the remaining diagonals, going either from the upper-right to the lower-left or from the lower-left to the upper right.
**Completed Diagonal Method:**

![Latin Square](image)

*Note:* If students note other patterns, the class should examine and discuss them as well.

*Note:* Problems 4-6 may be given as homework. However, if these problems are assigned as homework, please discuss them in general with the class prior to dismissal so that any questions may be answered before students work on these problems at home.

### 2.3 Problem

Create any Latin Squares of order 2, 3, 4, 5, 6, and 7. Record these Latin Squares and the strategy you used to create them in your journal.

*Students may create any arrays of the appropriate order so that no row or column has two copies of any entry. Sample responses are provided below.*

**Order 2:**

```
1 2
2 1
```
Order 3:

\[
\begin{array}{ccc}
2 & 3 & 1 \\
1 & 2 & 3 \\
3 & 1 & 2 \\
\end{array}
\]

Order 4:

\[
\begin{array}{cccc}
2 & 4 & 3 & 1 \\
1 & 2 & 4 & 3 \\
3 & 1 & 2 & 4 \\
4 & 3 & 1 & 2 \\
\end{array}
\]

Order 5:

\[
\begin{array}{ccccc}
5 & 3 & 2 & 1 & 4 \\
1 & 4 & 3 & 2 & 5 \\
3 & 2 & 4 & 5 & 1 \\
2 & 5 & 1 & 4 & 3 \\
4 & 1 & 5 & 3 & 2 \\
\end{array}
\]

This project was funded in part by grant # from NSF.
**Order 6:**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

**Order 7:**

<table>
<thead>
<tr>
<th></th>
<th>7</th>
<th>3</th>
<th>2</th>
<th>4</th>
<th>1</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>2</td>
<td>6</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>3</td>
<td>7</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>5</td>
<td>7</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>7</td>
<td>2</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

### 2.4 Problem

Now, if possible, create Latin Squares of orders $n = 2, 3, 4, 5, 6,$ and $7$ that are both idempotent and symmetric. Record your results and conjectures that you form in the process in your journal.
This problem is only possible when \( n \) is odd. It can be proved that an idempotent, symmetric Latin Square of even order does not exist. Because of this, please instruct your students to spend no more than 30 minutes on this particular problem. Possible solutions for odd orders are given below.

**Order 3:**

\[
\begin{array}{ccc}
1 & 3 & 2 \\
3 & 2 & 1 \\
2 & 1 & 3 \\
\end{array}
\]

**Order 5:**

\[
\begin{array}{ccccc}
1 & 4 & 2 & 5 & 3 \\
4 & 2 & 5 & 3 & 1 \\
2 & 5 & 3 & 1 & 4 \\
5 & 3 & 1 & 4 & 2 \\
3 & 1 & 4 & 2 & 5 \\
\end{array}
\]
Order 7:

<table>
<thead>
<tr>
<th>1</th>
<th>5</th>
<th>2</th>
<th>6</th>
<th>3</th>
<th>7</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
<td>6</td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>2</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>5</td>
<td>2</td>
<td>6</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

2.5 Problem

Jeff, Kyle, Seth, and Ryan are going to go on dates with Katie, Andrea, Keri, and Desiree. Each individual will go on four dates on four consecutive nights— one date each on Wednesday, Thursday, Friday, and Saturday. Additionally, all eight people will be on their dates at the same time. Using Latin Squares, show how the dates should be scheduled so that each boy goes on exactly one date with each girl?
The strategy for solving this problem is similar to the strategy used to solve the problem at the beginning of Section 1.

<table>
<thead>
<tr>
<th></th>
<th>Desiree</th>
<th>Katie</th>
<th>Andrea</th>
<th>Keri</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jeff</td>
<td>W</td>
<td>T</td>
<td>F</td>
<td>S</td>
</tr>
<tr>
<td>Kyle</td>
<td>T</td>
<td>F</td>
<td>S</td>
<td>W</td>
</tr>
<tr>
<td>Ryan</td>
<td>F</td>
<td>S</td>
<td>W</td>
<td>T</td>
</tr>
<tr>
<td>Seth</td>
<td>S</td>
<td>W</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

A Latin Square similar to the one above should be found by students. It is interpreted the same way the Latin Square in the first problem of the section was interpreted; however, instead of numbers going into the Latin Square, we used letters, which represent the days of the week. For example the Latin Square above indicated that Kyle would go out with Keri on Wednesday night. Before assigning this problem, suggest to your students that it might benefit them more to create a Latin Square with letters instead of numbers, as is shown in this solution.

Section 3: Half-Idempotent Latin Squares

3.1 Discussion and Review

With your instructor and classmates, discuss your results and strategies for finding Latin Squares along with how you found idempotent and symmetric Latin Squares of orders 2-7.

Find a strategy that can be extended to all Latin Squares of a certain order (odd or even).

If not already discussed, share ideas on how to solve the dating and round-robin problems from Sections 1 and 2.

Through their research and the class discussion, the students should be able to use both the clicking and diagonal methods. They may also have observed that the diagonal method only works for odd-order arrays. Any other strategies that they discover through experimentation or web search are welcome here, but the two identified methods are needed for later exercises.

This project was funded in part by grant # from NSF.
3.2 Exercise

In your journal, fill in the blanks to make the following Latin Square symmetric.

Half-idempotent squares are created by placing the entries $1, 2, \ldots, n/2, 1, 2, \ldots, n/2$ along the main diagonal of the array of order $n$, where $n$ is even.

Once the diagonal is complete, the remaining cells can be filled in to complete the Latin Square in a variety of ways. One strategy is very similar to the diagonal method shown in Section 2, and will always create a symmetric, half-idempotent Latin Square of even-order. We will refer to this method as the “Half-Idempotent Symmetric Diagonal” method. This strategy is illustrated in the two examples below. The first step is identical to the diagonal method of Section 2.

Step 1:
Step 2 (two examples):

\[
\begin{array}{cccc}
1 & 4 & 2 & 3 \\
4 & 2 & 3 & 1 \\
2 & 3 & 1 & 4 \\
3 & 1 & 4 & 2 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 3 & 2 & 4 \\
3 & 2 & 4 & 1 \\
2 & 4 & 1 & 3 \\
4 & 1 & 3 & 2 \\
\end{array}
\]

Note that the above completed examples are the only two solutions for Exercise 3.3 below.

Also note: As the order of Latin Squares increases, the number of possible correct solutions will increase as well. This is because as the Latin Square’s order increases, there is a larger number of empty extended diagonals in which the numbers \((n/2) + 1\) through \(n\) can be put. In fact, for a half-idempotent Latin Square of order \(n\), there are exactly \((n/2)!\) different solutions. For example, in case of half-idempotent Latin Squares of order 4 (above), the numbers 3 and 4 can be placed in either one of the empty extended diagonals.

Students should begin to recognize the power of the diagonal method for creating symmetric idempotent and half-idempotent Latin Squares. Also, students should see that there are multiple solutions (2) for this particular exercise, and the number of correct solutions will increase as the order of the Latin Square increases.

