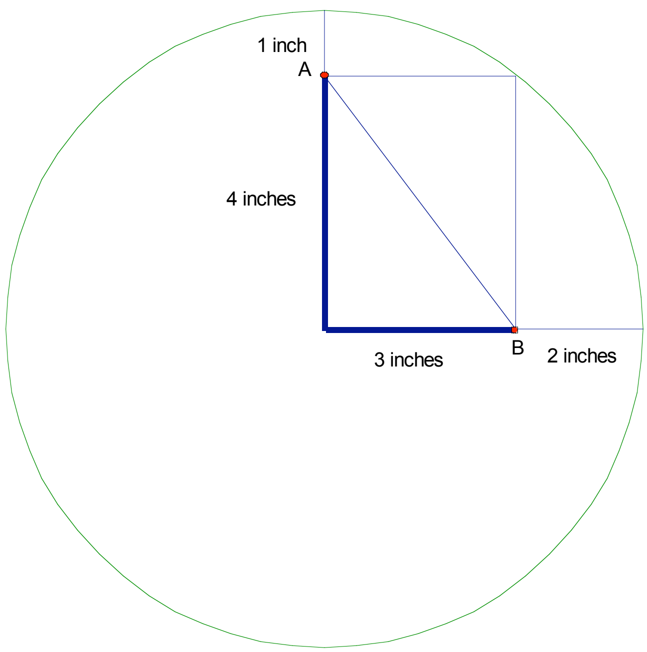
# **Pythagorean Triples Module**
# **Student Edition**

**Investigation I**

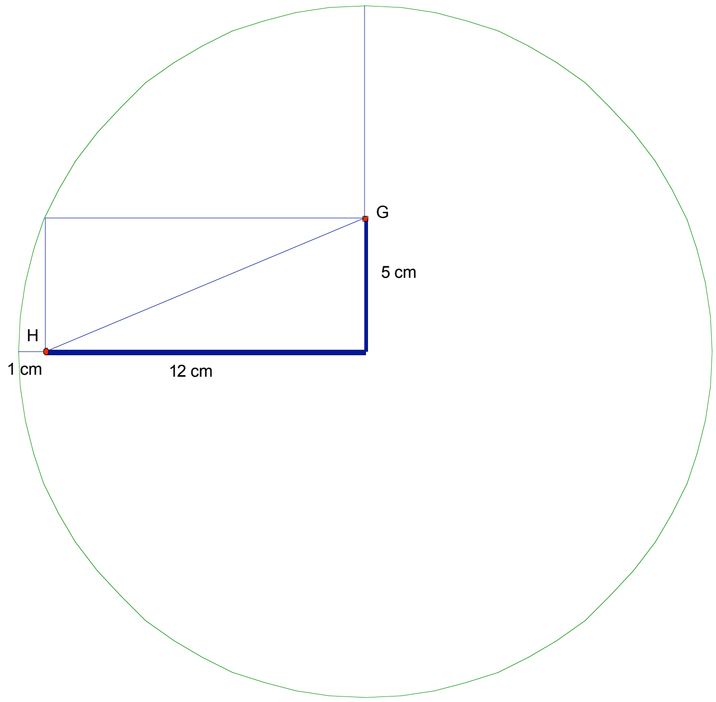
Problem One

Given the dimensions (in inches) shown in the illustration, what would be the length of the rectangle's diagonal that runs from corner A to corner B?




Explain your answer.

Now here is another circle that contains another rectangle. What would be the length of the rectangle's diagonal that runs from corner G to corner H?




Explain your answer.

