Investigating Decompositions of Cubic Graphs
Prepared by: 2013 ISU REU
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Objective

During this lesson students will become familiar with and explore the concepts of trees, cubic graphs, stars, and paths. Students will also decompose cubic graphs into **stars** and **simple paths** and experiment with finding patterns in the process. By the end, students will be able to articulate and generalize the patterns they have found. This document mimics the student version, but extra questions for educational purposes have been included.

Lesson Introduction

Below is a list of questions and points to consider. Each question should be discussed in small groups and then as a class until there is a common consensus. Presented here are the questions and the points that need to be garnered from the discussion.

Questions and Points to Cover

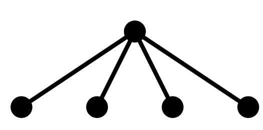
1. If a graph were to be called a **tree** what do you think it would look like?

   (a) What characteristics would the graph have?

   - Have students draw what they would call a tree. Have them share their graphs and explain why they would call them trees.
   - From here, provide examples of trees (possibly using student work). Ask if these graphs fit the notion of tree that they have developed. Make sure the examples are made up of stars, paths, and caterpillars.
   - Introduce formal definition, **Tree**: A graph composed of one component that contains no cycles.
   - Ask how this formal definition of **trees** differs from their original conception of **trees**.

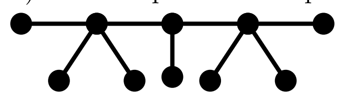
   (b) Sort the provided examples into groups and describe how the elements in these subgroups are the same.

   - May want to suggest the use of three groups if the students are struggling.
   - If they are still striggling, may suggest that they investigate the degree of each vertex and look for patterns.
   - In this way they can develop definitions together for stars, simple paths, and caterpillars.
     - Stars are noted by one vertex connected to every other vertex in the tree. These other vertices are connected to **ONLY** that one vertex. A star with four edges is provided below:



- Simple paths are noted by alternating sequence of edges and vertices, where no vertex is repeated. A path with 3 edges is provided below:

- Caterpillars are simple paths that have had branches added on. (The formal definition is that if you remove all the verticies with degree one, and the edges they are connected to, you are left with a simple path.) An example of a caterpillar is provided below:



2. If we were to take a caterpillar and add branches what do you think that graph would be called?
   - Allow them to discuss why different terms would be appropriate, but inform them that the "technical term" is a **lobster**.

Review

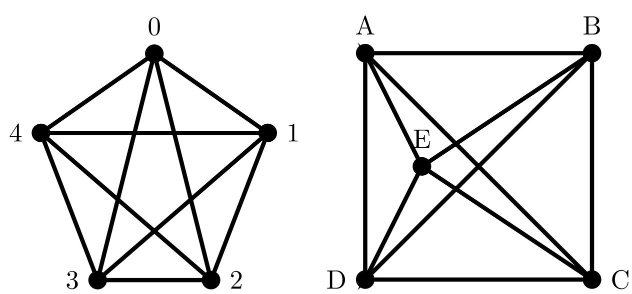
1. Does anyone remember what a **cubic graph**, or **3-regular graph**, is?
   - If they are drawing a blank have them reference their previous vocabulary packet.
   - Answer: A graph where every vertex has degree 3.
2. Recall: The concept of a **graph decomposition** is to break up a larger graph into, or to build a larger graph out of smaller graphs without duplicating edges (but you are allowed to reuse vertices).

Exploration

Pass out the manipulatives and discuss how the Play-Do© represents vertices and the pipe cleaners represent edges.

Questions and Points to Cover

1. How many different trees can be formed that have two edges? What about three edges? (Feel free to let the students use the manipulaives to explore this.)
   - There is only one with two edges (the simple path), and only two with three edges (the star and the simple path).
   - Make sure to point out isomorphic graphs that may look different at first but are really the same. Below is an example of two graphs that are isomorphic.

