On decompositions of complete multipartite graphs into the union of two even cycles*

A. Su, J. Buchanan, R. C. Bunge, S. I. El-Zanati,
E. Pelttari, G. Rasmuson, E. Sparks, S. Tagaris

Department of Mathematics
Illinois State University
Normal, IL 61790-4520 U.S.A.

Abstract

For positive integers c and d, let $K_{c\times d}$ denote the complete multipartite graph with c parts, each containing d vertices. Let G with n edges be the union of two vertex-disjoint even cycles. We use graph labelings to show that there exists a cyclic G-decomposition of $K_{(2n+1)\times t}$, $K_{(n/2+1)\times 4t}$, $K_{5\times(n/2)t}$, and of $K_{2\times 2nt}$ for every positive integer t. If $n \equiv 0 \pmod 4$, then there also exists a cyclic G-decomposition of $K_{(n+1)\times 2t}$, $K_{(n/4+1)\times 8t}$, $K_{9\times(n/4)t}$, and of $K_{3\times nt}$ for every positive integer t.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \varnothing$). Let $\mathbb{N}_0$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. Let K_k denote the complete graph on k vertices.

Let $V(K_k) = \mathbb{Z}_k$ and let G be a subgraph of K_k. The *length* of an edge $\{i, j\} \in E(G)$ is defined as $\min\{|i-j|, k-|i-j|\}$. By *clicking* G, we mean applying the isomorphism $i \to i+1$ to $V(G)$. Let H and G be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of pairwise edge-disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. If H is K_k, a G-decomposition Γ of H is *cyclic* if clicking is an automorphism of Γ. The

*Research supported by National Science Foundation Grant No. A1063038

decomposition is *purely cyclic* if it is cyclic and $|\Gamma| = |V(H)|$. If G is a graph and r is a positive integer, rG denotes the vertex disjoint union of r copies of G.

The study of graph decompositions, also known as the study of graph designs or G-designs, is a popular area of research. In particular, decompositions of complete graphs into cycles have attracted a great deal of attention. For relatively recent surveys on graph decompositions, we direct the reader to [2] and [5]. A popular method for obtaining graph decompositions is via graph labelings.

For any graph G, a one-to-one function $f\colon V(G) \to \mathbb{N}_0$ is called a *labeling* (or a *valuation*) of G. In [14], Rosa introduced a hierarchy of labelings. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{f}(E(G)) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

(ℓ1) $f(V(G)) \subseteq [0, 2n]$,

(ℓ2) $f(V(G)) \subseteq [0, n]$,

(ℓ3) $\bar{f}(E(G)) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,

(ℓ4) $\bar{f}(E(G)) = [1, n]$.

If in addition G is bipartite with vertex bipartition $\{A, B\}$, consider also

(ℓ5) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,

(ℓ6) there exists an integer λ such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

(ℓ1), (ℓ3) is called a *ρ-labeling*;

(ℓ1), (ℓ4) is called a *σ-labeling*;

(ℓ2), (ℓ4) is called a *β-labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. Suppose G is bipartite. If a ρ-, σ-, or β-labeling of G satisfies condition (ℓ5), then the labeling is *ordered* and is denoted by ρ^+, σ^+, or β^+, respectively. If in addition (ℓ6) is satisfied, the labeling is *uniformly ordered* and is denoted by ρ^{++}, σ^{++}, or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly ordered β-labeling is an *α-labeling* as introduced in [14]. Labelings of the types above are called *Rosa-type labelings* because of Rosa's original article [14] on the topic (see [10] for a comprehensive survey of Rosa-type labelings). A dynamic survey on general graph labelings is maintained by Gallian [11].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [14] and [9], respectively.

Theorem 1. *Let G be a graph with n edges. There exists a purely cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

Theorem 2. *Let G be a graph with n edges that admits a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

2 d-modular labelings and decompositions of $K_{c\times dt}$

For positive integers c and d, let $K_{c\times d}$ denote the complete multipartite graph with c parts, each containing d vertices. Note that $K_{c\times d}$ has cd vertices and $\binom{c}{2}d^2$ edges. We can consider $K_{c\times d}$ as a subgraph of the complete graph K_{cd}, with $V(K_{c\times d}) = \mathbb{Z}_{cd}$ and $E(K_{c\times d}) = \big\{\{u, v\} : u, v \in \mathbb{Z}_{cd}, u \not\equiv v \pmod c\big\}$, that is, the c parts of $K_{c\times d}$ are the congruence classes of $\mathbb{Z}_{cd}$ modulo c. Note that $K_{c\times d}$ has precisely the edges of K_{cd} whose lengths are not multiples of c.