3.3 Discussion

The array above is an example of a half-idempotent Latin Square. How is this different than an idempotent Latin Square? What pattern do you see that might lead to the name “half-idempotent?” What does it mean to have a half-idempotent Latin Square? Discuss your thoughts. Also, what difficulties did you have in completing Problem 2.5?

Encourage students to identify the relationship among the questions in this section. They should note that in Problem 2.5, the even-order arrays that they were asked to create are impossible because it can be proved that there are no even-order, idempotent Latin Squares. Their
explanation would be likely to be a proof by exhaustion indicating that they had tried all the possibilities.

For this reason, another strategy is used to create these even-order Latin Squares that have some of the characteristics of idempotent Latin Squares. These half-idempotent squares are created by placing the entries 1, 2, ..., n/2, 1, 2, ..., n/2 along the main diagonal of the array of order n, where n is even.

3.4 Problems

In your journal, create a symmetric, half-idempotent Latin Square of order 8. While doing so, attempt to find a pattern that you can use to create all symmetric, half-idempotent Latin Squares of even order.

If a pattern is found, try using it to create a symmetric, half-idempotent Latin Square of order 10.

Write what you have learned in your journal and be prepared to discuss your results in class.

One example of a half-idempotent Latin Square of order 8 that your students may find is given below:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>6</th>
<th>2</th>
<th>5</th>
<th>3</th>
<th>7</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>8</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>8</td>
<td>1</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>8</td>
<td>1</td>
<td>6</td>
<td>2</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>1</td>
<td>6</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>8</td>
<td>1</td>
<td>6</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>8</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>8</td>
<td>1</td>
<td>6</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>6</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

This project was funded in part by grant # from NSF.
Of course, the diagonals on which the 5’s, 6’s, 7’s, and 8’s lie can be interchanged, which creates multiple (24 in this case) solutions.

The method that we intend for the students to discover through this problem is the “Half-Idempotent Symmetric Diagonal” method, which was described earlier. This strategy should then be used to create a half-idempotent, symmetric Latin Square of order 10. One example of this is given below:

\[
\begin{array}{cccccccccc}
1 & 8 & 2 & 6 & 3 & 7 & 4 & 9 & 5 & 10 \\
8 & 2 & 6 & 3 & 7 & 4 & 9 & 5 & 10 & 1 \\
2 & 6 & 3 & 7 & 4 & 9 & 5 & 10 & 1 & 8 \\
6 & 3 & 7 & 4 & 9 & 5 & 10 & 1 & 8 & 2 \\
3 & 7 & 4 & 9 & 5 & 10 & 1 & 8 & 2 & 6 \\
7 & 4 & 9 & 5 & 10 & 1 & 8 & 2 & 6 & 3 \\
4 & 9 & 5 & 10 & 1 & 8 & 2 & 6 & 3 & 7 \\
9 & 5 & 10 & 1 & 8 & 2 & 6 & 3 & 7 & 4 \\
5 & 10 & 1 & 8 & 2 & 6 & 3 & 7 & 4 & 9 \\
10 & 1 & 8 & 2 & 6 & 3 & 7 & 4 & 9 & 5 \\
\end{array}
\]

Similar to the Latin Square of order 8, this particular problem has multiple solutions because the diagonals on which the 6’s, 7’s, 8’s, 9’s, and 10’s lie can be interchanged. In this case, there are 120 correct solutions.
Section 4: Mutually Orthogonal Latin Squares

4.1 Discussion and Summary

Discuss the strategies that have been discovered in order to create symmetric half idempotent Latin Squares. Review the strategies for finding symmetric, non-symmetric, idempotent, and half-idempotent Latin Squares. Be aware of the various strategies that can be employed to create such Latin Squares.

*During this discussion, all students should become aware of the “Half-idempotent Symmetric Diagonal” method. Additionally, the diagonal and clicking methods from Section 2 should be reviewed.*

*Note: If your class is excelling with the material, you may want to ask them how many possible solutions there are when using the “half-idempotent symmetric diagonal” method. As discussed earlier, the solution to this is \((n/2)!\) for a Latin Square of order \(n\), where \(n\) is even.*

4.2 Activity

Use the playing cards provided and a 3x3 grid. Arrange the cards on the grid in such a way that each row and column has no two cards of the same rank (e.g., not both kings) and no two cards of the same suit. Attempt to find a solution to this using Latin Squares.

*Please copy the attached playing card worksheet (Appendix B) and distribute the copies to the class. Give the students at least 10 minutes to complete this activity. After this is done, have a short class discussion in which some students share their solutions to this activity. Also have a few students share their solution methods with the class.*

*Through this activity, students will probably discover that there are multiple solutions. More importantly, students, by the end of this section, should realize that the key to solving this problem and other similar problems is to create mutually orthogonal Latin Squares (in this case, one for the rank of the cards and one for the suit of the cards), and superimpose them over one another.*

*Two Latin Squares of order \(n\) with labels 1 through \(n\) are mutually orthogonal if, when superimposed over one another, they yield exactly one of each ordered pair \{(1,1), (1,2), ..., (1,n), (2,1), (2,2), ..., (2,n), ...,(n,1), (n,2), ..., (n,n)\}.*
Note: For purposes of this text we will assume that the labels for mutually orthogonal Latin Squares are 1 through n. However, any n different symbols can be used to create two mutually orthogonal Latin Squares just as long as each possible ordered pair of symbols is accounted for when the two Latin Squares are superimposed on one another.

For this specific problem, the following two mutually orthogonal Latin Squares can be used to find a solution.

**Latin Square 1:**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

*In Latin Square 1, let 1 represent a Jack, 2 represent a Queen, and 3 represent a King.*

**Latin Square 2:**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

*In Latin Square 2, let 1 represent clubs, 2 represent diamonds, and 3 represent spades.*
If we superimpose Latin Square 2 onto Latin Square 1, then we obtain the following:

<table>
<thead>
<tr>
<th>1,1</th>
<th>2,2</th>
<th>3,3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2,3</td>
<td>3,1</td>
<td>1,2</td>
</tr>
<tr>
<td>3,2</td>
<td>1,3</td>
<td>2,1</td>
</tr>
</tbody>
</table>

This yields the solution:

<table>
<thead>
<tr>
<th>Jack of Clubs</th>
<th>Queen of Diamonds</th>
<th>King of Spades</th>
</tr>
</thead>
<tbody>
<tr>
<td>Queen of Spades</td>
<td>King of Clubs</td>
<td>Jack of Diamonds</td>
</tr>
<tr>
<td>King of Diamonds</td>
<td>Jack of Spades</td>
<td>Queen of Clubs</td>
</tr>
</tbody>
</table>

Note: This is only one solution. Please be aware that others exist.

