

Pythagorean Triples

Teacher's Guide

Introduction

This module is devoted to promoting student discovery and understanding of Pythagorean triples. The module is divided into separate investigations; some of these investigations may be used in less than one class period, whereas other investigations may take an entire class period or more. The length of each investigation is dependent on the amount of class time a teacher wants to devote to the module in addition to the level of students. One of the main goals is to provide as much flexibility for the teacher as possible.

Notes for the teacher including possible solutions, discussion topics, and questions for promoting student understanding and discovery are incorporated in *italics* for easy referencing by the teacher. Before each investigation you will find a list of the objectives for the investigation.

Module Outline

- I. Investigation I provides an introduction to the Pythagorean Theorem through a student-led discovery problem. The problem and its solutions are also used to discuss problem-solving strategies and methods that your students will find useful through-out the rest of the module. Additionally, an opportunity to explore the history of the Pythagoreans and Pythagorean Theorem is provided.
- II. Investigation II focuses on the proof and reasoning behind the Pythagorean Theorem. This investigation explores numerous proofs of the theorem; each proof has multiple solutions and representations.
- III. In Investigation III, an application of the Pythagorean Theorem that generates Pythagorean triples is given. There are also visual puzzles using squares with dimensions from Pythagorean triples that decomposed in to L-shaped regions.
- IV. Investigation IV centers around the generation of different Pythagorean triples. In groups the students will use different formulas to generate the triples, inspect and prove why the various formulas generate triples, and then also look at other relationships that exist for the values of a Pythagorean triple.
- V. Investigation V contains an application problem to direct the focus of the investigation to scatter plots with values of the triples, lines of best fit in the scatter plots, symmetry that may exist, and other methods for predicting future values or plots.

- VI. Investigation VI provides an opportunity for the students to ask “what if” by extending the idea of Pythagorean Triples to other possible areas. A few exploration topics are provided, but students are encouraged to pose questions to explore on their own.

Investigation I

Teacher's Notes –

Objectives:

Upon completion of this investigation, students will be able to

- *Explain the Pythagorean Theorem*
- *Apply the Pythagorean Theorem to various right triangles*
- *Describe a brief history of the Pythagorean Theorem and/or the Pythagoreans*

This investigation includes two problems to introduce/re-introduce the Pythagorean Theorem. Depending on the level of students and their background, more information may be added throughout or parts of this may be deleted from the module altogether.

The first problem may lead into a discussion of the Pythagorean Theorem, but begins in a little different manner than many problems. The students are first given a problem with a circle containing a rectangle and asked to find the diagonal length. This information can be found without using the Pythagorean Theorem. Yet, this provides an opportunity to bring the theorem into the discussion.

Secondly, there are two problems which have been adapted from materials by Agida Manizade. The students can use tangrams to solve the problems then again another discussion of the Pythagorean Theorem may follow.

This investigation may be done in a class period without loss of continuity or generality, but if a class is not familiar with the theorem more information and practice problems with the theorem should be added.

Lastly, there are some research questions which the students can investigate on the Internet or in the library to explore the history of the Pythagorean Theorem and the Pythagoreans.

Classroom Set-up:

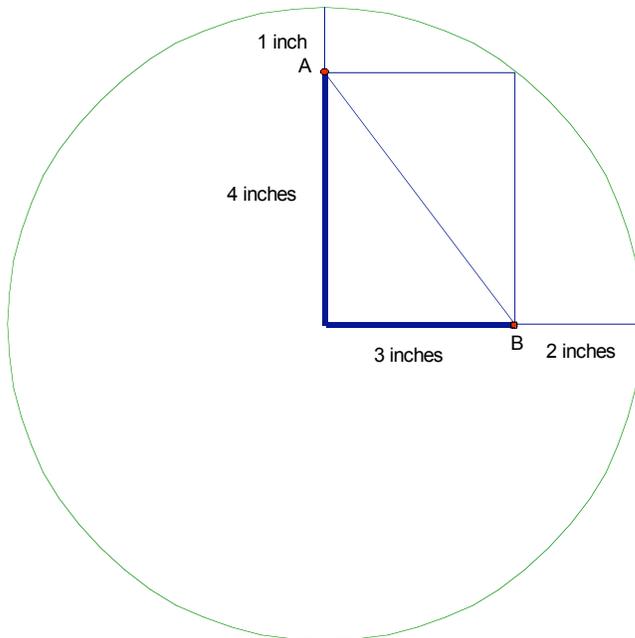
Problems One and Two may be solved individually or in groups. Problem Two does not require the use of tangrams, but the students may benefit from having them at their disposal to aid in the solution process in number two. A template has been added to Appendix A that may be used for an overhead or to cut out the pieces.

Problem Three can be completed outside of class and may be quickly accomplished via the Internet where there is a huge resource of information about the Pythagorean Theorem and the Pythagoreans. Be sure to clarify how the students are expected to represent the information that they find in response to the questions asked.

Investigation I

Problem One

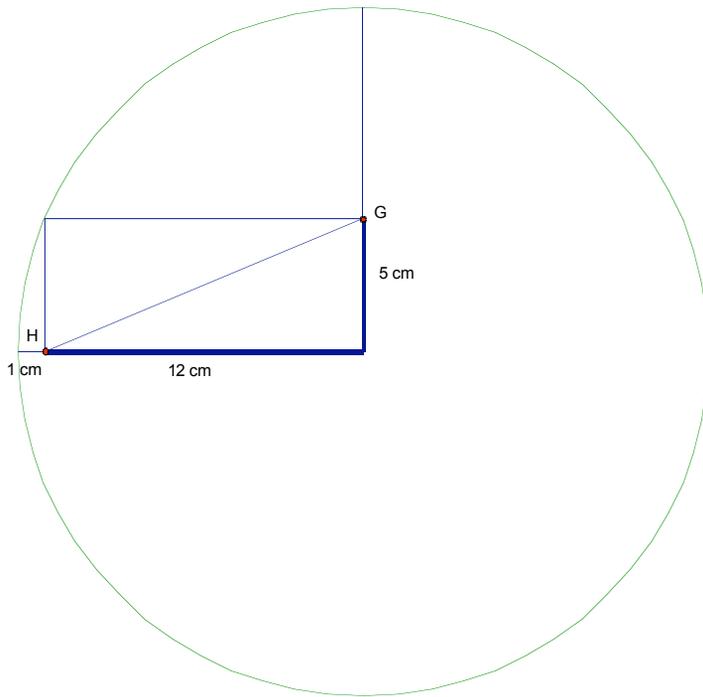
Given the dimensions (in inches) shown in the illustration, what would be the length of the rectangle's diagonal that runs from corner A to corner B?



Explain your answer.