Let G be a graph and let $\{G_1, G_2, \ldots, G_t\}$ be a G-decomposition of $K_{c\times d}$ (with $V(K_{c\times d}) = \mathbb{Z}_{cd}$ as defined above). If clicking permutes the graphs in the decomposition, then we say that it is a *cyclic G-decomposition* of $K_{c\times d}$, and if clicking G_1 $cd-1$ times produces each graph in the decomposition exactly once, then we say the decomposition is *purely cyclic*. In the latter case if G has n edges, we must have $\binom{c}{2}d^2 = ncd$, and so $c = 2n/d + 1$.

Suppose that G is a graph with n edges and d is a positive integer such that d divides $2n$. Set $c = 2n/d + 1$, so that $cd = 2n + d$. By a *d-modular ρ-labeling* of G we mean a one-to-one function $f : V(G) \to [0, cd-1]$ such that

$$\big\{\min\big\{|f(u) - f(v)|, cd - |f(u) - f(v)|\big\} : \{u, v\} \in E(G)\big\} = \big[1, \lfloor \tfrac{cd}{2} \rfloor\big] \setminus c\mathbb{Z}.$$

In other words, a d-modular ρ-labeling of a graph with n edges has every edge length in K_{2n+d} exactly once except for any multiples of $2n/d + 1$.

Figure 1 shows an example of a 3-modular ρ-labeling of a 6-cycle. As a subgraph of K_{15}, the edge length 5 is missing. Thus this C_6 has one edge of each length in $K_{5\times 3}$ and clicking it 14 times would produce a purely cyclic C_6-decomposition of $K_{5\times 3}$. Thus from the definition of d-modular ρ-labelings, it is straightforward to see that the following holds.

Theorem 3. *If the graph G with n edges admits a d-modular ρ-labeling and $c = 2n/d + 1$, then $K_{c\times d}$ has a purely cyclic G-decomposition.*

We observe that a ρ-labeling of G is necessarily a 1-modular ρ-labeling. Moreover, a σ-labeling of G is necessarily a 2-modular ρ-labeling. We also note the following.

Theorem 4. *Let G be a bipartite graph with n edges. If G admits a ρ^+-labeling, then G admits a $2n$-modular ρ-labeling.*

Proof. Let $\{A, B\}$ be a bipartition of $V(G)$ and let f be a ρ^+-labeling of G such that $f(a) < f(b)$ for every $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$. Define a labeling $g\colon V(G) \to [0, 4n-1]$ by $g(a) = 2f(a)$ for $a \in A$ and $g(b) = 2f(b) - 1$ for $b \in B$. It is easy to verify that g is a $2n$-modular ρ-labeling of G. ■

Next we note that if every vertex of a graph G has even degree, then in a d-modular labeling of G, the number of edges with an odd label must be even. This is known as the *parity condition.*

Lemma 5. *Let G be a graph with all even degrees and let f be a d-modular labeling of G. Let $O = \{e \in E(G)\colon \bar{f}(e)$ is odd$\}$. Then $|O|$ is even.*

Proof. For $e = \{u, v\} \in E(G)$, either $\bar{f}(e) = f(u) - f(v)$ or $\bar{f}(e) = f(v) - f(u)$. Let $S = \sum_{e\in E(G)} \bar{f}(e)$. Let $v \in V(G)$. Since $\deg(v)$ is even, the sum of the number of occurrences of $f(v)$ and of $-f(v)$ in S is even. Therefore S is even and hence $|O|$ must be even. ■

The concept of a d-modular ρ-labeling relates very closely to the concepts of difference families and difference matrices developed by Buratti and several co-authors over the last several years. See for example, Buratti [6], Buratti and Gionfriddo [7], and Buratti and Pasotti [8]. Another related concept is that of a d-graceful labeling as introduced by Pasotti in [13]. Rather than define these additional concepts here, we state a powerful result on d-modular ρ-labelings that can be obtained from the main result on graph decompositions with the use of difference matrices in [8].