Also note: If one solves this activity simply by guessing and checking or any other method not using Latin Squares, he or she can work backwards from the solution to this problem to create two mutually orthogonal Latin Squares by first assigning each suit and rank some number 1 – 3, and then creating a 3 x 3 array with ordered pairs where the first element in the ordered pair is the number assigned to the rank of that specific card and the second number in the ordered pair is the number assigned to the suit of that specific card. Then, two 3 x 3 mutually orthogonal Latin Squares can be created by putting the first element of each ordered pair into one Latin Square and putting the second element of each ordered pair into another Latin Square.

Also Note: If students are having trouble with the concept of superimposing two Latin Squares over one another, an overhead projector could be used to help the students visualize this concept. To do so, create two mutually orthogonal Latin Squares on two separate sheets of transparency (putting the entries of one Latin Square on the left side of each box and the entries
of the other Latin Square on the right side of each box), and simply overlap the two transparencies to show that no ordered pair is repeated.

4.3 Exercise/Discussion

Discuss strategies for finding a solution to the Playing Card Activity.

Now, find the definition for the following term and record it in your journal.

-Mutually Orthogonal Latin Squares

Discuss how mutually orthogonal Latin Squares can be used to solve the Playing Card Activity. Also discuss strategies that can be used to develop mutually orthogonal Latin Squares.

The definition of mutually orthogonal Latin Squares is given in the solution to the Playing Card Activity above. More information is given in the Appendix.

Unfortunately, as the order of Latin Squares increases, there is no known strategy to create any two mutually orthogonal Latin Squares. However, there is a strategy for creating two mutually orthogonal Latin Squares of order 3. This strategy is as follows:

Given any Latin Square of order 3, to create a second Latin Square that is mutually orthogonal to the first, simply keep the first row the same and interchange the second and third rows.

4.4 Problem/Discussion

Nine golfers are going to meet the next four Sundays to play rounds of 18 holes. They will always golf in groups of three people each. How should the nine golfers arrange themselves so that each golfer plays with every other golfer?

Discuss any ideas on ways to generate a solution to the golfer problem.

This project was funded in part by grant # from NSF.
At the end of this discussion students should understand how to solve this problem using mutually orthogonal Latin Squares.

The solution desired for the purposes of this module begins with each golfer being represented by a number one through nine. Then, two mutually orthogonal Latin Squares of order 3 should be created using the strategy described earlier. Also, each cell of the two mutually orthogonal Latin Squares should be labeled in the same manner. This process will result in the following:

\[
\begin{array}{ccc}
1_1 & 2_2 & 3_3 \\
2_4 & 3_5 & 1_6 \\
3_7 & 1_8 & 2_9 \\
\end{array}
\]

\[
\begin{array}{ccc}
1_1 & 2_2 & 3_3 \\
3_4 & 1_5 & 2_6 \\
2_7 & 3_8 & 1_9 \\
\end{array}
\]

Note that the subscripts represent the cell labeling.
The next step to solving this problem involves creating two basic grids that look like the following, again with the cells labeled 1 through 9:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

Now, four 3 x 3 arrays have been created and each cell has been labeled in the same manner. From these four arrays the desired scheduling can be derived. The scheduling is found by examining each array individually and creating the threesomes based on the cell labels that have the same entries. Each array (grid) above represents the groupings for one day of play.

For example the first array (which was a Latin Square) that we found yields the following:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>
This indicates that on day one, golfers 1, 6, and 8 will play together (because the number is in the cells labeled 1, 6, and 8). Similarly, golfers 2, 4, and 9 will play together, and golfers 3, 5, and 8 will play together.

If we take into account all four arrays the following correct scheduling is yielded:

<table>
<thead>
<tr>
<th>Threesome of Golfers</th>
<th>Sunday 1</th>
<th>Sunday 2</th>
<th>Sunday 3</th>
<th>Sunday 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,6,8</td>
<td>1,5,9</td>
<td>1,4,7</td>
<td>1,2,3</td>
<td></td>
</tr>
<tr>
<td>2,4,9</td>
<td>2,6,7</td>
<td>2,5,8</td>
<td>4,5,6</td>
<td></td>
</tr>
<tr>
<td>3,5,8</td>
<td>3,4,8</td>
<td>3,6,9</td>
<td>7,8,9</td>
<td></td>
</tr>
</tbody>
</table>

Note that there are multiple solutions because the arrays can be created in a different order and the labeling of the cells can be done differently.

4.5 Problem

Sixteen golfers are going to meet the next five Sundays to play rounds of 18 holes. They will always golf in groups of four people each. How should the sixteen golfers arrange themselves so that each golfer plays with every other golfer?

This problem can be done much like the previous problem. However, in this problem one must find three Latin Squares of order 4 that are all mutually orthogonal to one another along with two basic grids of order four (similar to the basic grids in the previous problem).

Unless your class is comfortable creating mutually orthogonal Latin Squares, you may want to help your students develop two or three mutually orthogonal Latin Squares of order 4 to get them started on this problem. It is very difficult to find mutually orthogonal Latin Squares of order four. The solution to this problem, done in the same manner as the previous problem, is given below:
Three mutually orthogonal Latin Squares (note that no matter which two are picked they are mutually orthogonal to one another):

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Basic Grids:

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>
These 4 x 4 arrays yield the following solution:

<table>
<thead>
<tr>
<th></th>
<th>Sunday 1</th>
<th>Sunday 2</th>
<th>Sunday 3</th>
<th>Sunday 4</th>
<th>Sunday 5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Foursome of Golfers</strong></td>
<td>1,6,11,16</td>
<td>1,7,12,14</td>
<td>1,8,10,15</td>
<td>1,5,9,13</td>
<td>1,2,3,4</td>
</tr>
<tr>
<td><strong>Foursome of Golfers</strong></td>
<td>2,5,12,15</td>
<td>2,8,11,17</td>
<td>2,7,9,16</td>
<td>2,6,10,14</td>
<td>5,6,7,8</td>
</tr>
<tr>
<td><strong>Foursome of Golfers</strong></td>
<td>3,8,9,14</td>
<td>3,5,10,16</td>
<td>3,6,12,13</td>
<td>3,7,11,15</td>
<td>9,10,11,12</td>
</tr>
<tr>
<td><strong>Foursome of Golfers</strong></td>
<td>4,7,10,13</td>
<td>4,6,9,15</td>
<td>4,5,11,14</td>
<td>4,8,12,16</td>
<td>13,14,15,16</td>
</tr>
</tbody>
</table>

Just like the previous problem this is not the only solution. It is crucial for students to realize that this problem is mathematically equivalent to the culminating problem that was given at the beginning of the module. Now students should be able to easily solve the culminating problem by using the same arrays used in this problem along with the same process.
4.6 Challenge Problem

Attempt to find two mutually orthogonal Latin Squares of order 6. Using the internet, you may want to research the findings of L. Euler on this topic.

In your journal, either write down the two mutually orthogonal Latin Squares of order 6 or provide an explanation of why two such Latin Squares do not exist.

*Students should discover that Euler found that there does not exist two mutually orthogonal Latin Squares of order six.*

Section 5: Affine Planes and Applications

5.1 Exercise

Find the definitions for the following terms and record them in your journal.