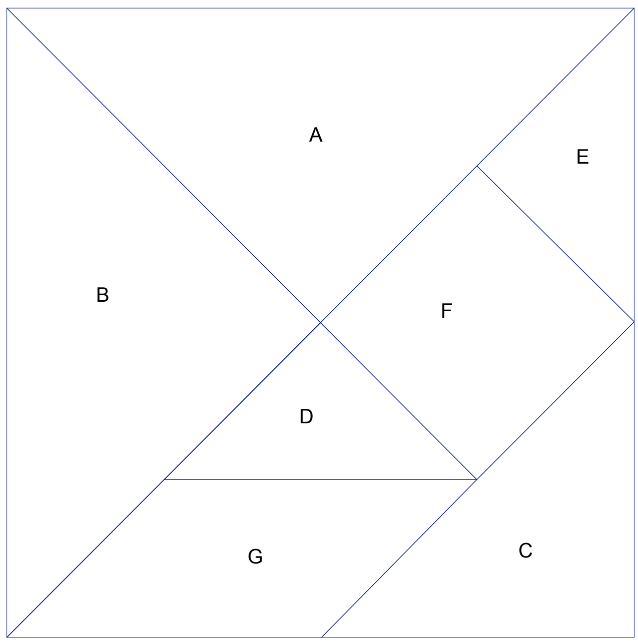
Do you see any patterns between the two problems?  How about the answers?

How could you obtain the answers in different ways?  Discuss this with another person or in your groups.

Problem Two

Consider the set of tangrams given below. Answer the following questions:
If the area of tangram piece B is 8 square units, what are the individual areas of the pieces A, C, D, E, F, and G?

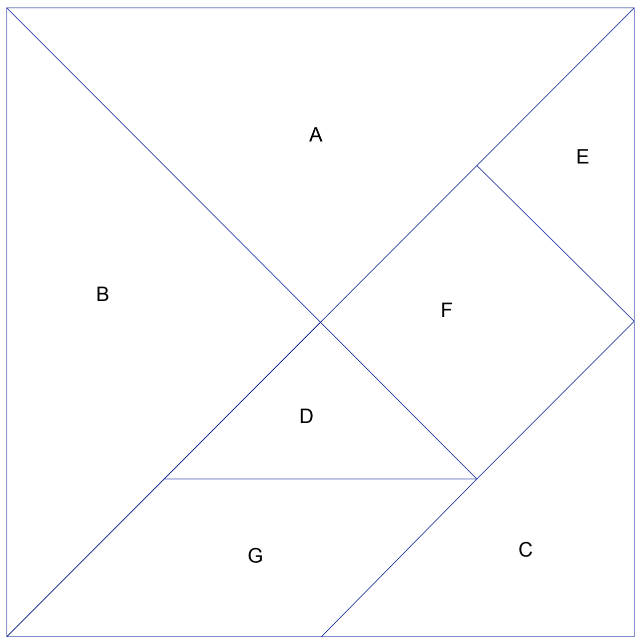
After finding the area of each of the pieces, identify the lengths of the sides of each of the polygons A, B, C, D, E, F, and G.




Discuss your answers with another person or in groups.

If the tangram piece F is 2 square units, what are the individual areas of pieces C, B, A, D, E, and G?

After finding the area of each of the pieces, identify the lengths of the sides of each of the polygons A, B, C, D, E, F, and G.




Discuss your answers with another person or in groups.

Problem Three

Use the Internet and/or the library to find information about the history of the Pythagorean Theorem and the Pythagoreans.

You need to report at least fifteen interesting pieces of information related to either the theorem or the Pythagoreans with a minimum of five facts for each. When you report your findings, be sure to write them in your own words and cite where you found each piece of information.

Lastly, write a short paragraph (minimum of five sentences) on what you found most interesting from your research and why.

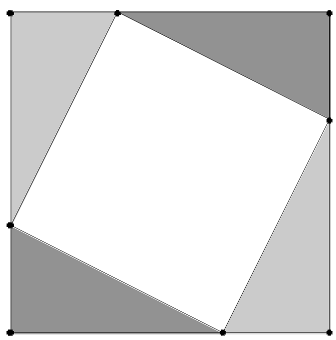
## **Investigation II**

This investigation focuses on the many available proofs of the Pythagorean Theorem. Below are three different proofs of the Theorem. Your group will be assigned one of the following three proofs to prove. Try and find as many solutions as you can.