Theorem 6. *If a z-partite graph G with n edges has a d-modular ρ-labeling and $c = 2n/d+1$, then $K_{c\times td}$ has a cyclic G-decomposition for every positive integer t such that* $\gcd(t, (z-1)!) = 1$.

Thus if G is bipartite, then we have the following corollary to Theorem 6.

Corollary 7. *If a bipartite graph G with n edges has a d-modular ρ-labeling and $c = 2n/d+1$, then $K_{c\times td}$ has a cyclic G-decomposition for every positive integer t.*

We illustrate how the result in Corollary 7 works. Let $\{A, B\}$ be a bipartition of $V(G)$ and let f be a d-modular ρ-labeling of G. Let $A = \{u_1, u_2, \ldots, u_r\}$ and $B = \{v_1, v_2, \ldots, v_s\}$. Let x be a positive integer. For $1 \leq i \leq x$, let G_i be a copy of G with bipartition (A, B_i) where $B_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,s}\}$ and $v_{i,j}$ corresponds to v_j in B. Let $G(x) = G_1 \cup G_2 \cup \cdots \cup G_x$. Thus $G(x)$ is bipartite with bipartition $\{A, B_1 \cup B_2 \cup \cdots \cup B_x\}$. Define a labeling f' of $G(x)$ as follows: $f'(a) = f(a)$ for each $a \in A$ and $f'(v_{i,j}) = f(v_j) + (i-1)(2n+d)$ for $1 \leq i \leq x$ and $1 \leq j \leq s$. It is easy to see that f' is a d-modular ρ-labeling of $G(x)$ and thus Theorem 3 applies.

Figure 1 shows a 3-modular ρ-labeling of C_6 and the three starters for a cyclic C_6-decomposition of $K_{5\times 9}$ that can be obtained from that 3-modular ρ-labeling of C_6.

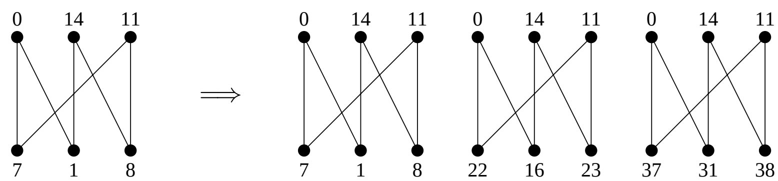

Figure 1: A 3-modular ρ-labeling of C_6 and three starters for a cyclic C_6-decomposition of $K_{5\times 9}$.

In this article, we investigate the existence of d-modular ρ-labelings for the graph G consisting of the vertex-disjoint union of two even cycles. In light of Corollary 7, these labelings lead to cyclic G-decompositions of various infinite classes of complete multipartite graphs. In [13], Pasotti produces labelings of C_{4k} that lead to cyclic C_{4k}-decompositions of $K_{(2k+1)\times 4n}$ and of $K_{(k+1)\times 8n}$ for all positive integers k and n. She also produces labelings that lead to cyclic C_{2k}-decompositions of $K_{(k+1)\times 4n}$ for all odd integers $k \geq 1$ and all positive integers n. In [3], Benini and Pasotti refine the results from [13] to produce labelings of C_{4k} that yield cyclic C_{4k}-decompositions of $K_{(\frac{4k}{d}+1)\times 2dn}$ for any positive integers k, n and any positive divisor d of $4k$. Numerous other authors have studied decompositions (not necessarily cyclic ones) of complete multipartite graphs into cycles. Particular focus has been placed on C_3-decompositions of complete multipartite graphs. Such decompositions fall under the umbrella of the study of group divisible designs (see [12] for a summary). The problem of C_{2k}-decompositions of the complete bipartite graph $K_{m,n}$ was settled completely by Sotteau in [15].