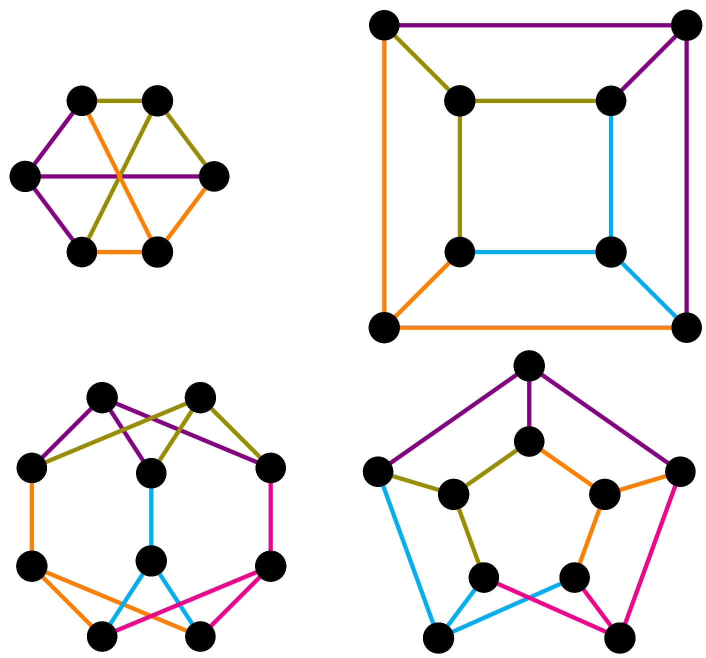


These two graphs are isomorphic. They look differnt, but you can see that if you relabeled the vertices of the second one (A→0, B→1, C→2, D→3, and E→4) and move them into the same posistions that they are the same graph.

2. Construct stars of degree three.

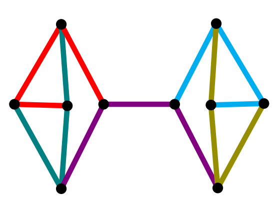
- Suggest that they make the base of the star one color of Play-Do© and the ends a different color.
- Suggest that they make the edges of a star the same color, and the edges of different stars to be differnet colors.

3. Using the **3-stars** (stars with three edges), you have made, create **cubic graphs**, or **3-regular graphs**.
   - Let them explore in groups creating graphs with the Play-Do© and pipe cleaners.
   - When you bring them back together as a class, have them share their graphs and examine which ones are isomorphic to each other. Encourage them to rearrange their vertices and edges (without breaking thier graphs apart) until they see how their graphs are isomorphic.
   - There are only four cubic graphs of order at most 10 that decompose into 3-stars. They are given below, with one possible star decomposition shown for each.

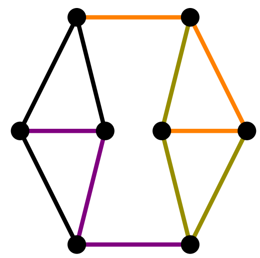


Exploration 2

Explain to the students that you are giving them all the cubic graphs with 4 to 10 vertices and would like them to use **simple paths** with 3 edges to decompose it. Demonstrate this on C25 (one decomposition for C25 is given below). Provide them with C6, and have them attempt to decompose it on thier own. Bring them back together as a class to check the decompositions. Tell the students that they will be decomposing the remainder of the cubic graphs. Encourage them to look for strategies that they may find benificial. From here break the students into small groups and give each group a set of **Cubic Graph Cards**. The students can choose which graph they want from the pile and work on decomposing it. Have them make a "done" pile and a "not done" pile. Note that the students do not need to have all of the graphs finished before you wrap up this activity. When appropriate, pull the class back together for discussion.

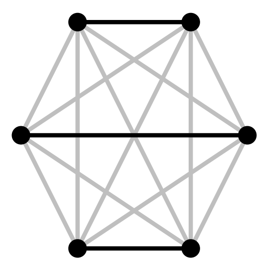


A possible decomposistion of C6:

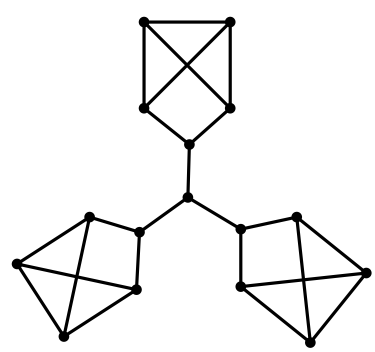


Questions and Points to Cover

1. Are there certain strategies to finding a decomposition?
2. Will they all decompose into paths, why or why not?
3. Could we create a cubic graph that would **NOT** work?
   - Yes. Those that were passed out will all decompose into simple paths.
   - They all work because they all have a 1-factor (a set of disconnected edges that span the graph). An example of a 1-factor is given below, the gray edges are the whole graph, and the black edges are the 1-factor.



