Stanton Graph Families

Review

Define "path" in your own words. Draw some paths below.

Suggestion: class discussion after students have some time to write their own. Have a student share their definition, then ask for edits/expansions until the class is satisfied with the definition they have. Draw some examples and ask if they would be paths under the definition they've given.

Define "star" in your own words. Draw some stars below.

Please note the following definitions/notations:

C_n: a cycle with n vertices.

S_n: a star with $n + 1$ vertices.

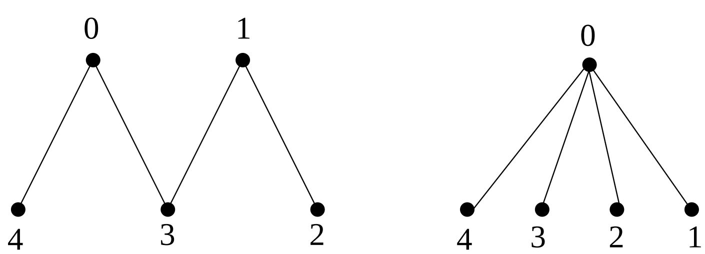
P_n: a path with $n + 1$ vertices.

IMPORTANT NOTE: This is not the conventional definition/notation for paths and stars. Usually, an S_n or P_n has n vertices. However, this definition/notation will make working with Stanton paths and stars much more logical, because this way the n corresponds with the maximum edge multiplicity present in the Stanton graph. If you've used the conventional notation with your students, make sure to note and discuss this change.

Depending on how comfortable your students are with cyclic decomposition and the other following concepts, you may want to give them time to work individually on these questions. If they are very comfortable with them, a whole-class discussion would be more brief and probably more fitting.

Find a cyclic decomposition for K_9 using a path with four edges and then a star with four edges.

possible answers:



How many different edge lengths are present in K_{19}? What complete graphs have 31 different edge lengths?

9 edge lengths, K_{62} or K_{63}

If students are stuck, prompt them to try drawing out smaller cases to remember the general rule

Lets talk λ: Decomposing λ-fold multigraphs

Let's review first.

Definition: The **edge multiplicity** is the number of edges between two verticies.

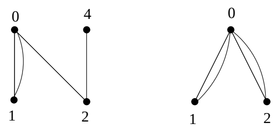
In your own words, what does it mean for K_n to be decomposed into a graph H cyclically? What requirements must H satisfy?

Possible answers: H has to have one of each edge length present in K_n. You click H around to get a complete K_n.

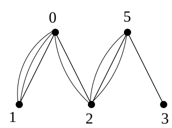
Based on this, if we were to increase the edge multiplicity of K_n, how would the restrictions on H change? Explore this a bit on a separate paper. One way to do this would be to draw a ${}^{\lambda}K_n$ and create a path, star, or cycle, which can cyclically decompose the complete graph. First try ${}^{2}K_5$, and once you feel that you understand how to cyclically decompose ${}^{2}K_5$, try ${}^{3}K_5$, ${}^{3}K_7$, or a choose a higher value for λ or a different K_n and try it out!

Suggestion: Let students work on this question individually, then let them discuss with a partner or small group.

Possible answers: H could have higher edge multiplicity. See examples:



${}^{2}K_5$



${}^{3}K_7$

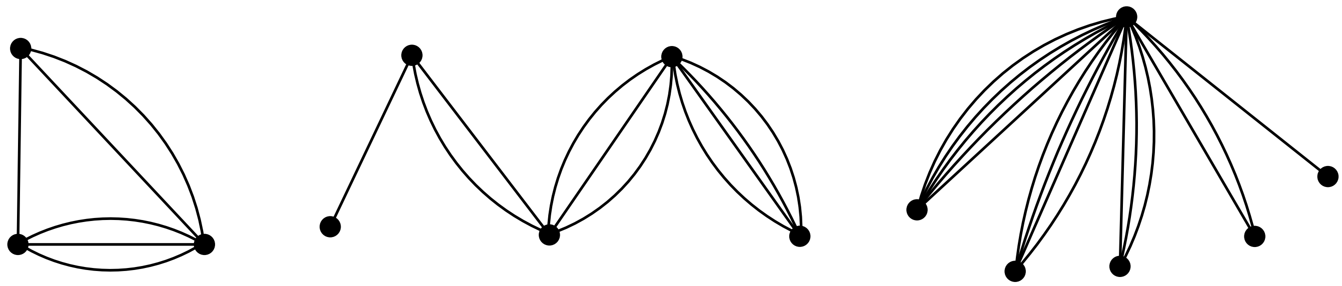
At this point, you have probably come to the conclusion that to decompose ${}^{\lambda}K_n$ into a graph H, H must have a maximum edge multiplicity of λ. Hence, we have added a new restriction for decomposition when using graphs that have multiple-edges. Based on your findings, what would you expect the smallest value for λ to be, such that a graph G, with maximum edge multiplicty x, so that G decomposes ${}^{\lambda}K_n$?