3 Additional Notation

We denote the directed path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \le i \le k-1$, by $(x_0, x_1, \ldots, x_k)$. The *first vertex* of this path is x_0, the *second vertex* is x_1, and the *last vertex* is x_k. If $x_0, x_1, \ldots, x_k$, are distinct vertices, then the path $(x_0, x_1, \ldots, x_k, x_0)$ is necessarily a cycle on $k+1$ vertices. If $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$ are directed paths with $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$.

Let $P(k)$ be the path with k edges and $k+1$ vertices $0, 1, \ldots, k$ given by $(0, k, 1, k-1, 2, k-2, \ldots, \lceil k/2 \rceil)$. Note that the set of vertices of this graph is $A \cup B$, where $A = [0, \lfloor k/2 \rfloor]$, $B = [\lfloor k/2 \rfloor + 1, k]$, and every edge joins a vertex of A to one of B. Furthermore, the set of labels of the edges of $P(k)$ is $[1, k]$.

Now let a and b be nonnegative integers with $a \le b$ and let us add a to all the vertices of A and b to all the vertices of B. We will denote the resulting graph by $P(a, b, k)$. Note that this graph has the following properties.

(P1) $P(a, b, k)$ is a path with first vertex a and second vertex $b + k$. Its last vertex is $a + k/2$ if k is even and $b + (k+1)/2$ if k is odd.

(P2) Each edge of $P(a, b, k)$ joins a vertex of $A' = [a, \lfloor k/2 \rfloor + a]$ to a larger vertex of $B' = [\lfloor k/2 \rfloor + 1 + b, k + b]$.

(P3) The set of edge labels of $P(a, b, k)$ is $[b - a + 1, b - a + k]$.

Now consider the directed path $Q(k)$ obtained from $P(k)$ replacing each vertex i with $k - i$. The new graph is the path $(k, 0, k-1, 1, \ldots, k - \lfloor k/2 \rfloor)$. The set of vertices of $Q(k)$ is $A'' \cup B''$, where $A'' = k - B = [0, k - \lfloor k/2 \rfloor - 1]$ and $B'' = k - A = [k - \lfloor k/2 \rfloor, k]$, and every edge joins a vertex of A'' to one of B''. The set of edge labels is still $[1, k]$. The last vertex of $Q(k)$ is $k/2 \in B''$ if k is even and $(k-1)/2 \in A''$ if k is odd.

We add a to the vertices of A'' and b to vertices of B'', where a and b are integers, $0 \le a \le b$. This graph is $(k+b, a, k+b-1, a+1, \ldots)$ which we will denote by $Q(a, b, k)$. Note that this graph has the following properties.

(Q1) $Q(a, b, k)$ is a path with first vertex $k + b$. Its last vertex is $b + k/2$ if k is even and $a + (k-1)/2$ if k is odd.

(Q2) Each edge of $Q(a, b, k)$ joins a vertex of $A' = [a, a + k - \lfloor k/2 \rfloor - 1]$ to a larger vertex of $B' = [b + k - \lfloor k/2 \rfloor, b + k]$.

(Q3) The set of edge labels of $Q(a, b, k)$ is $[b - a + 1, b - a + k]$.

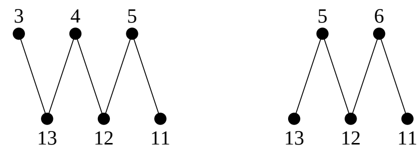

Figure 2: Examples of the path notations with an even number of edges.

4 Main Results

Lemma 8. *A d-modular ρ-labeling of $C_{4r} \cup C_{4s}$ exists for $1 \le r \le s$ and $d \in \{1, 2, 4, 8, r+s, 2(r+s), 4(r+s), 8(r+s)\}$.*

Proof. Let $G = C_{4r} \cup C_{4s}$ where $r, s \ge 1$. The cases $d = 1$, $d = 2$, and $d = 8(r+s)$ can be obtained from the fact that such a G necessarily admits an α-labeling (see [1]).
Case 1: $d = 4$.
Let $c = 2(4r+4s)/4+1$, so the complete multipartite graph we are working in is $K_{c\times d} = K_{(2r+2s+1)\times 4}$. Let $C_{4r} = G_1 + G_2 + (2r-1, 4r+4s+1)$ and $C_{4s} = G_3 + G_4 + (4r+6s+1, 6r+8s+3)$ where

$$
\begin{aligned}
G_1 &= Q(0, 2r+4s+2, 2r-1),\\
G_2 &= P(r-1, r-1, 2r),\\
G_3 &= Q(4r+4s+2, 6r+6s+4, 2s-1),\\
G_4 &= P(4r+5s+1, 6r+5s+1, 2s).
\end{aligned}
$$