   - These will work for all cubic graphs that have a 1-factor.
   - Below is an example of the smallest cubic graph **not** containing a one factor. Thus a 3-path will not decompose it.



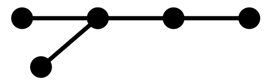
Exploration 3

Acknowledge the accomplishments of working with the cubic graphs and explain that the natural extension of this is to move to **quartic graphs**, or **4-regular graphs**. Discuss how, since we are now working with 4-regular graphs, we are interested in decomposing into trees with 4 edges.

Questions and Points to Cover

1. How many different trees can you come up with that have four edges?

   - There are 3 different trees with 4 edges: the star, the simple path, and the tree shown below which we will denote as Y.



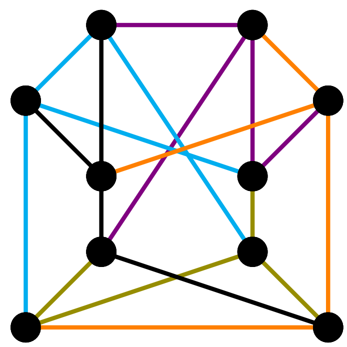
2. Draw a 4-regular graph.

   - Make sure your students understand what you are asking them to draw.
   - Ensure that the graphs they draw are indeed 4-regular and, if time permits, discuss how their different *looking* representations could be isomorphic.

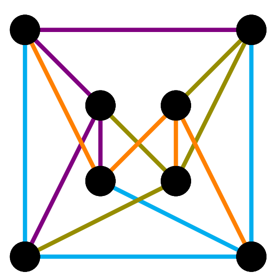
3. There exist simple path decompositions on quartic graphs and star decompositions on those with no odd cycles. However, it is unknown which quartic graphs Y decompose.

   Look at the **Quartic Graph Cards** and explore Y decompositions of quartic graphs. Remember to look for patterns that make finding the decompositions easier. If you feel that a graph is not decomposable into copies of Y make sure to justify why.

   - Graphs of even degree and odd order will not decompose into trees in one fold. You can justify this, by explaining that 4 does not divide the number of edges.
   - If you feel they need assistance decompose Q46, a decomposition is shown below:



   - If you feel they need additional assistance, have them try Q5 on thier own, and then come together as a class and discuss the solution. A decomposition is shown below:

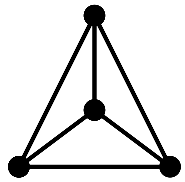


   - Encourage them to check each other's answers.

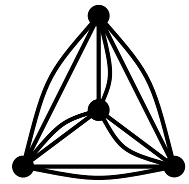
4. Once the students have aquired a significant number of findings or their progress has slowed, have them explain thier observations and justifications.

   - Will all quartic graphs decompose into copies of Y?
   - If not, which quartic graphs will decompose into copies of Y and which will not?
   - What strategies did you find that will allow you to decompose the graphs more easily?

5. If there are quartic graphs that Y does not decompose, ask "Would they decompose the graph if we made it a 2-fold version of that graph (if we gave each edge a 'parallel' edge)?" This is shown below.

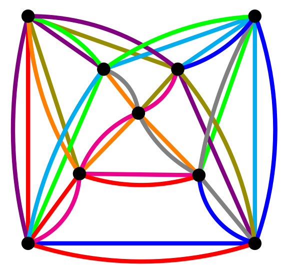


Here is C1



Here is ^{2}C1

- Make sure that they understand what you are asking them to do. Below is a Y-decomposition of a 2-fold version of Q13. We know Y will not decompose a 1-fold Q13, because there are 18 edges which is not a multiple of 4, the size of Y.



Conclusion

These concluding points can be used as an assignment, or to wrap up this investigation.

- Present the findings of the class and discuss the significance of them.
- Discuss the importance of finding patterns in mathematics.
- Discuss how mathematics research is about finding, articulating, and justifying patterns.