- Affine Planes
- Parallel Classes of Affine Planes
- Euclid’s Parallel Postulate

*Precise definitions of these terms are given in the Appendices.*
The diagram above is a drawing of an Affine Plane with 9 points labeled 1 to 9. Know that each encircled number is a point in the Affine Plane, and each closed figure (for example, the figure connecting the points 1, 4, and 7, or the triangle connecting the points 2, 6, and 7) represents one of the 12 “lines” in this Affine Plane. Also see that each parallel class is represented by a different set of lines (thin, thick, thin-dotted, and thick-dotted). If desired, confirm that all the properties of an Affine Plane are indeed satisfied by this diagram.

*Students should know that Affine Planes have the following properties:*

- *Each pair of distinct points is on a unique line.*
• For any point p and any line l not through p, there is exactly one line m through p that does not intersect l.
• There are three distinct points not on a line.
• Students should also understand that an Affine Plane can be created with a finite number of points.

An example of an Affine Plane with a finite number of points is given below (Note: this listing here and the diagram above are two representations of the same Affine Plane):

\((1,2,3);(4,5,6);(7,8,9);(1,4,7);(2,5,8);(3,6,9);(1,6,8);(2,4,9);(3,5,8);(1,5,9);(2,6,7);(3,4,8)\)

In this example a line is made up of the three points within parentheses. Note how each condition is satisfied. For example the distinct points 3 and 6 are only on the same line one time (line \((3,6,9)\)). Also, when considering the point 4 and the line \((7,8,9)\) there is only one line \((4,5,6)\) that does not intersect \((7,8,9)\) (a line like \((2,4,9)\) contains 4 but intersects \((7,8,9)\) at 9). Also, there clearly exist the points 1, 3, and 9, all of which are not on the same line.

Students should note that these properties are all satisfied by the above diagram.

Students should also know basic ideas about parallel classes of affine planes.

Parallel classes of affine planes are sets of lines which together contain all the distinct points of the plane such that no two of them intersect. In the above example the parallel classes are as follows (Again note that these are the same parallel classes that are illustrated in the above diagram):

Parallel class one: \((1,2,3);(4,5,6);(7,8,9)\)
Parallel class two: \((1,4,7);(2,5,6);(3,6,9)\)
Parallel class three: \((1,6,8);(2,4,9);(3,5,7)\)
Parallel class four: \((1,5,9);(2,6,7);(3,4,8)\)

Clearly, each parallel class has no lines that intersect, and each parallel class contains all nine points of the plane. One should also note that all the parallel classes combined contain every line in the entire affine plane.

Students should also realize that affine planes follow Euclid’s parallel postulate. This is true because: given a line and a point there is always exactly one line that is parallel to the given line that contains the given point.
5.2 Discussion

Discuss how the golfer problem from Section 1.4 relates to Affine Planes. Furthermore, discuss how parallel classes of Affine Planes also relate to the golfer problem of Section 1.4 and to Euclid’s Parallel Postulate.

*Students should realize that the scheduling for the golfer problem forms an affine plane when the golfers are labeled and then written in groups. Specifically, each group of golfers forms a line in the affine plane and the way in which they are scheduled in a single day forms the parallel classes of the affine plane.*

5.3 Possible Project Topics

1. Alphabet Latin Squares

Create three different Latin Squares of order 26, using each letter of the English alphabet. Describe the different strategies used to generate each Latin Square.

*This type of Latin Square can be easily created by using the Clicking method and then simply moving around the rows. One should also note that the half idempotent symmetric diagonal strategy can be employed.*

*Three examples of this are given below:*
1. Clicking Method

This project was funded in part by grant # from NSF.
2. Half-Idempotent Method

This project was funded in part by grant # from NSF.
3. Clicking Method with rows 25 and 26 interchanged

```
   a b c d e f g h i j k l m n o p q r s t u v w x y z
   z a b c d e f g h i j k l m n o p q r s t u v w x y
   y z a b c d e f g h i j k l m n o p q r s t u v w x
   x y z a b c d e f g h i j k l m n o p q r s t v w
   w x y z a b c d e f g h i j k l m n o p q r s t u v
   v w x y z a b c d e f g h i j k l m n o p q r s t u
   u v w x y z a b c d e f g h i j k l m n o p q r s t
   t u v w x y z a b c d e f g h i j k l m n o p q r
   s t u v w x y z a b c d e f g h i j k l m n o p q r
   r s t u v w x y z a b c d e f g h i j k l m n o p q
   q r s t u v w x y z a b c d e f g h i j k l m n o p
   p q r s t u v w x y z a b c d e f g h i j k l m n o
   o p q r s t u v w x y z a b c d e f g h i j k l m n
   n o p q r s t u v w x y z a b c d e f g h i j k l m
   m n o p q r s t u v w x y z a b c d e f g h i j k l
   l m n o p q r s t u v w x y z a b c d e f g h i j k
   k l m n o p q r s t u v w x y z a b c d e f g h i j
   j k l m n o p q r s t u v w x y z a b c d e f g h i
   i j k l m n o p q r s t u v w x y z a b c d e f g h
   h i j k l m n o p q r s t u v w x y z a b c d e f g
   g h i j k l m n o p q r s t u v w x y z a b c d e f
   f g h i j k l m n o p q r s t u v w x y z a b c d e
   e f g h i j k l m n o p q r s t u v w x y z a b c d
   d e f g h i j k l m n o p q r s t u v w x y z a b c
   b c d e f g h i j k l m n o p q r s t u v w x y z a
   c d e f g h i j k l m n o p q r s t u v w x y z a b
```
2. Scheduling a Round-Robin Tournament

Use Latin Square(s) to design a schedule for 7 tennis players to play a round robin tournament in which every player plays one match with every other player in 6 rounds, 3 simultaneous matches per round (Note that one player will have a by in each round).

For a small number of players, say 4, students can use guess and check and find a solution:

round 1: 1 plays 2, 3 plays 4
round 2: 2 plays 3, 1 plays 4
round 3: 3 plays 1, 2 plays 4

Now think of this schedule as a kind of 'multiplication': \( ab = c \) means "a plays b in round c". Rewrite the schedule as a multiplication table.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 1 & 3 & 2 \\
2 & 1 & 2 & 3 \\
3 & 3 & 2 & 1 \\
4 & 2 & 3 & 1
\end{array}
\]

If the blank entries were filled with the symbol 4, then we have a Latin Square. Also, the square is symmetric about the blank diagonal. These facts remain true when any schedule is converted to a 'multiplication' table. The table must be symmetric about the blank diagonal because "a plays b in round c" means the same thing as "b plays a in round c". The 'multiplication' table is also a Latin Square (when 2n fills the blank diagonal) because each player plays exactly one other player in a given round. Hence no row or column of the table can repeat an entry.
With seven players the problem is slightly more difficult. The student must think of some way to indicate a by, which will occur once for each player. In the solution below the blank space indicates a by:

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th>Player 2</th>
<th>Player 3</th>
<th>Player 4</th>
<th>Player 5</th>
<th>Player 6</th>
<th>Player 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Player 2</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>1</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Player 3</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Player 4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Player 5</td>
<td>6</td>
<td>7</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Player 6</td>
<td>7</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Player 7</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

Clearly this Latin Square will produce the successful round robin scheduling. If students struggle with this project encourage them to examine smaller cases like four in order to discover the solution.