Pythagorean Theorem Proof 1

Although the author of this proof is unknown, it is estimated that this proof came about near 200 B.C.!

Consider the following image constructed by inscribing one square inside another. The triangles surrounding the inner square are congruent. Use the image and this given information to prove the Pythagorean Theorem. Write your solution (and all your work) in the space provided.

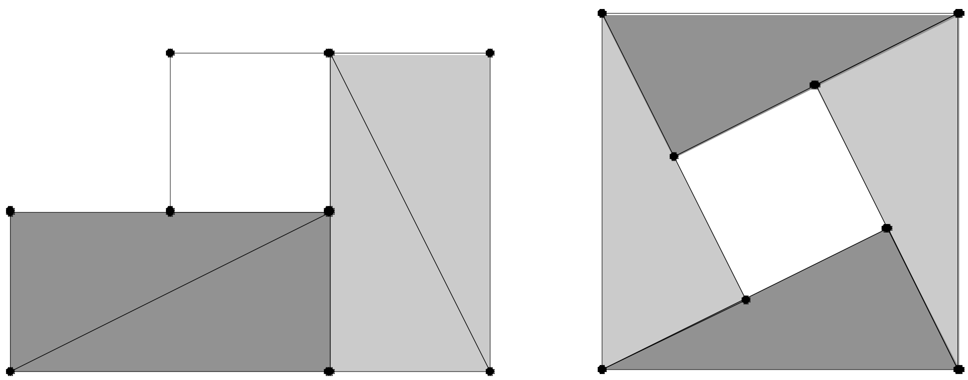


Can you think of other ways to use this image to prove the Theorem? Provide any additional solutions your group finds on the back of this page.

Pythagorean Theorem Proof 2

This proof is attributed to a 12th century mathematician by the name of Bh□skara. Bh□skara made significant contributions to numerous branches of mathematics (including calculus and algebra), in addition to the following proof of the Pythagorean theorem.

Consider the following image constructed using squares and congruent triangles. Use the image and this given information to prove the Pythagorean Theorem.

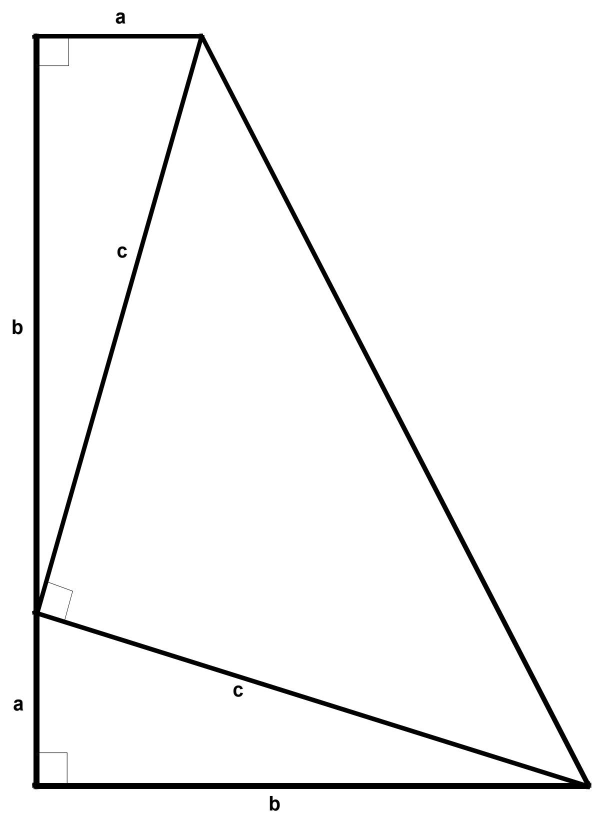


Can you think of other ways to use this image to prove the Theorem? Provide any additional solutions your group finds on the back of this page.

Pythagorean Theorem Proof 3

The following proof was completed by President James A. Garfield, the 20th President of the United States, around 1876.

Consider the following trapezoid divided up into various triangles. Use the image and this given information to prove the Pythagorean Theorem.