The answer should be that the diagonal is 5 inches. The students may reach this by using the Pythagorean Theorem or from their knowledge of circles and rectangles. By drawing in the other diagonal, which would have to be the radius of the circle, one can see the length is 5 inches. Be aware some students may simply guess five without making either connection.

Now here is another circle that contains another rectangle. What would be the length of the rectangle's diagonal that runs from corner G to corner H?



Explain your answer.

If the students assume that the two sides intersect at the center of the circle, then they should reason that the diagonal is 13 centimeters. Again, this could follow from the use of the Pythagorean Theorem or the knowledge of circles and rectangles.

Do you see any patterns between the two problems? How about the answers?

Here the students may comment on the circle properties with the radii or make comments about the lengths of the diagonals within rectangles having to be congruent. Or some students may comment on the Pythagorean Theorem with or without reference to its name.

How could you obtain the answers in different ways? Discuss this with another person or in your groups.

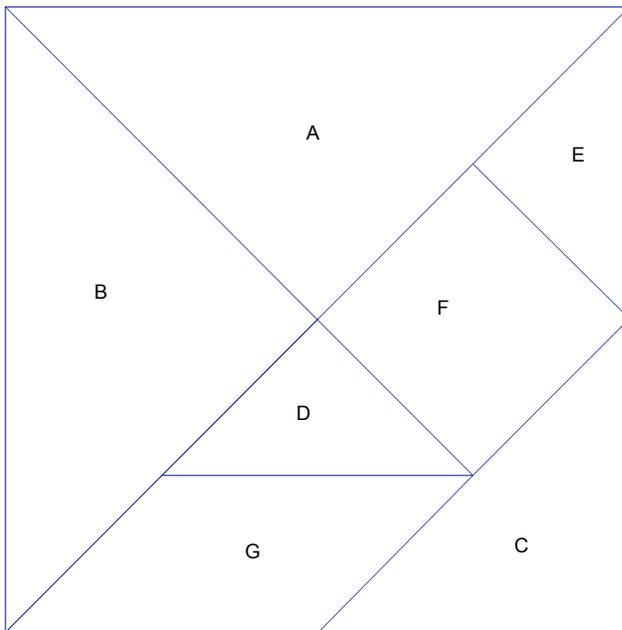
*After the students have had time to discuss this, as a class discuss:
How the answer may be found with the circle and rectangle and
How the answer may be found using the Pythagorean Theorem
Perhaps give more examples of right triangles given two lengths and ask the students to find a missing side*

Problem Two

Consider the set of tangrams given below. Answer the following questions:
If the area of tangram piece B is 8 square units, what are the individual areas of the pieces A, C, D, E, F, and G?

Note: Students that haven't had much work with square roots may need some help throughout this problem to find the side lengths. One way to do this is to discuss the area of the big square and then what its side lengths may be and/or to focus on solutions with the Pythagorean Theorem.

After finding the area of each of the pieces, identify the lengths of the sides of each of the polygons A, B, C, D, E, F, and G.



Answers:

For Areas:

A 8 square units

B 8 square units

C 4 square units

D 2 square units

E 2 square units

F 4 square units

G 4 square units

For Sides:

4, 4, $4\sqrt{2}$

4, 4, $4\sqrt{2}$

$2\sqrt{2}$, $2\sqrt{2}$, 4

2, 2, $2\sqrt{2}$

2, 2, $2\sqrt{2}$

2, 2, 2, 2

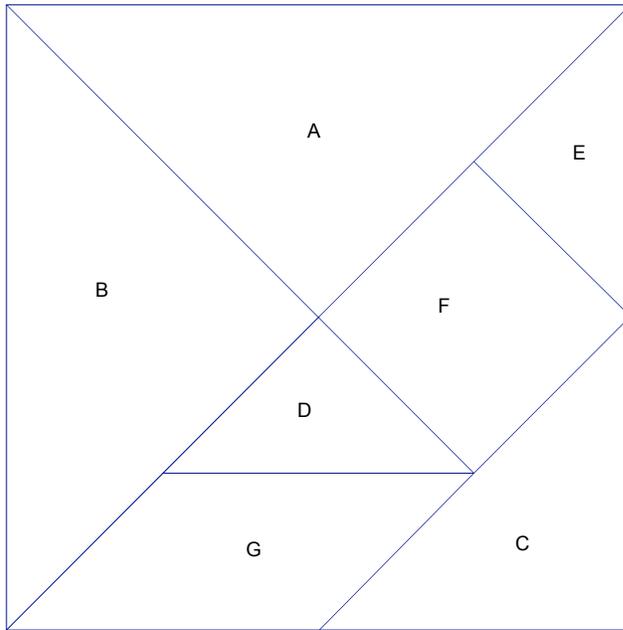
2, $2\sqrt{2}$, 2, $2\sqrt{2}$

Discuss your answers with another person or in groups.

After the students have had an opportunity to discuss their answers, be sure to have a class discussion that focuses on how the areas can be found and how the side lengths can be found. Relate how the lengths can be found using the Pythagorean Theorem, but remember to allow multiple representations or solution paths.

If the tangram piece F is 2 square units, what are the individual areas of pieces C, B, A, D, E, and G?

After finding the area of each of the pieces, identify the lengths of the sides of each of the polygons A, B, C, D, E, F, and G.



Answers:

For Areas:

A 4 square units

B 4 square units

C 2 square units

D 1 square units

E 1 square units

F 2 square units

G 2 square units

For Sides:

$2\sqrt{2}, 2\sqrt{2}, 4$

$2\sqrt{2}, 2\sqrt{2}, 4$

$2, 2, 2\sqrt{2}$

$\sqrt{2}, \sqrt{2}, 2$

$\sqrt{2}, \sqrt{2}, 2$

$\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}$

$2, \sqrt{2}, 2, \sqrt{2}$

Discuss your answers with another person or in groups.

Problem Three

Use the Internet and/or the library to find information about the history of the Pythagorean Theorem and the Pythagoreans.

You need to report at least fifteen interesting pieces of information related to either the theorem or the Pythagoreans with a minimum of five facts for each. When you report your findings, be sure to write them in your own words and cite where you found each piece of information.

Lastly, write a short paragraph (minimum of five sentences) on what you found most interesting from your research and why.

Investigation II

Objectives:

Upon completion of this investigation, students will be able to

- *Explain a proof of the Pythagorean Theorem*
- *Identify a historical connection of a proof of the Pythagorean Theorem*
- *Recognize that a variety of proofs of the Pythagorean Theorem exist*

This investigation focuses on exploring different proofs of the Pythagorean Theorem. Depending on the level the class, the entire investigation or only one proof may be used or this investigation may be omitted from the module without loss of continuity.

Although the Pythagorean Theorem was attributed to the mathematician by the name of Pythagoras, the proofs under examination today are not Pythagoras' proof of the theorem. The activity contains three different proofs of the theorem spanning from approximately 200 B.C. to the 1876. Each page contains a brief description of the background of the different proofs.