Note that there may be variations of doing this problem with Latin Squares. For example students may choose to give an actual label for what it means to have a by. Therefore, all solutions must be carefully examined in order to discover if the scheduling will be successful.

3. Field Problem

In what ways do Latin Squares apply to finite fields? How do the addition tables of finite fields relate to Latin Squares? Provide examples and explain your reasoning. To complete this project, you must understand finite fields. A finite field is a finite set of elements with four operations: addition, subtraction, multiplication, and division with the exception of division by zero. In a
field, the result of any combination of operations on members of the field results in a member of the field.

_Students completing this project should find that the addition table of any finite field will result in a Latin Square._

_Note that only highly motivated students should attempt this project._

_One simple example that you may want to share is the finite field containing the elements zero and one. In the finite field with the elements zero and one, the addition properties must be defined as follows to satisfy all the properties of a finite field:_

\[
\begin{align*}
0+0 &= 0 \\
0+1 &= 1 \\
1+0 &= 1 \\
1+1 &= 0 \\
\end{align*}
\]

_The addition table is as follows:_

\[
\begin{array}{|c|c|c|}
\hline
+ & 0 & 1 \\
\hline
0 & 0 & 1 \\
\hline
1 & 1 & 0 \\
\hline
\end{array}
\]

_Note that the addition table forms a two-by-two Latin Square. Students will notice that a finite field with three elements will have an addition table that forms a three-by-three Latin Square and so on. A challenging aspect of this project is correctly defining addition for larger finite fields._
Appendix A: Internet Search Results

Latin Square

A Latin square of order \( k \) is defined as a \( k \) by \( k \) square grid, the \( k^2 \) cells of which are occupied by \( k \) distinct symbols of a set \( X = 1, 2, \ldots, k \), such that each symbol occurs once in each row and each column. Two Latin squares are said to be orthogonal if, when superposed, any symbol of the first square occurs exactly once with each symbol of the second square.

Latin square - a square matrix of \( n \) rows and columns; cells contain \( n \) different symbols so arranged that no symbol occurs more than once in any row or column

http://www.thefreedictionary.com/Latin+square

Latin square (ˈlat-ən ˈskwer)

(mathematics) An \( n \times n \) square array of \( n \) different symbols, each symbol appearing once in each row and once in each column; these symbols prove useful in ordering the observations of an experiment.

http://www.answers.com/topic/latin-square-2?cat=technology

Latin Squares

The applet below offers you two problems: one simple and one less simple. In the simple one, you are requested to arrange numbers in a square matrix so as to have every number just once in every row and every column. The second problem imposes one additional condition: the arrangement must be symmetric with respect to the main diagonal (the one from the North-West to the South-East corner.)

There are two ways to manipulate rows of the matrix:

1. You can cycle rows left or right by clicking just outside the matrix
2. You can swap two squares in the same row by clicking first on one and then the other
Remark

Both problems fall into the framework of Latin Squares originating with L.Euler. Professor W.MeWorter who wrote his Masters Thesis on (orthogonal) latin squares kindly offered his assistance in preparing pages on this entertaining topic. His introduction follows the applet.

A latin square of order n is an n by n array of n symbols in which every symbol occurs exactly once in each row and column of the array. Here are two examples.

\[
\begin{array}{cc}
\text{Latin square of order 2} & \text{Latin square of order 3} \\
\begin{bmatrix}
a & b \\
b & a \\
\end{bmatrix} & \begin{bmatrix}
x & y & z \\
z & x & y \\
y & z & x \\
\end{bmatrix}
\end{array}
\]

The great mathematician Leonhard Euler introduced latin squares in 1783 as a "nouveau espece de carres magiques", a new kind of magic squares. He also defined what he meant by orthogonal latin squares, which led to a famous conjecture of his that went unsolved for over 100 years. In 1900, G. Tarry proved a particular case of the conjecture. It was shown in 1960 by Bose, Shrikhande, and Parker that, except for this one case, the conjecture was false.

http://www.cut-the-knot.org/arithmetic/latin.shtml

A Latin square of order \( n \) is an \( n \times n \) array containing symbols from some alphabet of size \( n \), arranged so that each symbol appears exactly once in each row and exactly once in each column. The best introduction here is in Bose and Manvel.

It seems that Latin squares were originally mathematical curiosities, but serious statistical applications were found early in the 20th century, as "experimental designs." The classic example is the use of a Latin square configuration to place 3 or 4 different grain varieties in test patches. Having multiple patches for each variety helps to minimize localized soil effects. Similar statements can be made about medical "treatments."

http://www.ciphersbyritter.com/RES/LATSQ.HTM

This project was funded in part by grant # from NSF.
Latin square:

n : a square matrix of n rows and columns; cells contain n different symbols so arranged that no symbol occurs more than once in any row or column [syn: Latin square]

http://dict.die.net/latin%20square/

**Definition.** A *latin square of order* n is an n-by-n matrix in which each entry is chosen from a set of n distinct symbols, in which each symbol appears once in each row and once in each column.

Alternatively: A latin square of order n is a set of n^2 ordered triples (r, c, s), where r is an element of the n-set R, c is an element of the n-set C, and s is an element of the n-set S, in which every possible combination of RxC, RxC, and CxS is represented. The three sets R, C, and S are called the *constraints* of the latin square. (This definition shows the symmetry of a latin square on its constraints.)

Examples:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

A latin square with symbols chosen from an ordered set (such as the integers), written so that the symbols appear in sequence in the top row and the leftmost column, is said to be in *reduced form*. The four examples above are in reduced form.

The ordered pairs of RxC are called *cells*.

http://home.flash.net/~markthom/html/latin_square_math.html
Order. In a Latin square, the number of entries in a row, or in a column, or the number of unique symbols.

http://www.ciphersbyritter.com/GLOSSARY.HTM#Orthogonal

Diagonal

The diagonal in a square matrix that goes from the upper left corner to the lower right corner.

http://www.yourdictionary.com/ahd/p/p0563300.html

Symmetric

In linear algebra, a symmetric matrix is a square matrix, $A$, that is equal to its transpose

$$A = A^T,$$

The entries of a symmetric matrix are symmetric with respect to the main diagonal (top left to bottom right). So if the entries are written as $A = (a_{ij})$, then

$$a_{ij} = a_{ji}$$

for all indices $i$ and $j$. The following $3 \times 3$ matrix is symmetric:

$$\begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & -5 \\
3 & -5 & 6
\end{bmatrix}$$

http://www.answers.com/topic/symmetric-matrix?cat=technology
In many areas in mathematics, matrices with certain structure arise. A few important examples are

- **Symmetric matrices** are such that elements symmetric about the *main diagonal* (from the upper left to the lower right) are equal, that is, \( a_{i,j} = a_{j,i} \iff A^T = A \).
- A square matrix \( A \) is called **idempotent** if \( A^2 = AA = A \).

http://www.answers.com/topic/matrix-mathematics?cat=technology

---

**Even/Odd Order, Order, Diagonal, Latin Square**

**Latin Square**
An \( m \times m \) array of \( m \) symbols in which each symbol appears exactly once in each row and each column of the array. A set of two Latin squares are frequently used for generating **magic squares**. See **Graeco-Latin square**.