First, we show that $G_1 + G_2 + (2r-1, 4r+4s+1)$ is a cycle of length $4r$ and $G_3 + G_4 + (4r+6s+1, 6r+8s+3)$ is a cycle of length $4s$. Note that by (Q1) and (P1), the first vertex of G_1 is $4r+4s+1$, and the last is $r-1$; the first vertex of G_2 is $r-1$, and the last is $2r-1$; the first vertex of G_3 is $6r+8s+3$, and the last is $4r+5s+1$; and the first vertex of G_4 is $4r+5s+1$, and the last is $4r+6s+1$. For $1 \le i \le 4$, let A_i and B_i denote the sets labeled A' and B' in (Q2) and (P2) corresponding to the path G_i. Then using (Q2) and (P2), we compute

$$
\begin{aligned}
A_1 &= [0, r-1], & B_1 &= [3r+4s+2, 4r+4s+1],\\
A_2 &= [r-1, 2r-1], & B_2 &= [2r, 3r-1],\\
A_3 &= [4r+4s+2, 4r+5s+1], & B_3 &= [6r+7s+4, 6r+8s+3],\\
A_4 &= [4r+5s+1, 4r+6s+1], & B_4 &= [6r+6s+2, 6r+7s+1].
\end{aligned}
$$

Thus, $A_1 \le A_2 < B_2 < B_1 < A_3 \le A_4 < B_4 < B_3$. Note that $V(G_1) \cap V(G_2) = \{r-1\}$ and $V(G_3) \cap V(G_4) = \{4r+5s+1\}$; otherwise, G_i and

G_j are vertex-disjoint for $i \neq j$. Therefore, $G_1 + G_2 + (2r-1, 4r+4s+1)$ is a cycle of length $4r$ and $G_3 + G_4 + (4r+6s+1, 6r+8s+3)$ is a cycle of length $4s$.

Next, let E_i denote the set of edge labels in G_i for $1 \leq i \leq 4$. By (Q3) and (P3), we have edge labels

$$E_1 = [2r+4s+3, 4r+4s+1], \qquad E_2 = [1, 2r],$$
$$E_3 = [2r+2s+3, 2r+4s+1], \qquad E_4 = [2r+1, 2r+2s].$$

Moreover, the path $(2r-1, 4r+4s+1)$ consists of an edge with label $2r+4s+2$, and the path $(4r+6s+1, 6r+8s+3)$ consists of an edge with label $2r+2s+2$. Thus, the edge set of G has one edge of each label i where $1 \leq i \leq 4r+4s+1$ except $2r+2s+1$. That is, the set of edge labels is $[1, \lfloor cd/2 \rfloor] \setminus c\mathbb{Z}$. Therefore, we have a 4-modular ρ-labeling of G.

Case 2: $d = 8$.
Let $c = 2(4r+4s)/8+1$, so the complete multipartite graph we are working in is $K_{c\times d} = K_{(r+s+1)\times 8}$. Without loss of generality, we can assume that $r \leq s$.

Case 2.1: $r+s$ is even.
Let $C_{4r} = G_1 + G_2 + (2r-1, 4r+4s+3)$ and $C_{4s} = G_3 + G_4 + G_5 + G_6 + (4r+6s+4, 6r+8s+7)$ where

$$G_1 = Q(0, 2r+4s+4, 2r-1),$$
$$G_2 = P(r-1, r-1, 2r),$$
$$G_3 = Q(4r+4s+4, 7r+7s+7, s-r),$$
$$G_4 = Q(\tfrac{r+s}{2} + 3r+4s+5, \tfrac{r+s}{2} + 5r+6s+8, r+s-1),$$
$$G_5 = P(4r+5s+4, 5r+6s+5, r+s),$$
$$G_6 = P(\tfrac{r+s}{2} + 4r+5s+4, \tfrac{r+s}{2} + 6r+5s+4, s-r).$$

If we continue as in the proof for Case 1, we can see that we have an 8-modular ρ-labeling of G.