Can you find any other ways to use this image to prove the Theorem? Provide any additional solutions your group finds on this page.

**Investigation III**

Problem One

Desiree has 48 meters of wire fence to enclose a portion of her yard for her dog Rambo. Due to the vegetable garden that is already in her yard, the region she must enclose is in the shape of an isosceles triangle with an altitude of 12 meters. How many square meters of area will Rambo have to run once the triangle is enclosed by the entire 48 meters of fence?

What are the lengths of the sides of the triangle you formed?

Are these the only possible lengths for the sides? Why or why not?

Are any other areas for the enclosed region possible?  Why or why not?

Many times in life, it is advantageous to use whole numbers for calculations. Discuss this idea with another person. Write down two or three different times you discussed where using whole number values instead of decimal or irrational values is helpful.

One of the ways to answer the question dealing with the dimensions of the enclosed region for Rambo results from using whole number values for the sides of the isosceles triangle.

Specifically, let's look at the two right triangles formed by drawing the altitude in the isosceles triangle.

Sketch an isosceles triangle below that has an altitude of 12m, perimeter of 48m, and then has whole number values as the side lengths.

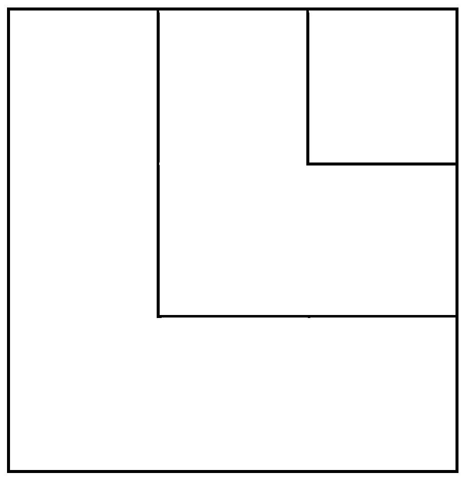
Each of the right triangles that are formed in this triangle has whole number values as the legs and hypotenuse lengths. In each triangle, these three numbers create **Pythagorean triples**. From the work you have done so far or by exploration, list at least three other Pythagorean triples below.

Problem Two

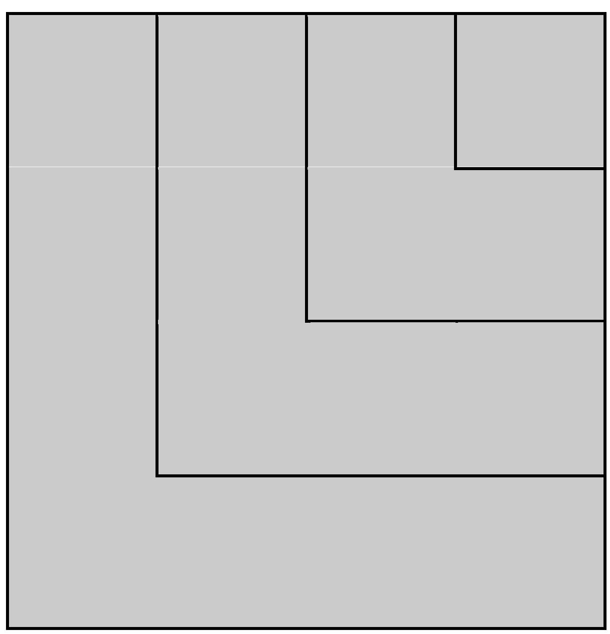
Recall the Pythagorean Theorem and think back to the proofs of the Pythagorean Theorem. One of the ways to prove the Pythagorean Theorem, included using squares with areas of $a^2$, $b^2$, and $c^2$ respectively.

You will now attempt to extend this idea.

Below is a 3 X 3 square decomposed into L-shaped regions.



Here is a 4 X 4 square decomposed into L-shaped regions.