λ **would need to be at least** x**.**

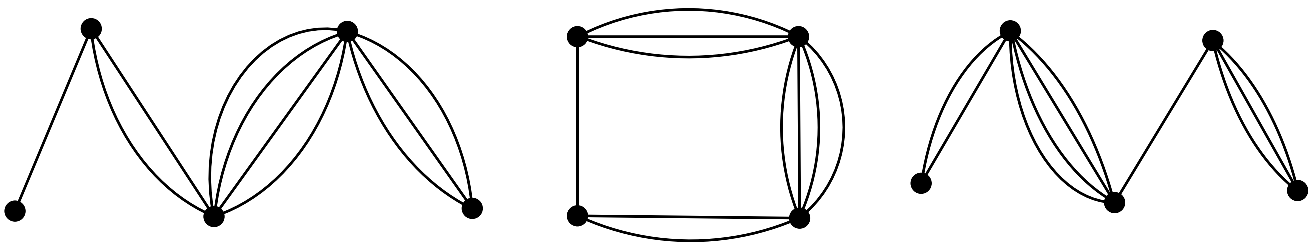
Stanton Graphs

Now we are going to explore some families of graphs. Look for patterns and compare the Stanton and non-Stanton graphs.

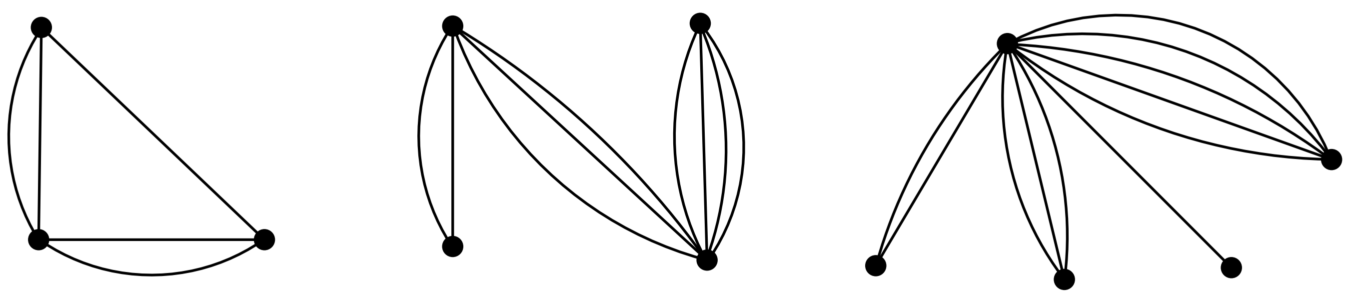
Standard Stanton Graphs



Non-Standard Stanton Graphs



Non-Stanton Graphs



Based on the example graphs above, define what a Stanton graph is in your own words.

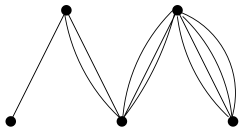
Again, let students study the graphs on their own and then bring it back to a whole class discussion. Make sure all the graphs from groups 1 and 2 fit within this definition and that none of the groups from the 3rd group do. Also draw some other example graphs and fit them based on your definition, and decide whether anything needs to be changed.

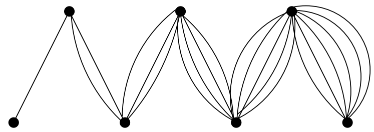
Based on the example graphs above, define what a standard Stanton graph is in your own words.

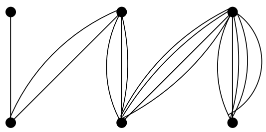
Focus on groups 1 and 2 here. Ask: What are the differences between graphs in these two groups? How could you change the graphs from the first group to make them "standard"?

Please note, from now on, when we refer to Stanton graphs, we are referring to standard Stanton graphs.

Draw a Stanton 4-path (SP_4), a Stanton 5-path(SP_5), and a Stanton 6-path (SP_6).







How many edges does each graph have?

SP_4:

10

SP_5:

15

SP_6:

21

How many edges would a SP_n have? (That is, a Stanton path on $n+1$ vertices). A SS_n? A SC_n?

Answer: $\frac{n(n+1)}{2}$

If students have difficulty generalizing the pattern, go through Gauss' famous problem of adding the numbers 1-100 as a whole class. (Pair 1 with 100, 2 with 99, etc. Each pair is 101, there are 50 pairs, 5,050 is the answer. Then generalize this to n numbers instead of 100)

You may find that students struggle with the concept of n or generalizing patterns into formulas. Going through the Gauss problem may help, but if your students have very little exposure to this type of task you may consider just modeling finding the formula rather than asking them to find it on their own.

Let's try to decompose some complete graphs using Stanton Graphs!

Try decomposing a K_5 using a Stanton-2 path. Can you do it? If not, why?