Students should try to prove the Pythagorean Theorem using the given images, some geometry and/or algebra, and their own ideas and creations. Possible solutions to the proofs have been included. Keep in mind that the possible solutions provided herein are not exhaustive, and students may find other solutions. Some students, with minimal experience with proofs, may have a difficulty finding a starting point for their proof. Suggested "hints" have been included which can be given struggling students, without stating the solution.

Classroom set-up:

For this investigation, assign each group (recommended size of 3 or 4 students) one of the given proofs to investigate. Each group should finish their proof and put their solution process on the board/overhead to share with the rest of the class. Have one student from each group explain their group's solution method and any strategies used in the process of trying to solve the proof.

As students are sharing their results, this is an excellent time to incorporate a class discussion on problem-solving/proof strategies that the students used to complete their proofs.

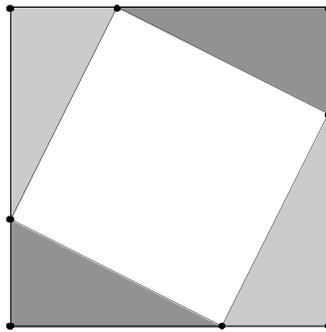
Investigation II

This investigation focuses on the many available proofs of the Pythagorean Theorem. Below are three different proofs of the Theorem. Your group will be assigned one of the following three proofs to prove. Try and find as many solutions as you can.

Pythagorean Theorem Proof 1

Although the author of this proof is unknown, it is estimated that this proof came about near 200 B.C.!

Consider the following image constructed by inscribing one square inside another. The triangles surrounding the inner square are congruent. Use the image and this given information to prove the Pythagorean Theorem. Write your solution (and all your work) in the space provided.



Hints:

- 1) Have the students assign variables as lengths for the sides of the triangles / squares.
- 2) Have students try using actual numbers as a small example.

Possible Solution:

$$\text{Area of large square} = (a + b)^2$$

$$\text{Sum of the Triangles and Inner Square} = 4 \times \frac{1}{2} (ab) + c^2$$

$$\text{Now, set the two equations equal: } (a + b)^2 = 4 \times \frac{1}{2} (ab) + c^2$$

$$\text{Simplify: } a^2 + 2ab + b^2 = 2ab + c^2$$

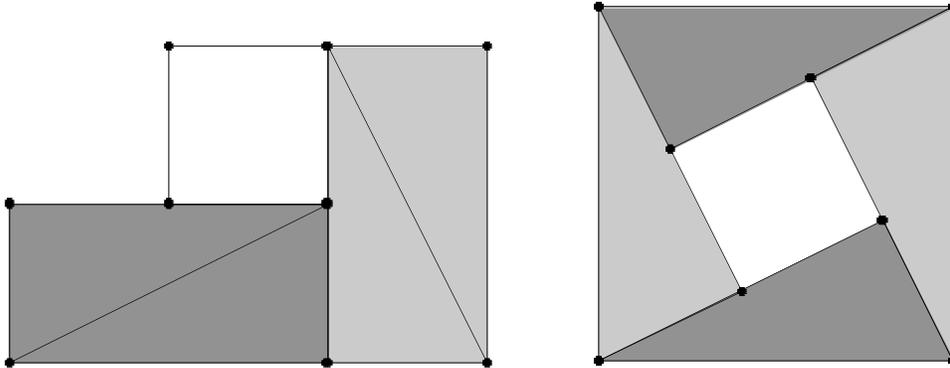
$$a^2 + b^2 = c^2$$

Can you think of other ways to use this image to prove the Theorem? Provide any additional solutions your group finds on the back of this page.

Pythagorean Theorem Proof 2

This proof is attributed to a 12th century mathematician by the name of Bhāskara. Bhāskara made significant contributions to numerous branches of mathematics (including calculus and algebra), in addition to the following proof of the Pythagorean theorem.

Consider the following image constructed using squares and congruent triangles. Use the image and this given information to prove the Pythagorean Theorem.



Hints:

- 1) Have the students assign variables as lengths for the sides of the triangles / squares.
- 2) Have students try using actual numbers as a small example.

Possible Solution:

From the image on the right, let the side of the square be length z , note this is the length of the diagonal of the rectangles in the image on the left. Also from the image on the left, let the side of the inner square be length x .

From the image on the right, we can deduce that the two rectangles have dimensions x y . By adding the bold blue line shown above, we produce two squares – one of area x^2 and one of area y^2 .

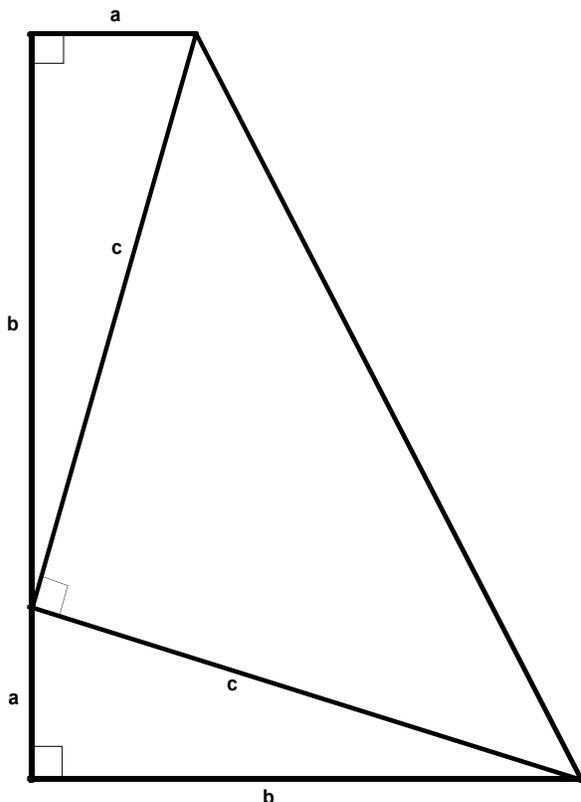
However, by rearranging these shapes as shown in the figure on the left – we see the sum of the areas also equals z^2 .

Can you think of other ways to use this image to prove the Theorem? Provide any additional solutions your group finds on the back of this page.

Pythagorean Theorem Proof 3

The following proof was completed by President James A. Garfield, the 20th President of the United States, around 1876.

Consider the following trapezoid divided up into various triangles. Use the image and this given information to prove the Pythagorean Theorem.