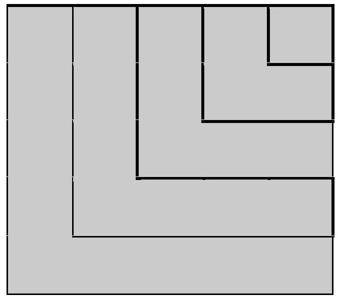
Rearrange the L-shaped regions of the 3 X 3 and 4 X 4 squares to create a new square. Draw your arrangement below.

Can you identify anything significant about the size of squares that you used and then composed?

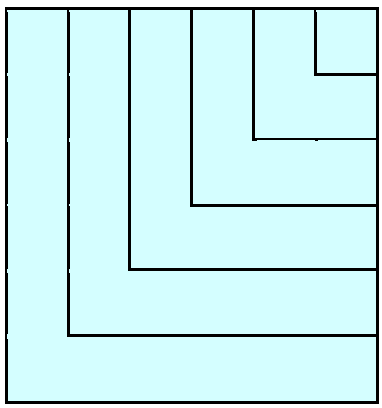
On the next couple of pages you have a 5 X 5, a 6 X 6, an 8 X 8, and a 12 X 12 square. Each is decomposed into L-shaped regions.

Is it possible to rearrange pairs of these squares to form a third square? If so, which squares and why? If not, why not?

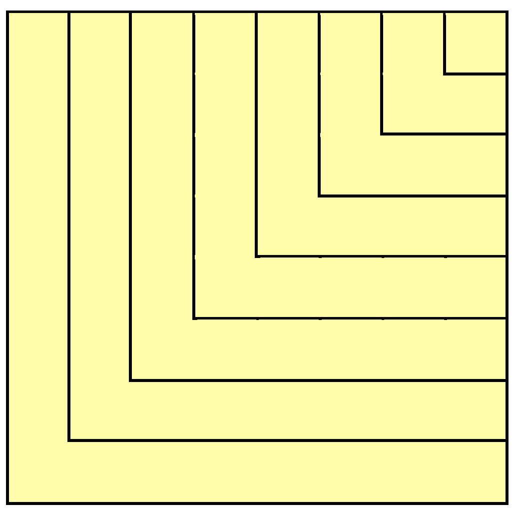
5 X 5 square decomposed into L-shaped regions.



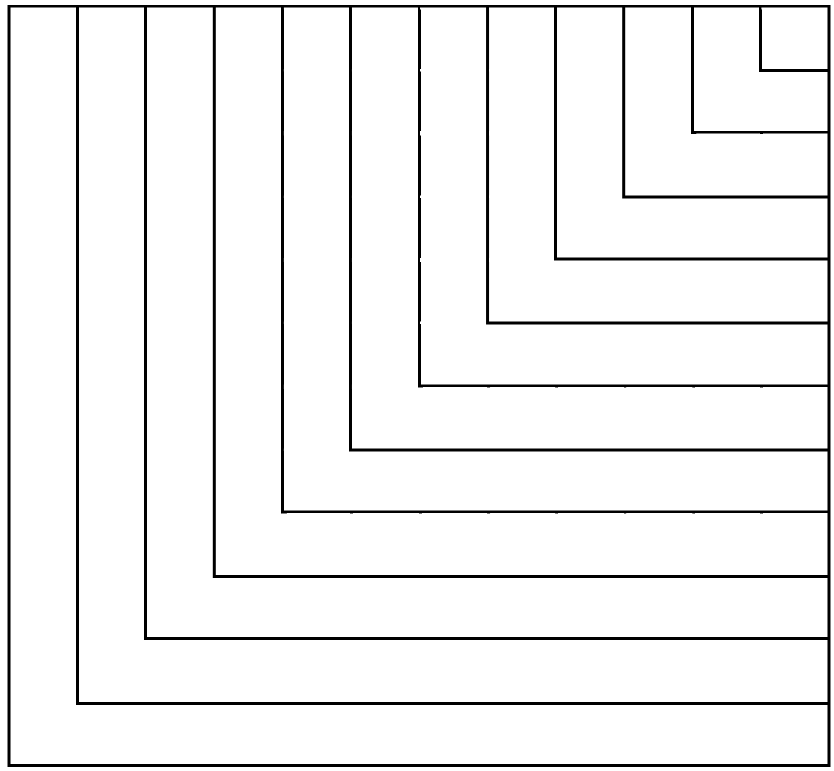
6 X 6 square decomposed into L-shaped regions.