Case 2.2: $r+s$ is odd.
Let $C_{4r} = G_1 + G_2 + (2r-1, 4r+4s+3)$ and $C_{4s} = G_3 + G_4 + G_5 + G_6 + (4r+6s+4, 6r+8s+7)$ where

$$G_1 = Q(0, 2r+4s+4, 2r-1),$$
$$G_2 = P(r-1, r-1, 2r),$$
$$G_3 = Q(4r+4s+4, 7r+7s+7, s-r),$$
$$G_4 = P(\tfrac{r+s-1}{2} + 3r+4s+4, \tfrac{r+s-1}{2} + 5r+6s+7, r+s-1),$$
$$G_5 = P(4r+5s+3, 5r+6s+4, r+s),$$
$$G_6 = Q(\tfrac{r+s-1}{2} + 4r+5s+5, \tfrac{r+s-1}{2} + 6r+5s+5, s-r).$$

If we continue as in the proof for Case 1, we can see that we have an 8-modular ρ-labeling of G.

Case 3: $d = r + s$.
Let $c = 2(4r + 4s)/(r + s) + 1$, so the complete multipartite graph we are working in is $K_{c\times d} = K_{9\times(r+s)}$.
Case 3.1: $r \equiv s \equiv 0 \pmod 4$.
Let $C_{4r} = G_1 + G_2 + (9 \cdot \frac{r}{4} - 2, 9 \cdot \frac{r}{2} + 9 \cdot \frac{s}{2} - 1)$ and $C_{4s} = G_3 + G_4 + (9 \cdot \frac{r}{2} + 27 \cdot \frac{s}{4} - 2, 27 \cdot \frac{r}{4} + 9s - 1)$ where

$$\begin{aligned}
G_1 &= \textstyle\sum_{i=1}^{\frac{r}{4}-1} \big(Q(5i - 5, 9 \cdot \frac{r}{2} + 9 \cdot \frac{s}{2} - 4i - 5, 8)\big)\\
&\quad + Q(5 \cdot \tfrac{r}{4} - 5, 7 \cdot \tfrac{r}{2} + 9 \cdot \tfrac{s}{2} - 4, 7),\\
G_2 &= \textstyle\sum_{i=1}^{\frac{r}{4}} \big(P(5 \cdot \frac{r}{4} + 4i - 6, 7 \cdot \frac{r}{2} - 5i - 6, 8)\big),\\
G_3 &= \textstyle\sum_{i=1}^{\frac{s}{4}-1} \big(Q(9 \cdot \frac{r}{2} + 9 \cdot \frac{s}{2} + 5i - 5, 27 \cdot \frac{r}{4} + 9s - 4i - 5, 8)\big)\\
&\quad + Q(9 \cdot \tfrac{r}{2} + 23 \cdot \tfrac{s}{4} - 5, 27 \cdot \tfrac{r}{4} + 8s - 4, 7),\\
G_4 &= \textstyle\sum_{i=1}^{\frac{s}{4}} \big(P(9 \cdot \frac{r}{2} + 23 \cdot \frac{s}{4} + 4i - 6, 27 \cdot \frac{r}{4} + 8s - 5i - 6, 8)\big).
\end{aligned}$$

First, we show that $G_1 + G_2 + (9 \cdot \frac{r}{4} - 2, 9 \cdot \frac{r}{2} + 9 \cdot \frac{s}{2} - 1)$ is a cycle of length $4r$ and $G_3 + G_4 + (9 \cdot \frac{r}{2} + 27 \cdot \frac{s}{4} - 2, 27 \cdot \frac{r}{4} + 9s - 1)$ is a cycle of length $4s$. Note that by (Q1) and (P1), the first vertex of G_1 is $9 \cdot \frac{r}{2} + 9 \cdot \frac{s}{2} - 1$, and the last is $5 \cdot \frac{r}{4} - 2$; the first vertex of G_2 is $5 \cdot \frac{r}{4} - 2$, and the last is $9 \cdot \frac{r}{4} - 2$; the first vertex of G_3 is $27 \cdot \frac{r}{4} + 9s - 1$, and the last is $9 \cdot \frac{r}{2} + 23 \cdot \frac{s}{4} - 2$; and the first vertex of G_4 is $9 \cdot \frac{r}{2} + 23 \cdot \frac{s}{4} - 2$, and the last is $9 \cdot \frac{r}{2} + 27 \cdot \frac{s}{4} - 2$. For $1 \le i \le 4$, let A_i and B_i denote the sets labeled A' and B' in (Q2) and