Hints:

- 1) *Have students try using actual numbers as a small example.*
- 2) *Ask students to identify the shapes within the figure; it may not be obvious to students that the three triangles also form a trapezoid.*
- 3) *Ask students to look at the relationships between the areas of the figures.*

Possible Solution:

$$\text{Area of the Trapezoid} = \frac{1}{2} (a+b)^2$$

$$\text{Sum of the Area of the Inner Triangles} = 2 \times \frac{1}{2} (ab) + \frac{1}{2} c^2$$

Now, set the two areas equal:

$$\frac{1}{2} (a+b)^2 = 2 \times \frac{1}{2} (ab) + \frac{1}{2} c^2$$

Simplify:

$$\frac{1}{2} a^2 + \frac{1}{2} \times 2ab + \frac{1}{2} b^2 = ab + \frac{1}{2} c^2$$

$$\frac{1}{2} a^2 + ab + \frac{1}{2} b^2 = ab + \frac{1}{2} c^2$$

$$\frac{1}{2} a^2 + \frac{1}{2} b^2 = \frac{1}{2} c^2$$

$$a^2 + b^2 = c^2$$

Can you find any other ways to use this image to prove the Theorem? Provide any additional solutions your group finds on this page.

The following are possible discussion questions:

1. *What are some strategies/tactics you used to develop your proof?*

The following are some possible problem solving/proof strategies:

- *Compute or Simplify*
- *Use a Formula*
- *Make a Model or Diagram*
- *Make a Table, Chart or List*
- *Guess, Check & Revise*
- *Consider a Simpler Case*
- *Eliminate*
- *Look for Patterns*

2. *What are the important components of a proof?*

Be sure to discuss the various types of proofs, the necessity for clarity, and the importance of a well-developed argument.

Investigation III

Objectives:

Upon completion of this investigation, students will be able to

- *Identify Pythagorean triples*
- *Make connections between different squares with side lengths from Pythagorean triples*
- *Extend the idea of areas of squares with side lengths given as values of a Pythagorean triple to another realm or geometric idea*

This investigation explores Pythagorean triples. Some or all of the investigation may be incorporated into the module. But in order to successfully complete the remaining investigations in the module, students will need an understanding of what a Pythagorean triple is.

Classroom set-up:

Students may begin the investigation individually or in groups. For the puzzle component of the investigation, the students should first attempt to create the squares individually before they compare solutions.

Investigation III

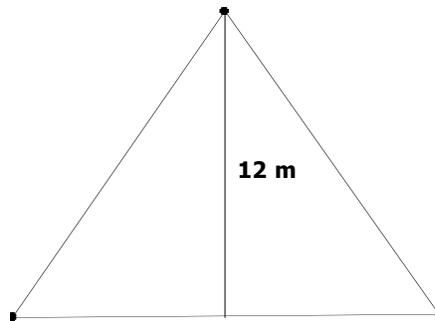
Problem One

Desiree has 48 meters of wire fence to enclose a portion of her yard for her dog Rambo. Due to the vegetable garden that is already in her yard, the region she must enclose is in the shape of an isosceles triangle with an altitude of 12 meters. How many square meters of area will Rambo have to run once the triangle is enclosed by the entire 48 meters of fence?

For this first problem, be sure to look at the students' different pictorial representations of the problem as they are working through. Most students will begin by putting the altitude of 12 meters perpendicular to the base of the isosceles triangle (see below).

As the students are asked at the end of the first page, be sure to ask about whether other lengths are possible and what types of other triangles can be made to fit some if not all of these specifications.

Possible Visual Representation:



Possible solution:

Side lengths: 15,15,18

$$\begin{aligned} \text{Triangle Area} &= \frac{1}{2} (18 \times 12) \\ &= 108 \text{ m}^2 \end{aligned}$$

What are the lengths of the sides of the triangle you formed?

Possible solution: 15, 15, 18

Are these the only possible lengths for the sides? Why or why not?

*No, two other possible solutions the students may identify by rounding would be:
17.55m, 17.55m, 12.91m (Note: The altitude is perpendicular to one of the congruent sides.)*

21.84m, 13.08m, 13.08m (Note: The altitude is perpendicular to one of the congruent sides and the triangle is obtuse.)

Are any other areas for the enclosed region possible? Why or why not?

The other areas corresponding to the two other triangles provided in the last question would be:

$$105.3 \text{ m}^2$$

$$78.48 \text{ m}^2$$

Many times in life, it is advantageous to use whole numbers for calculations. Discuss this idea with another person. Write down two or three different times you discussed where using whole number values instead of decimal or irrational values is helpful.

One of the ways to answer the question dealing with the dimensions of the enclosed region for Rambo results from using whole number values for the sides of the isosceles triangle.

Specifically, let's look at the two right triangles formed by drawing the altitude in the isosceles triangle.

Sketch an isosceles triangle below that has an altitude of 12m, perimeter of 48m, and then has whole number values as the side lengths.

Each of the right triangles that are formed in this triangle has whole number values as the legs and hypotenuse lengths. In each triangle, these three numbers create **Pythagorean triples**. From the work you have done so far or by exploration, list at least three other Pythagorean triples below.

Pythagorean triples is not defined here, and you may want to be sure the class agrees on the important components of Pythagorean triples before moving on to Problem Two.

Problem Two

This problem really draws the students in to a hands-on activity. Emphasize that the squares the students should make from the L-shaped regions must be solid squares, with no holes in the middle.

Templates for these squares have been included (see Appendix B). You may wish to copy and cut these in advance or have the students cut them apart before they construct their new squares.

Begin with the 3-by-3 and 4-by-4 squares. Before moving on to the other squares, discuss what type of square was formed and how this relates to the early proofs that have been studied and the Pythagorean triples.

Then discuss the other squares given and which ones they may want to choose to put together and why.

After the students have completed a second puzzle, discuss the strategies that the students used and whether unique solutions existed.

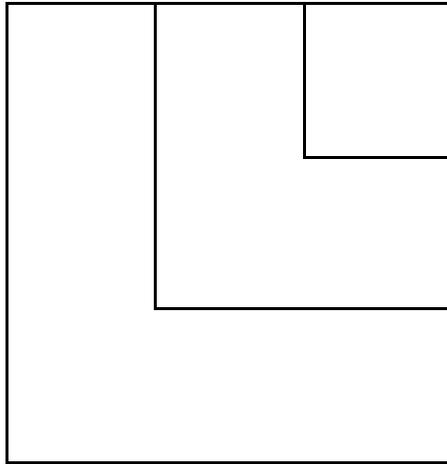
If you would like the students to continue with this idea or complete a challenge, some graph paper has been included on which they could draw another square made up of L-shaped decompositions from two other squares.