**Leading Diagonal**
Also called left diagonal. The line of numbers from the upper left corner of the magic square to the lower right corner. See **Main Diagonals**.

**Main Diagonals**
The two diagonal series of cells that go from corner to corner of the magic square. Each must sum to the constant in order for the array to be magic. The leading (or left) diagonal is the one from upper left to lower right. The right diagonal is the one from lower left to upper right.

**Order \( m \)**
Indicates the number of cells per side of the magic square, cube, tesseract, etc. (But see **order \( n \)**.)

**Order \( n \)**
\( n \) traditionally indicated the number of cells per side of the magic square, cube, tesseract, etc. \( m \) is now used increasingly for this purpose. For a magic star, \( n \) indicates the number of points in the star pattern.

**Order, Odd**
The order is not divisible by 2, i.e. 3 (the smallest possible magic square), 5, 7, etc.
**Order, even**

The order is evenly divisible by 2 but not by 4. i.e. 6, 10, 14, etc. This order is by far the hardest to construct.

**Associated Magic Squares**

A magic square where all pairs of cells diametrically equidistant from the center of the square equal the sum of the first and last terms of the series, or $m^2 + 1$. Also called Symmetrical or center-symmetric. The center cell of odd order associated magic squares is always equal to the middle number of the series. Therefore the sum of each pair is equal to 2 times the center cell. In an order-5 magic square, the sum of the 2 symmetrical pairs plus the center cell is equal to the constant, and any two symmetrical pairs plus the center cell sum to the constant. i.e. the two pairs do not have to be symmetrical to each other.

In an even order magic square the sum of any 2 symmetrical pairs will equal the constant (the sum of the 2 members of a symmetrical pair is equal to the sum of the first and last terms of the series).

W. S. Andrews, Magic squares & Cubes, 1917
Benson & Jacoby, New Recreations with Magic Squares, 1976


Many traditional mathematical topics are related to latin squares.

<table>
<thead>
<tr>
<th>1 2 3 4</th>
<th>1 2 3 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 3 4 1</td>
<td>2 4 1 3</td>
</tr>
<tr>
<td>3 4 1 2</td>
<td>3 1 4 2</td>
</tr>
<tr>
<td>4 1 2 3</td>
<td>4 3 2 1</td>
</tr>
</tbody>
</table>

Table A Table B

The addition table for the integers modulo $n$ is a latin square (latin square A above, the integers modulo 4 in disguise), the multiplication table for the integers modulo $n$ but relatively prime to $n$ is a latin square (latin square B above, the nonzero integers modulo 5), and, for more abstract examples, the multiplication tables for finite groups and finite quasigroups are latin squares. Indeed, there is hardly any difference between latin squares and quasigroups.
Mutually Orthogonal Latin Squares

Orthogonal Latin Squares

(\text{OLs}). Two Latin squares of order $n$, which, when superimposed, form each of the $n^2$ possible ordered pairs of $n$ symbols exactly once. At most, $n-1$ Latin squares may be mutually orthogonal.

\[
\begin{array}{cccc}
3 & 1 & 2 & 0 \\
0 & 2 & 1 & 3 \\
1 & 3 & 0 & 2 \\
2 & 0 & 3 & 1 \\
\end{array}
\begin{array}{cccc}
0 & 3 & 2 & 1 \\
2 & 1 & 0 & 3 \\
1 & 2 & 3 & 0 \\
3 & 0 & 1 & 2 \\
\end{array}
= \begin{array}{cccc}
30 & 13 & 22 & 01 \\
02 & 21 & 10 & 33 \\
11 & 32 & 03 & 20 \\
23 & 00 & 31 & 12 \\
\end{array}
\]

http://www.ciphersbyritter.com/GLOSSARY.HTM#Orthogonal

Two latin squares are said to be orthogonal if no pair of corresponding elements occurs more than once. For example,

\[
\begin{array}{cccc}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{array}
\text{is orthogonal to}
\begin{array}{cccc}
1 & 3 & 2 \\
2 & 1 & 3 \\
3 & 2 & 1 \\
\end{array}
\]

We can see this most easily by writing them together as follows, and observing that no pair appears twice:

\[
\begin{array}{cccc}
11 & 23 & 32 \\
22 & 31 & 13 \\
33 & 12 & 21 \\
\end{array}
\]

A set of $n$ latin squares is mutually orthogonal if every pair of latin squares from the set is orthogonal.

Euler studied orthogonal latin squares because they can be used to construct magic squares, and he found that, while orthogonal latin squares of any odd order are easy to generate, even orders
are not. He conjectured, but did not prove, that there are no orthogonal latin squares of order 4n+2, for any integer n. In 1960, this was proved incorrect; it turns out that there are orthogonal latin squares of any size except 1, 2, and, oddly enough, 6.

http://everything2.com/index.pl?node_id=1425590

---

**Euclid’s Parallel Postulate**

![Diagram of Euclid’s Parallel Postulate](image)

In geometry, the parallel postulate, also called Euclid's fifth postulate since it is the fifth postulate in Euclid's Elements, is a distinctive axiom in what is now called Euclidean geometry. It states that:

If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

Euclidean geometry is the study of geometry that satisfies all of Euclid's axioms, including the parallel postulate. A geometry where the parallel postulate cannot hold is known as a non-Euclidean geometry. Geometry that is independent of Euclid's fifth postulate (i.e., only assumes the first four postulates) is known as absolute geometry (or, in some places, neutral geometry).

http://en.wikipedia.org/wiki/Parallel_postulate
Euclids Fifth Postulate

Besides 23 definitions and several implicit assumptions, Euclid derived much of the planar geometry from five postulates.

1. A straight line may be drawn between any two points.
2. A piece of straight line may be extended indefinitely.
3. A circle may be drawn with any given radius and an arbitrary center.
4. All right angles are equal.
5. If a straight line crossing two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which are the angles less than the two right angles.

The fifth postulate refers to the diagram on the right. If the sum of two angles $A$ and $B$ formed by a line $L$ and another two lines $L_1$ and $L_2$ sum up to less than two right angles then lines $L_1$ and $L_2$ meet on the side of angles $A$ and $B$ if continued indefinitely.