8 X 8 square decomposed into L-shaped regions.



12 X 12 square decomposed into L-shaped regions.



Draw at least one of the squares formed by rearranging two of the given pairs of decomposed squares. Be sure your drawing clearly shows each L-shaped piece.

Problem Three

Now that you have explored a couple of different cases, what types of generalizations can you make about your findings regarding squares decomposed into L-shaped regions and/or Pythagorean Triples? List at least two.

What other questions could we explore that relate to these topics? List at least two.

Describe how you could investigate one of the related topics you identified.

## Investigation IV

Problem One

You are to generate up to twenty Pythagorean triples. Your group will be assigned one of the following four formulas to use to generate your triples.

1) $(2m, m^2 - 1, m^2 + 1)$ for $m > 1$
2) $(v^2 - u^2, 2uv, u^2 + v^2)$, where $v > u > 1$
3) $(2k + 1, (2k + 1)k + k, (2k + 1)k + k + 1)$ for $k \geq 1$
4) $(F_n F_{n+3}, 2F_{n+1} F_{n+2}, F^2_{n+1} + F^2_{n+2})$ where $F_n$ is the $n^{th}$ Fibonacci number

List the twenty triples below.

| ***Formula*** | ***#'s used to generate*** | ***a*** | ***b*** | ***c*** |
|---|---|---|---|---|
| 2 | 2, 5 | 21 | 20 | 29 |
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| ***Formula*** | ***#'s used to generate*** | ***a*** | ***b*** | ***c*** |
|---|---|---|---|---|
| 1 | 3 | 6 | 8 | 10 |
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Do you notice any patterns? Name at least two characteristics about the triples you have created.

Problem Two

First, let's compare the different triples that have been created using different formulas. Share your results with the class.

Take a few moments to write down what you believe each of the following terms mean. Then discuss in groups what the words mean.

Prime

Relatively Prime

Exhaustive

Parity

Primitive

Looking at the numbers used to generate the triples and the triples that have been created, identify how and where you these terms (prime, relatively prime, exhaustive, parity, primitive) are involved in the generation of the triples. Describe what you found for each formula below.

Formula:

1)

2)

3)

4)

Problem Three

Put twenty different **primitive** triples in the table below. Let *a* and *b* represent the lengths of the legs of a right triangle while *c* is the hypotenuse.

| ***a*** | ***b*** | ***c*** | **Even** | **Odd** | ***a x b*** |
|---|---|---|---|---|---|
| 3 | 4 | 5 | b | a,c | 12 |
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After placing twenty triples in the table given, identify which values (a, b, or c) are even or odd in each triple. Then find the product of the legs.

Problem Four

What generalization/s can you state about the parity of the Pythagorean triples? Identify at least one example of your generalization and explain what led you to this conclusion.

Do you believe your generalization is always true? Why or why not? How could you be certain of your generalization?

What generalization/s can you state about the product of *a* and *b*? Identify at least one example of your generalization and explain what led you to this conclusion.

Do you believe your generalization is always true? Why or why not? How could you be certain of your generalization?

Problem Five

Many other properties of primitive Pythagorean triples exist. Using the Pythagorean triples that have been generated up to this point or by generating more, explore different triples and look for patterns or characteristics that exist. You are to identify three more properties of Pythagorean triples and explain your thought process.

| **Property** | **Explanation** |
|---|---|
| | |
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| | |