Problem Two

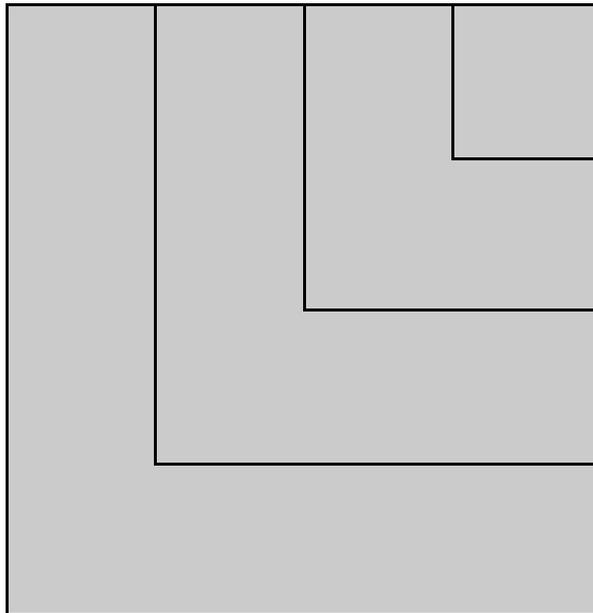
Recall the Pythagorean Theorem and think back to the proofs of the Pythagorean Theorem. One of the ways to prove the Pythagorean Theorem, included using squares with areas of a^2 , b^2 , and c^2 respectively.

You will now attempt to extend this idea.

Below is a 3 X 3 square decomposed into L-shaped regions.

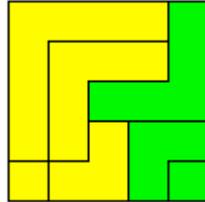


Here is a 4 X 4 square decomposed into L-shaped regions.



Rearrange the L-shaped regions of the 3 X 3 and 4 X 4 squares to create a new square. Draw your arrangement below.

A possible solution is shown below.

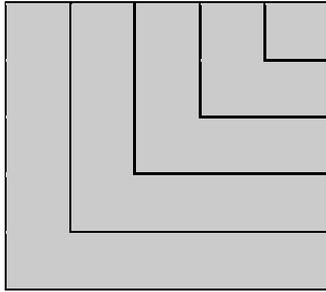


Can you identify anything significant about the size of squares that you used and then composed?

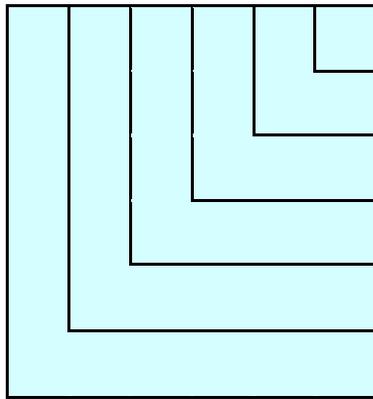
On the next couple of pages you have a 5 X 5, a 6 X 6, an 8 X 8, and a 12 X 12 square. Each is decomposed into L-shaped regions.

Is it possible to rearrange pairs of these squares to form a third square? If so, which squares and why? If not, why not?

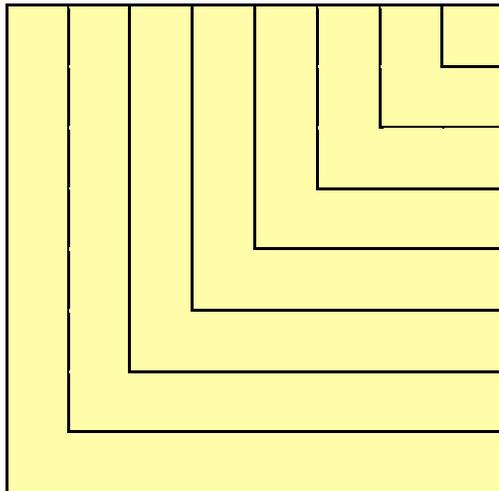
5 X 5 square decomposed into L-shaped regions.



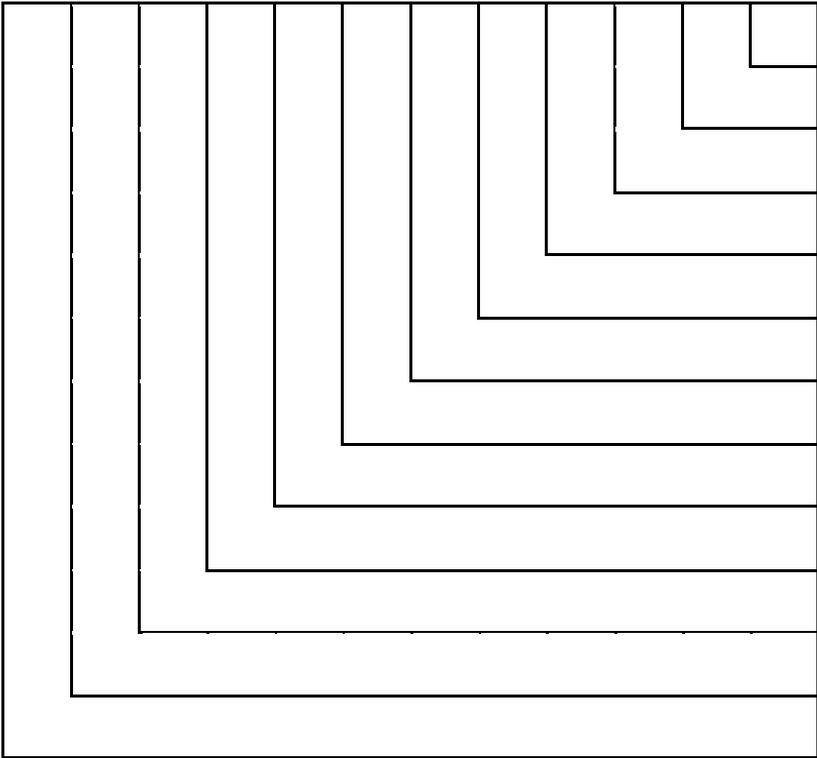
6 X 6 square decomposed into L-shaped regions.



8 X 8 square decomposed into L-shaped regions.

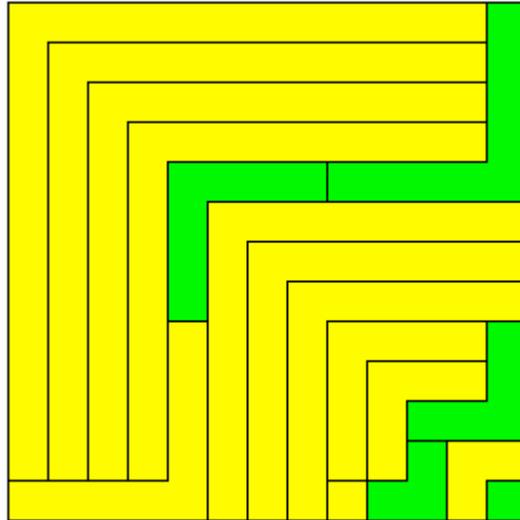


12 X 12 square decomposed into L-shaped regions.



Draw at least one of the squares formed by rearranging two of the given pairs of decomposed squares. Be sure your drawing clearly shows each L-shaped piece.