Postulates 1 and 3 set up the "ruler and compass" framework that was a standard for geometric constructions until the middle of the 19th century. They may be said to be based on man's practical experience. The second postulate gives an expression to a commonly held belief that straight lines may not terminate and that the space is unbounded. By the Definition 10, an angle is right if it equals its adjacent angle. Thus the fourth postulate asserts homogeneity of the plane: in whatever directions and through whatever point two perpendicular lines are drawn, the angle they form is one and the same and is called right. We may think of the fourth postulate as having been justified by the everyday experience acquired by man in the finite, inhabited portion of the universe which is our world and extrapolated (much as the Postulate 2) to that part of the world whose existence (and infinite expense) we sense and believe in.

Elaborateness of the fifth postulate stands in a stark contrast to the simplicity of the first four. Euclid himself, probably, had mixed feelings about it as he did not make use of it until Proposition I.29. The postulate looks more like another proposition than a basic truth. Here's, for example, Proposition I.27 which, combined with Proposition I.13, claims that, if (in the diagram above) angles $A$ and $B$ sum up to two right angles, then the lines $L_1$ and $L_2$ are parallel.

If a straight line crossing two straight lines makes the alternate angles equal to one another, the straight lines will be parallel to one another.

(By Definition 23, two straight line in the same plane are parallel if they do not meet even when produced indefinitely in both directions.)
Proposition I.17 is actually a converse of the fifth postulate:

In any triangle two angles taken together in any manner are less than two right angles.

The postulate (also known as the Parallel Postulate) attracted immediate attention. The commentator Proclus (c. 410-485) tells us that the postulate was attacked from the very start. He wrote, "This postulate ought even to be struck out of Postulates altogether; for it is a theorem..." Now we know that it is impossible to derive the Parallel Postulate from the first four. The numerous (and failed) attempts to do that gave rise to a slew of statements equivalent to the postulate itself. Several are cited by S.Brodie. Following are a few more:

1. The exists a pair of similar noncongruent triangles.
2. There exists a pair of straight lines everywhere equidistant from one another.
3. For any three noncollinear points, there exists a circle passing through them.
4. If three angles of a quadrilateral are right angles, then the fourth angle is also a right angle.
5. If a straight line intersects one of two parallels it will intersect the other.
6. Straight lines parallel to a third line are parallel to each other.
7. Two straight lines that intersect one another cannot be parallel to a third line.
8. There is no upper limit to the area of a triangle.

The last one seems especially intuitive. The reverse holds in non-Euclidean geometries of Lobachevsky and Riemann. Coxeter mentions the fact that Lewis Carroll could not accept this assertion and considered it as a proof of the contradictory nature of non-Euclidean geometries.

In one of his books R.Smullyan tells of an experiment he ran in a remedial geometry class. He drew the famous Pythagoras' diagram and asked whether the two small squares are bigger or smaller than the square on the hypotenuse. Half the class thought the sum was bigger, another half thought it was smaller. By all accounts, the Pythagorean Theorem is far from obvious. It is amazing that the Parallel Postulate, being equivalent to such intuitive statements as 1 and 8 above, is also equivalent to the Pythagorean Theorem.

http://www.cut-the-knot.org/triangle/pythpar/Fifth.shtml

Euclid's Parallel Postulate. Also known as "Euclid's Fifth Postulate" Given any straight line and a point not on it, there "exist one and only on straight line which passes" through that point and never intersects the first line, no matter how far they are extended.

Postulate 5
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Of course, this is a postulate for plane geometry. It should include the condition that the two straight lines lie in a plane, otherwise, skew lines in space would satisfy the hypotheses. Also, without an ambient plane, the term "that side [of the straight line]" has no meaning.

This postulate is usually called the "parallel postulate" since it can be used to prove properties of parallel lines. Euclid develops the theory of parallel lines in propositions through I.31.

The parallel postulate is historically the most interesting postulate. Geometers throughout the ages have tried to show that it could be proved from the remaining postulates so that it wasn't necessary to assume it. The process tried was to assume its falsehood, then derive a contradiction. Many strange conclusions follow from denying the parallel postulate, and several geometers found such great absurdities that they concluded that the parallel postulate did follow from the rest.

Nevertheless, these apparent absurdities are not contradictions. In the early nineteenth century, Bolyai, Lobachevsky, and Gauss found ways of dealing with this "non-Euclidean" geometry by means of analysis and accepted it as a valid kind of geometry, although very different from Euclidean geometry. This hyperbolic geometry, as it is called, is just as consistent as Euclidean geometry and has many uses.

Thus, we know now that we must include the parallel postulate to derive Euclidean geometry. For more on noneuclidean geometries, see the notes on hyperbolic geometry after I.29 and elliptic geometry after I.16.

Euclid does not use this parallel postulate until Proposition I.29, but nearly all of the rest of Book I depends on it.

http://aleph0.clarku.edu/~djoyce/java/elements/bookI/post5.html
### Affine Planes

**Affine Plane**

A two-dimensional affine geometry constructed over a finite field. For a field $F$ of size $n$, the affine plane consists of the set of points which are ordered pairs of elements in $F$ and a set of lines which are themselves a set of points. Adding a point at infinity and line at infinity allows a projective plane to be constructed from an affine plane. An affine plane of order $n$ is a block design of the form $(n^2, n, 1)$. An affine plane of order $n$ exists iff a projective plane of order $n$ exists.

http://mathworld.wolfram.com/AffinePlane.html

An incidence structure $\pi = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ is called an affine plane if the following three axioms hold.

- Each two distinct points are on a unique line.
- Given any point $p$ and any line $l$ not through $p$, there is exactly one line $g$ through $p$ disjoint from $l$.
- There are three distinct points not on a line.

Two lines $a$ and $b$ of an affine plane are defined to be parallel if either $a=b$ or $a \cap b = \emptyset$. We will write $a \parallel b$ in this case. The relation of parallelism defines a parallelism on $\pi$. In fact, this is the unique parallelism on $\pi$.

http://www.math.uni-kiel.de/geometrie/klein/math/geometry/affine.html

In geometry, **affine geometry** is geometry not involving any notions of origin, length or angle, but with the notion of subtraction of points giving a vector.

It occupies a place intermediate between Euclidean geometry and projective geometry. It is the geometry of **affine space**, of a given dimension $n$, coordinatized over a field $K$.

There is also (in two dimensions) a combinatorial generalization of coordinatized affine space, as developed in synthetic finite geometry. However, this article concentrates on coordinatized affine geometry, in the dominant tradition of the nineteenth and twentieth centuries.

Affine geometry can be explained as the geometry of vectors, not involving any notions of coordinate, length or angle. An affine space is distinguished from a vector space of the same dimension by 'forgetting' the origin $0$. That way of thinking was in older texts sometimes talked about as a theory of free vectors. A contemporary and more abstract way of putting it is mentioned at the end of this page, completing a formal reduction of most affine geometry to linear algebra.
http://en.wikipedia.org/wiki/Affine_geometry

Affine planes

Imagine an infinitely large sheet of paper, a straight ruler of infinite length and a pencil with an infinitely sharp point (a tall order). With these tools you may draw infinitely small points and infinitely long straight lines which are infinitely thin. These are the points and lines of 'classical' geometry. They have the following properties:

- Two points are joined by exactly one straight line.
- Two lines intersect in at most one point.