## Investigation V

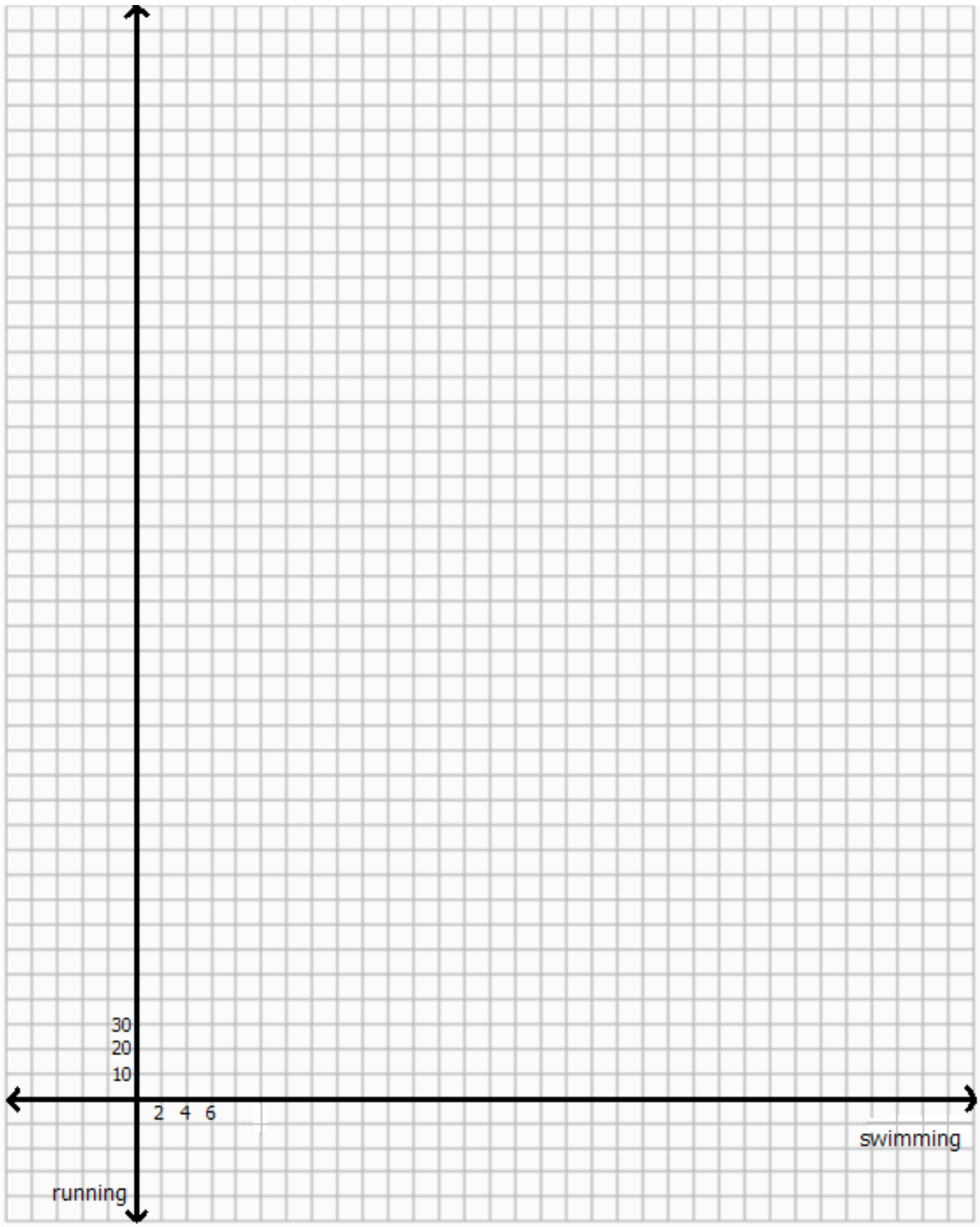
### Problem 1

People in Triangleville are preparing for the annual Triangular Triathlon. Just like a regular triathlon, participants will swim the first part of the race, bike the second, and run the third. Unlike a regular triathlon, the course is set up in the shape of a right triangle so the competitors will begin the race where they finish. The only stipulation for the race is that the swimming distance is less than the running distance, which is less than the biking distance. What are the possible distances of each part of the race? Come up with at least ten possibilities on your own, then ten different ones with a partner. Be sure to express your solutions in a way that is easy to understand, and include the method(s) you used to find them.