A possible solution to the 5-12-13 triple is given below.



Problem Three

This last problem is to help the students think "What if..." They should try to think of other types of ideas they could explore that are similar in nature to the L-shaped decompositions or relating to Pythagorean triples.

Share the ideas for exploration with the entire class. These topics may be areas of investigation to assign as an ongoing project.

Problem Three

Now that you have explored a couple of different cases, what types of generalizations can you make about your findings regarding squares decomposed into L-shaped regions and/or Pythagorean Triples? List at least two.

- 1. For any two L-shaped decompositions (that are part of a Pythagorean triple) and can be reassembled into a single larger square, there are multiple ways to reassemble the shapes into the larger square.*
- 2. It seems as though for any two L-decompositions (that are part of a Pythagorean triple) the two decompositions can be reassembled into a single larger square.*

What other questions could we explore that relate to these topics? List at least two.

- 1. How many solutions exist for each puzzle? Is there a maximum/minimum number of solutions for each puzzle?*
- 2. What other types of shapes can one explore with these side lengths?*

Describe how you could investigate one of the related topics you identified.

Investigation IV

Objectives:

Upon completion of this investigation, students will be able to

- *Generate at least twenty Pythagorean triples*
- *Justify why at least one method of generating triples is valid*
- *Identify primitive Pythagorean triples*
- *Determine whether a method of generating Pythagorean triples results in primitive triples and if the method is exhaustive*
- *Identify various relationships amongst the values of each Pythagorean triple*

This investigation focuses on the properties of Pythagorean triples. But before these properties can be explored, the students must first generate sets of triples. Four different formulas for generating triples are given and may be used to explore the ideas. The students will also be asked to discuss why the different formulas work, if the methods are exhaustive, and whether the methods produce primitive triples.

Classroom Setup:

For the first problem, set up groups and assign each group one of the formulas to use so that the entire class will have examples from each during the discussion.

Investigation IV

Problem One

You are to generate up to twenty Pythagorean triples. Your group will be assigned one of the following four formulas to use to generate your triples.

- 1) $(2m, m^2 - 1, m^2 + 1)$ for $m > 1$
- 2) $(v^2 - u^2, 2uv, u^2 + v^2)$, where $v > u > 1$
- 3) $(2k + 1, (2k + 1)k + k, (2k + 1)k + k + 1)$ for $k \geq 1$
- 4) $(F_n F_{n+3}, 2F_{n+1} F_{n+2}, F_{n+1}^2 + F_{n+2}^2)$ where F_n is the n^{th} Fibonacci number

List the twenty triples below.

<i>Formula</i>	<i>#'s used to generate</i>	<i>a</i>	<i>b</i>	<i>c</i>
2	2, 5	21	20	29

<i>Formula</i>	<i>#'s used to generate</i>	<i>a</i>	<i>b</i>	<i>c</i>
1	3	6	8	10

Do you notice any patterns? Name at least two characteristics about the triples you have created.

Problem Two

As a class, have the students record their results for the different formulas. At this point, it may be helpful to have poster boards or large sheets of paper for each group to record and display. That way all groups will have the information later in the investigation.

After the triples have been shared, have the students quietly write their description of the five terms listed: prime, relatively prime, exhaustive, parity, and primitive. Then have the students discuss these in groups. Following the group discussions, talk about these terms as a class and be sure to identify the important characteristics for each term, and you may wish to give examples.

Next have the students work on identifying how these terms relate to the Pythagorean triples and/or to the values used to generate each triple.

You may wish to conduct a class discussion after the individuals or groups have completed problem two. Just be cautious not to give too much information away before moving on to Problem Three.

Problem Two

First, let's compare the different triples that have been created using different formulas. Share your results with the class.

Take a few moments to write down what you believe each of the following terms mean. Then discuss in groups what the words mean.

Prime

A prime number is a counting number which has only two distinct factors 1 and the prime number itself.

Relatively Prime

Two integers are relatively prime if they have no two divisors in common, except for 1.

Exhaustive

Refers to the fact that all options or events have been studied or explored.

Parity

The parity of an integer indicates whether the number is even or odd.

Primitive

In terms of Pythagorean triples, each pair in the primitive triples has a GCF of 1.

Either individually or in groups, have the students complete the questions in Problem Four. A discussion of their findings would be very beneficial before moving onto Problem Five.

Problem Four

What generalization/s can you state about the parity of the Pythagorean triples? Identify at least one example of your generalization and explain what led you to this conclusion.

Do you believe your generalization is always true? Why or why not? How could you be certain of your generalization?

What generalization/s can you state about the product of a and b ? Identify at least one example of your generalization and explain what led you to this conclusion.

Do you believe your generalization is always true? Why or why not? How could you be certain of your generalization?

Problem Five:

This last problem asks the students to look for patterns in the triples and to explain their findings. Monitoring of this problem is essential and students may need some hints as they are working through the problem. Additionally, many of these could be great exploratory topics for continued work or for the students to prove.

Below is a list of properties obtained from Wikipedia
http://en.wikipedia.org/wiki/Pythagorean_triple

- *Exactly one of a , b is odd; c is odd.*
- *The area ($A = ab/2$) is an integer.*
- *Exactly one of a , b is divisible by 3.*
- *Exactly one of a , b is divisible by 4.*
- *Exactly one of a , b , c is divisible by 5.*
- *Exactly one of a , b , $(a + b)$, $(b - a)$ is divisible by 7.*
- *At most one of a , b is a square.*
- *Every integer greater than 2 that is not congruent to 2 mod 4 is part of a primitive Pythagorean triple. Examples of integers not part of a primitive Pythagorean triple: 6,10,14,18*
- *Every integer greater than 2, is part of a primitive or non-primitive Pythagorean triple, for example, the integers 6,10,14, and 18 are not part of primitive triples, but are part of the non-primitive triples 6,8,10; 14,48,50 and 18,80,82.*
- *There exist infinitely many Pythagorean triples whose hypotenuses are squares of natural numbers.*
- *There exist infinitely many Pythagorean triples in which one of the legs is the square of a natural number.*
- *For each natural number n , there exist n Pythagorean triples with different hypotenuses and the same area.*
- *For each natural number n , there exist at least n different Pythagorean triples with the same leg a , where a is some natural number*
- *For each natural number n , there exist at least n different triangles with the same hypotenuse.*
- *In every Pythagorean triple, the radius of the [incircle](#) and the radii of the three [excircles](#) are natural numbers.*
- *There is no Pythagorean triple in which the hypotenuse and one leg are the legs of another Pythagorean triple.*

Problem Five

Many other properties of primitive Pythagorean triples exist. Using the Pythagorean triples that have been generated up to this point or by generating more, explore different triples and look for patterns or characteristics that exist. You are to identify three more properties of Pythagorean triples and explain your thought process.