In a simple K_5, anything with edge multiplicity higher than 1 will not decompose it.

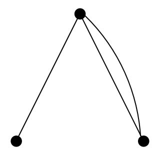
How could we change K_5 so that it could possibly be decomposed using a Stanton-2 path?

Increase the edge multiplicity to a ${}^2K_5$ or higher

Investigation

We want to decompose λ-fold complete graphs into Stanton stars or paths.

Draw the Stanton 2-star and Stanton 2-path below. What do you notice about these two graphs?



They are the same graph. This is the only size Stanton that this is true for.

What is the smallest λ you will need to be able to decompose a complete graph into copies of the Stanton 2-star?

2

If a graph G is going to decompose ${}^{\lambda}K_n$ cyclically, how many of each edge length will G need to have? (Hint: there are 2 cases, so you should have two answers.)

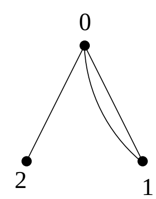
To decompose an even complete graph, you would need 2 of each edge length except the longest, which you would need 1 of. To decompose an odd complete graph, you would need 2 of each edge length

Questions for students: What do you think the 2 cases are? Draw some complete graphs and note any differences to determine the cases

Can you get that many of each edge length with a Stanton 2-star or 2-path? (Hint: How many total edges are in this Stanton graph?) Why or why not?

You can decompose ${}^{2}K_4$, with 1 of length 2 and 2 of length 1.

If possible, find a decomposition of a λ-fold complete graph into Stanton 2-stars.



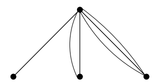
Can you decompose a higher λ complete graph using the Stanton 2-star? For example, could $\lambda = 3$? 4? What values of λ work?

note: You may have more than one copy of the 2-star in your cyclic decomposition. For example, with 2 copies, you could have 3 edges of length 1 and 3 of length 2 to decompose ${}^{3}K_5$.

With 3 copies, you could have 4 of length 1 and 2 of length 2 to decompose ${}^{4}K_4$.

Student answers will vary greatly, so students may benefit from a small group or whole class discussion of this question and the possibilities it presents

We will continue the investigation with Stanton stars. Draw the Stanton 3-star (SS_3) below.



Decide (i) which λ you should be working with to decompose into copies of SS_3 cyclically and (ii) how many of each edge length you need. Is this possible with this Stanton graph?

i. You can work in $\lambda = 3$, but there are also other possibilities that students could come up with

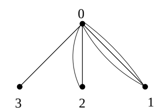
ii. In $\lambda = 3$, you need 3 of each edge length

If it is possible, decide what order of complete multigraph you will be decomposing. How many different edge lengths do you have?

For $\lambda = 3$, you would be decomposing ${}^3K_5$ because you can get 3 of length 1 and 3 of length 2 with 6 edges.

Try to cyclically decompose your complete multigraph into this Stanton 3-star.

possible answer:



If students are struggling, it may be beneficial to remind them about wraparound edges here.

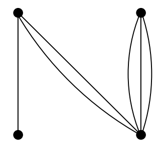
Continue investigating Stanton stars on a separate sheet of paper. See if you can find a pattern for decomposing complete graphs into Stanton stars based on their order. Is there a pattern in the order of complete multigraph you are using? Is there a pattern in the labeling you are using? Investigate whick ${}^{\lambda}K_n$ can be decomposing by SS_4, SS_5, SS_6... If you can find any patterns, give justifications for why your patterns work.

It is suggested that students work individually for a time on this, then share strategies and results with partners or small groups. Students may need help developing an organized method of investigation. If your class particularly struggles on this point, you could develop more scaffolded questions to lead their investigation. After a few cycles of individual investigation and sharing with partners or small groups, the discussion should return to the whole class.

For a Stanton n-star, students should be able to decompose ${}^{n}K_{n+2}$ and ${}^{n+1}K_{2n+1}$. These are not the only complete graphs that may be able to be decomposed, but these will always be able to be decomposed into SS_n.

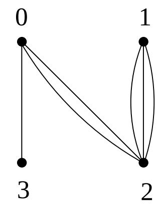
Students will also come up with various answers here and should share with the whole class. Finding the patterns should be emphasized, but make sure to encourage justification of the patterns.

Next, you will investigate decomposing λ-fold complete graphs into Stanton *paths*. Draw the Stanton 3-path below.



Decide what value for λ you should be working with, how many different edge lengths you would need, and how many different edge lengths you have. Decide what ${}^{\lambda}K_n$ you are working in, and find a cyclic decomposition.

possible answer:



Try decomposing using the Stanton 4-path and greater.

Note any patterns you find. Could you find a decomposition for any Stanton n-path? Describe what you would do to create this decomposition, and give a justification for why this pattern holds.

Conduct this investigation in a similar manner to the stars investigation. Be sure to draw connections between the two investigations!