Are these distances realistic? Why or why not?

What other ways could you represent your results?

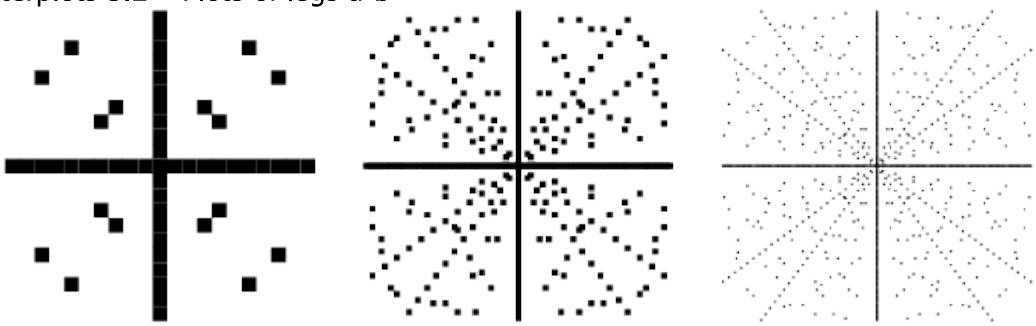
Another way to represent this data is by using a scatter plot. Using the graphing lines, create a coordinate plane and plot pairs of your solutions. In order aid future discussion, plot the swimming and running. Let the swimming distance be represented by the x-axis, and let the running distance represent the y-axis. Plot the points you and your partner came up with on the graph below, and on the transparency given to you by your teacher. Graph 5.1



First, copy any points onto your graph that the other groups had that are different from your group's. Looking at the overhead transparencies, what do you notice about the points? Are there any patterns you see? How does your graph compare to the transparencies? Write down a minimum of five different comparisons and/or patterns you notice.

Now, compare your graph and the transparency to the Scatterplots 5.2 below. What differences do you notice? What are the similarities? Do you see any patterns emerging? Be specific and record at least three findings for each heading below. Be sure to continue taking notes during the discussion.

Scatterplots 5.2 – Plots of legs *a-b*



**<u>Notes:</u>**
**Similarities**

**Differences**

**Patterns/Other Findings**

Now think about this question:  Can you predict any more numbers based on what you already know about these graphs?  How?  Write down any ideas you have.

One way we can do this is to *generalize* a pattern.  In other words, find a way to express this pattern so that the expression will always give us numbers or points in the pattern.  Pick at least one pattern you found from our transparency, and at least one from a pattern you found from 5.2 and generalize them.

**Pattern 1**

**Pattern 2**

Look at your generalizations. If they are correct, they should give you points that are numbers in a Pythagorean triple. Check your generalizations below by finding at least 10 points with your generalizations.

Can you find the third number to make the Pythagorean triple complete? List them below.

## Investigation VI

Now that you have spent some time with the Pythagorean Theorem, proofs of the Pythagorean Theorem, Pythagorean triples, and properties of triples or graphs of triples, what other types of explorations could you investigate?

Can you think of any ideas that are similar or related to the Pythagorean theorem that deal with algebra, geometry, or general number sense that you could explore? If so, list some of your ideas.

Here are a couple of ideas.

- Are there values for which $a^2 + b^2 + c^2 = d^2$ ?
  Try different values for the variables.
  Show the work that you have tried.

  What types of conclusions can you make about the idea of a Pythagorean quadruple (a, b, c, d) for which $a^2 + b^2 + c^2 = d^2$?

- Are there values for which $a^n + b^n = c^n$ ?
  Try different values for the variables.
  Show the work that you have tried.

Why types of conclusions can you make based upon your calculations?

Now try to extend these ideas or another idea from the module. Think of a question to explore that relates to one of the prior topics. Identify the question that you are asking and then in small groups work to try to identify some conclusions about the question you asked.