Property	Explanation

Investigation V

Objectives:

Upon completion of this investigation, students will be able to:

- *Generate Pythagorean triples.*
- *Graph pairs of values from Pythagorean triples.*
- *Develop patterns based on Pythagorean triples.*
- *Generalize patterns based on Pythagorean triples.*

The goal of this investigation is to give students a visual representation of Pythagorean triples. This will allow them to prove and discover patterns with the help of visualizations of these triples.

*This investigation was developed with the intention to give the teacher as much flexibility as possible. Therefore, the majority of discussion questions are found **only** in the teacher's guide. This means more responsibility will be on the student to take notes and record their findings. Teachers are encouraged to use the given discussion questions to enrich students' understanding of the topics. Teachers should use personal judgment as to the length and depth of the discussions. Teachers are encouraged to use their knowledge about their students, as well as their own pedagogy when furthering discussion questions.*

Classroom Setup:

*Students should be split into groups **different** from the previous investigation. This will allow students to be exposed to all the different ways to generate Pythagorean triples. Each group will need one transparency copy of the transparency graph found in Appendix C. Each group should also be given a different colored marker. Be sure no two groups have the same color.*

Investigation V

Problem 1

People in Triangleville are preparing for the annual Triangular Triathlon. Just like a regular triathlon, participants will swim the first part of the race, bike the second, and run the third. Unlike a regular triathlon, the course is set up in the shape of a right triangle so the competitors will begin the race where they finish. The only stipulation for the race is that the swimming distance is less than the running distance, which is less than the biking distance. What are the possible distances of each part of the race? Come up with at least ten possibilities on your own, then ten different ones with a partner. Be sure to express your solutions in a way that is easy to understand, and include the method(s) you used to find them.

The goal in this first problem is to encourage students to use knowledge from the previous investigation on generating Pythagorean triples to develop their solutions. Encourage students to use their knowledge, not only about formulas to generate triples, but also what they know about primitive and non-primitive triples. Be sure to have the students explain how they found their solutions, and that the answers are clear.

Possible questions to pose:

--Can the triangle be isosceles? What makes you think so?

--How do you know which distance is the hypotenuse?

Possible answers, given as: (swimming, running, biking)

<i>(3, 4, 5)</i>	<i>(6, 8, 10)</i>	<i>(12, 16, 20)</i>	<i>(24, 32, 40)</i>
<i>(48, 64, 80)</i>	<i>(11, 60, 61)</i>	<i>(5, 12, 13)</i>	<i>(8, 15, 17)</i>
<i>(10, 24, 26)</i>	<i>(12, 35, 37)</i>	<i>(14, 48, 50)</i>	<i>(7, 24, 25)</i>
<i>(15, 112, 113)</i>	<i>(17, 144, 145)</i>	<i>(19, 180, 181)</i>	<i>(9, 40, 41)</i>
<i>(28, 195, 197)</i>	<i>(34, 288, 290)</i>	<i>(25, 312, 313)</i>	<i>(27, 364, 365)</i>

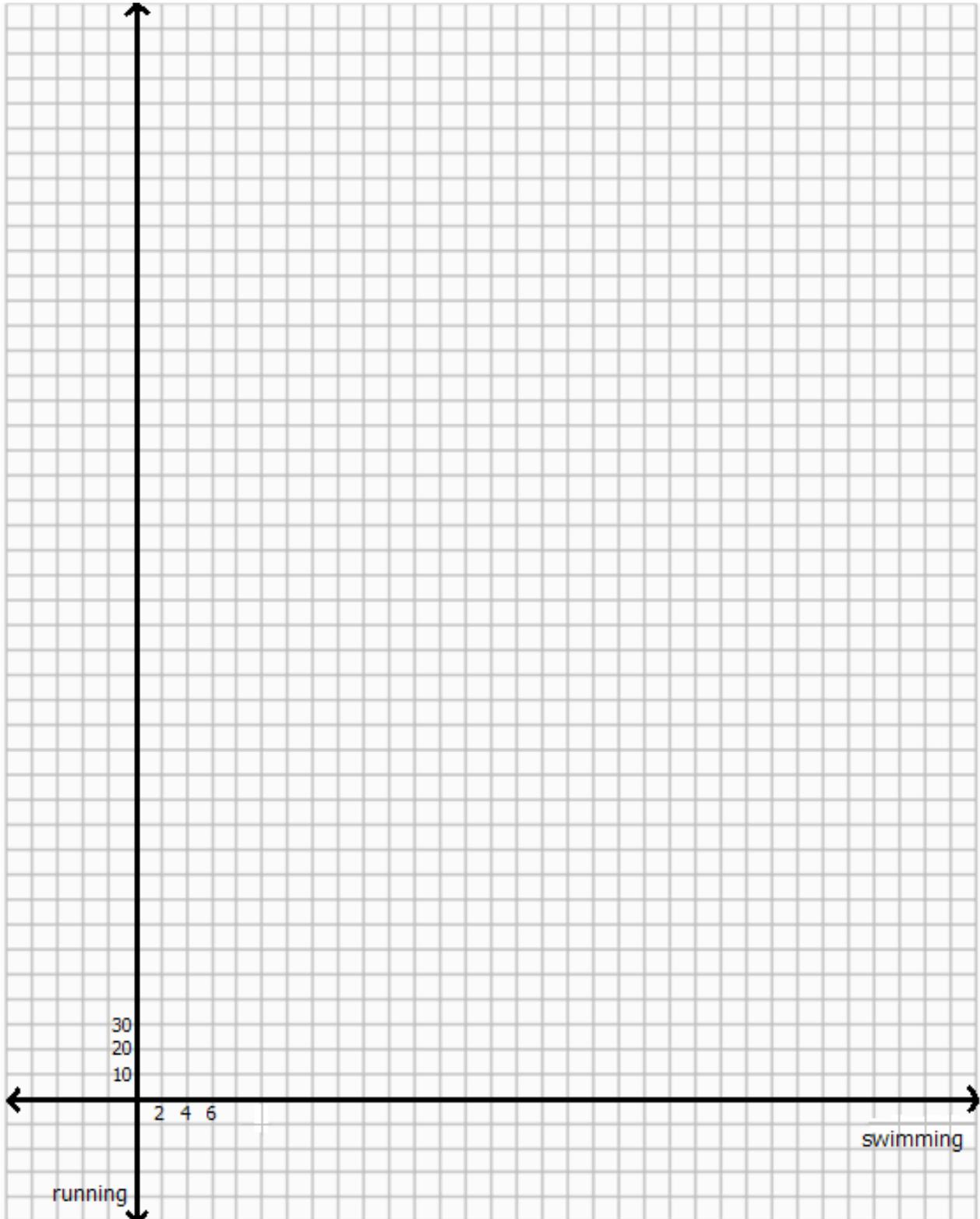
Are these distances realistic? Why or why not?