Lines that do not intersect are called parallel. There is a third property which is slightly more involved. It is often called 'the axiom of parallelism':

- Given a line and a point not on that line, there is exactly one line through that point which is parallel to the original line. In other words, all lines through that point intersect the first line in a single point, except one line which is parallel to the first line.

Semi-Linear spaces which satisfy these axioms are called affine planes. The 'classical' plane is an example of an infinite affine plane. We shall be more interested in finite affine planes.

Affine and semi-affine planes

Imagine an infinitely large sheet of paper, a straight ruler of infinite length and a pencil with an infinitely sharp point (a tall order). With these tools you may draw infinitely small points and infinitely long straight lines which are infinitely thin. These are the points and lines of 'classical' geometry. They have the following properties:

- Two points are joined by exactly one straight line.
- Two lines intersect in at most one point.

Lines that do not intersect are called parallel. There is a third property which is slightly more involved. It is often called 'the axiom of parallelism':

- Given a line and a point not on that line, there is exactly one line through that point which is parallel to the original line. In other words, all lines through that point intersect the first line in a single point, except one line which is parallel to the first line.

Semi-Linear spaces which satisfy these axioms are called affine planes. The 'classical' plane is an example of an infinite affine plane. We shall be more interested in finite affine planes.
Parallel Classes of Affine Planes

Finite affine planes
Affine planes with only a finite number of points are called finite. Every line in such a plane has only a finite number of points, and it can be proved that this number is the same for every line. This number (traditionally written as q) is called the order of the plane.

An affine plane of order q has q points on every line, q+1 lines through every point, q^2 (q squared) points in total and q^2+q lines. Parallel lines come in groups of q that are all parallel to each other. Such a group is called a parallel class. The lines of an affine plane can be partitioned into q+1 disjoint parallel classes.

Some examples
The smallest affine planes have order two and three. We present them in the pictures below. Note that we have used different colours for the lines. Each colour corresponds to a different parallel class. Remember that lines need not be straight and that an intersection of two lines only counts when there is a point (a circle) on it.

The affine plane of order 2.
There are four points and six lines. Each line contains two points and each point lies on three lines. There are three parallel classes (black, red and blue) with two lines to each class.

The affine plane of order 3.
There are nine points and twelve lines. Each line contains three points and each point lies on four lines. There are four parallel classes (black, red, blue and green) with three lines to each class.

http://www.inference.phy.cam.ac.uk/cds/part7.htm
A term in combinatorics. Over any finite field $F$ one can construct a two-dimensional affine geometry. If the field is of size $n$, the affine plane consists of a set $P$ of points which are ordered pairs of elements of the field (thus there are $n^2$ points), and a set $L$ of lines which are themselves sets of points. The points on a given line consist of all pairs $(x,y)$ which are solutions of a linear equation ($y=mx+b$, or $x=k$, with $m,b$ and $k$ in $F$). The lines can be further divided up into "parallel classes" based on the value of $m$ in their defining equation, (the lines of the form $x=k$ are said to be in the "infinite slope" parallel class). By adding an additional "point at infinity" for each of the parallel classes and an additional "line at infinity" which connects the points at infinity, one can construct a projective plane from an affine plane.

http://www.everything2.com/index.pl?node_id=891396
Appendix B: Playing Cards

This project was funded in part by grant # from NSF.

64

July, 2007
Glossary

**Affine Planes:** Affine geometry requires that a plane have the following properties:

- Each pair of distinct points are on a unique line.
- For any point p and any line l not through p, there is exactly one line m through p that does not intersect l.
- There are three distinct points not on a line.

**Array:** a set of numbers or other symbols arranged in rows and columns so that each row has the same number of elements (n) and each column has the same number of elements (m)

```
1 2
Example -- 1 a is an array with 3 rows and 2 columns.
? #
```

**Cell:** An individual location of an entry within a Latin Square. For example, an n by n Latin Square would have n^2 cells.

**Clicking Method:** This method for creating Latin Squares will always produce a symmetric Latin Square for any order.

**Diagonal:** The main diagonal of a Latin Square would be formed by listing n selected elements in order, beginning with the element in the first row, first column, followed by the element in the second row, second column, continuing to the element in the nth row, nth column.

**Diagonal Method:** This strategy for creating Latin Squares will produce idempotent, symmetric Latin Squares for arrays of odd order. To do this for an array of order n begin at the upper left cell and place consecutive numbers 1 through n along the main diagonal. Now look at the diagonals that run from upper right to lower left (using the wrap-around concept) copy the number from the main diagonal into the other cells along each upper right to lower left diagonal.
**Euclid’s Parallel Postulate:** This is also known as the Fifth Postulate. If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

**Half-Idempotent:** A Latin Square of order n, where n is even, is this if the main diagonal consists of the entries 1, 2, …, n/2, 1, 2, …, n/2.

**Half-Idempotent Symmetric Diagonal Method:** This strategy should be used to create half-idempotent, symmetric Latin Squares.

**Idempotent:** A Latin Square is idempotent if each element along the diagonal is the number or symbol of the row and column where it is located.

Example – If we label the rows top to bottom and the columns left to right by a, b, c, … then the Latin Square below is idempotent.

```
 a   b   c
 a  a   c   b
 b  c   b   a
 c  b   a   c
```

**Latin Square:** an array with n rows and n columns with n different symbols so that no row nor column has two or more copies of any symbol.

**Mutually Orthogonal Latin Squares:** Two Latin Squares of order n with labels 1 through n are mutually orthogonal if, when superimposed over one another, they yield exactly one of each ordered pair \{(1,1), (1,2), …, (1,n), (2,1), (2,2), …, (2,n), …(n,1), (n,2), …, (n,n)\}.

This project was funded in part by grant # from NSF.
Order: The order of a Latin Square is the number of rows (or columns) in the square. A Latin Square is of even order if the number of rows is even. Similarly, a Latin Square is of odd order if the number of rows is odd. For some of the work with Latin Squares, strategies vary if the Latin Square is even or odd.

Parallel Classes of Affine Planes: An affine plane can be partitioned into groups of lines so that all lines within a group are parallel to one another. These groups are called parallel classes.

Sudoku: A puzzle based upon a 9 x 9 Latin Square with constraints on the contents of smaller squares in which some of the entries, either consecutive numbers or letters, are missing. The puzzle-doer uses the properties of Latin Squares to determine the missing numbers or letters.

Symmetric: A Latin Square is symmetric if, for every entry in the ith row, jth column, the same entry is in the jth row, ith column.
Works Cited


This project was funded in part by grant # from NSF.
"Euclid's Parallel Postulate." 13 July 2007


This project was funded in part by grant # from NSF.
This project was funded in part by grant # from NSF.
"Web Sudoku - Billions of Free Sudoku Puzzles to Play Online." WebSudoku. 13 July 2007