This discussion is intended to make the problem realistic for students. It should NOT take up significant amount of investigation time. Be sure to discuss the possibility of using other units of measure, such as meters, kilometers, etc.

What other ways could you represent your results?

Encourage students to think creatively. If students are having trouble, brainstorm as a class and have students record the possibilities.

Another way to represent this data is by using a scatter plot. Using the graphing lines, create a coordinate plane and plot pairs of your solutions. In order aid future discussion, plot the swimming and running. Let the swimming distance be represented by the x-axis, and let the running distance represent the y-axis. Plot the points you and your partner came up with on the graph below, and on the transparency given to you by your teacher. Graph 5.1



Place students into groups different from the previous investigation in order to ensure that all students have experience with the other formulas for generating Pythagorean triples. Give each group a transparency of the transparency graph in Appendix C. Then, have them plot their points with a specific color of marker. Using an overhead projector, have each group lay their transparency on top of the overhead, so that the axes overlap. The effect should be a wider range of points plotted on the graph. If the students' transparencies have many of the same points plotted, or do not extend to the higher numbers, generate a few more as a class and plot them on the transparencies.

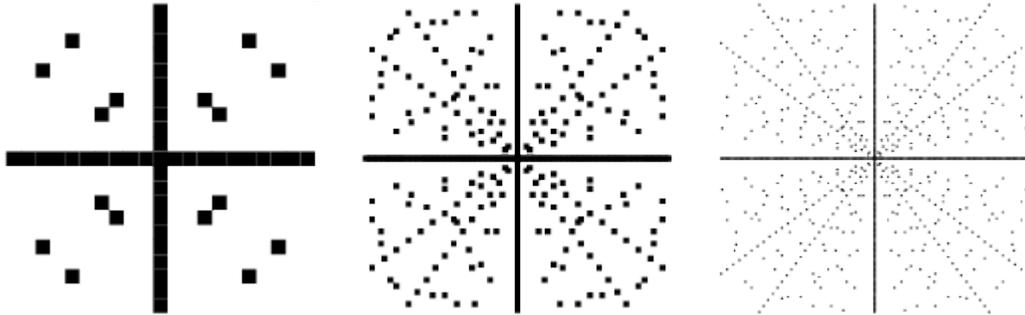
First, copy any points onto your graph that the other groups had that are different from your group's points. Looking at the overhead transparencies, what do you notice about the points? Are there any patterns you see? How does your graph compare to the transparencies? Write down a minimum of five different comparisons and/or patterns you notice.

When leading class discussion, write down students' findings to help aid in generalizing the patterns discovered for the next question. Discussion ideas:

- *Lines the points create*
- *Shapes created*
- *Distances between points*

Now, compare your graph and the transparency to the Scatterplots 5.2 below. What differences do you notice? What are the similarities? Do you see any patterns emerging? Be specific and record at least three findings for each heading below. Be sure to continue taking notes during the discussion.

Scatterplots 5.2 – Plots of legs a - b



Notes:
Similarities

Possible discussion questions:

- *What do you notice about lines our graphs make compared to these graphs?*
- *What do you notice about the shapes of our graph compared to these graphs?*

Differences

- *Are there points on Scatterplots 5.2 that would fit the criteria for our graph?*
- *Are there points on Scatterplots 5.2 that would **NOT** fit the criteria?*
 - *For the above two questions, be sure to take into consideration the fact that swimming < running < biking, or $a < b < c$, for our graph.*
- *What do you notice about the symmetry of the graphs in 5.2? Why do you think this symmetry exists?*
 - *This will tie into the above two questions, because in 5.2, $a < b < c$ is not a necessary condition.*
- *Why are there all four quadrants in 5.2, but not our graph? Are all the points in 5.2 Pythagorean triples? Why or why not?*
 - *They are **NOT** all Pythagorean triples because they are not all positive. But all the points **DO** provide integer values for which $a^2 + b^2 = c^2$.*

Patterns/Other Findings

- *Do the patterns we found for our transparency graph hold true for 5.2?*
- *Are there patterns that are true for 5.2 but not our graph?*

Now think about this question: Can you predict any more numbers based on what you already know about these graphs? How? Write down any ideas you have.

These ideas can lead to formal generalizations.

One way we can do this is to *generalize* a pattern. In other words, find a way to express this pattern so that the expression will always give us numbers or points in the pattern. Pick at least one pattern you found from our transparency, and at least one from a pattern you found from 5.2 and generalize them.

Pattern 1

Help students by suggesting using lines of best fit, equations for the lines by finding the slope, and equations of curves (use discretion based on the mathematics your students already know).

Pattern 2

Look at your generalizations. If they are correct, they should give you points that are numbers in a Pythagorean triple. Check your generalizations below by finding at least 10 points with your generalizations.

Can you find the third number to make the Pythagorean triple complete? List them below.

Investigation VI

Objectives:

Upon completion of this investigation, students will be able to

- *Generalize the idea of a Pythagorean theorem to explore Pythagorean quadruples*
- *Extend the concept of Pythagorean triples to other realms*

This investigation focuses on expanding the students' understanding of Pythagorean triples and encourages them to pose questions in new directions. To aid in this exploration, the students are first asked to look at Pythagorean quadruples. Then they are asked to try different values in $a^n + b^n = c^n$. Lastly, the students can extend these ideas for further explorations.

Classroom Set-up:

This investigation should be completed in groups to facilitate the development of discussion.

Investigation VI

Now that you have spent some time with the Pythagorean Theorem, proofs of the Pythagorean Theorem, Pythagorean triples, and properties of triples or graphs of triples, what other types of explorations could you investigate?

Can you think of any ideas that are similar or related to the Pythagorean theorem that deal with algebra, geometry, or general number sense that you could explore? If so, list some of your ideas.

Here are a couple of ideas.

- Are there values for which $a^2 + b^2 + c^2 = d^2$?
Try different values for the variables.
Show the work that you have tried.

What types of conclusions can you make about the idea of a Pythagorean quadruple (a, b, c, d) for which $a^2 + b^2 + c^2 = d^2$?

- Are there values for which $a^n + b^n = c^n$?
Try different values for the variables.
Show the work that you have tried.

Why types of conclusions can you make based upon your calculations?

This would be a good opportunity to discuss Fermat's Last Theorem and that there are no values for which this works where $n > 2$.

Now try to extend these ideas or another idea from the module. Think of a question to explore that relates to one of the prior topics. Identify the question that you are asking and then in small groups work to try to identify some conclusions about the question you asked.

This final question may be assigned as an ongoing project for the students to work on and then report their findings to the class a few weeks after the investigation has been completed.