(P2) corresponding to the path G_i. Then using (Q2) and (P2), we compute

$$
\begin{aligned}
A_1 &= \bigcup_{i=1}^{\frac{r}{4}-1} ([5i-5, 5i-2]) \cup [5\cdot\tfrac{r}{4}-5, 5\cdot\tfrac{r}{4}-2] \subseteq [0, 5\cdot\tfrac{r}{4}-2],\\
B_1 &= \bigcup_{i=1}^{\frac{r}{4}-1} \left([9\cdot\tfrac{r}{2}+9\cdot\tfrac{s}{2}-4i-1, 9\cdot\tfrac{r}{2}+9\cdot\tfrac{s}{2}-4i+3]\right)\\
&\qquad \cup [7\cdot\tfrac{r}{2}+9\cdot\tfrac{s}{2}, 7\cdot\tfrac{r}{2}+9\cdot\tfrac{s}{2}+3]\\
&= [7\cdot\tfrac{r}{2}+9\cdot\tfrac{s}{2}, 9\cdot\tfrac{r}{2}+9\cdot\tfrac{s}{2}-1],\\
A_2 &= \bigcup_{i=1}^{\frac{r}{4}} \left([5\cdot\tfrac{r}{4}+4i-6, 5\cdot\tfrac{r}{4}+4i-2]\right) = [5\cdot\tfrac{r}{4}-2, 9\cdot\tfrac{r}{4}-2],\\
B_2 &= \bigcup_{i=1}^{\frac{r}{4}} \left([7\cdot\tfrac{r}{2}-5i-1, 7\cdot\tfrac{r}{2}-5i+2]\right) \subseteq [9\cdot\tfrac{r}{4}-1, 7\cdot\tfrac{r}{2}-3],\\
A_3 &= \bigcup_{i=1}^{\frac{s}{4}-1} \left([9\cdot\tfrac{r}{2}+9\cdot\tfrac{s}{2}+5i-5, 9\cdot\tfrac{r}{2}+9\cdot\tfrac{s}{2}+5i-2]\right)\\
&\qquad \cup [9\cdot\tfrac{r}{2}+23\cdot\tfrac{s}{4}-5, 9\cdot\tfrac{r}{2}+23\cdot\tfrac{s}{4}-2]\\
&\subseteq [9\cdot\tfrac{r}{2}+9\cdot\tfrac{s}{2}, 9\cdot\tfrac{r}{2}+23\cdot\tfrac{s}{4}-2],\\
B_3 &= \bigcup_{i=1}^{\frac{s}{4}-1} \left([27\cdot\tfrac{r}{4}+9s-4i-1, 27\cdot\tfrac{r}{4}+9s-4i+3]\right)\\
&\qquad \cup [27\cdot\tfrac{r}{4}+8s, 27\cdot\tfrac{r}{4}+8s+3]\\
&= [27\cdot\tfrac{r}{4}+8s, 27\cdot\tfrac{r}{4}+9s-1],\\
A_4 &= \bigcup_{i=1}^{\frac{s}{4}} \left([9\cdot\tfrac{r}{2}+23\cdot\tfrac{s}{4}+4i-6, 9\cdot\tfrac{r}{2}+23\cdot\tfrac{s}{4}+4i-2]\right)\\
&= [9\cdot\tfrac{r}{2}+23\cdot\tfrac{s}{4}-2, 9\cdot\tfrac{r}{2}+27\cdot\tfrac{s}{4}-2],\\
B_4 &= \bigcup_{i=1}^{\frac{s}{4}} \left([27\cdot\tfrac{r}{4}+8s-5i-1, 27\cdot\tfrac{r}{4}+8s-5i+2]\right)\\
&\subseteq [27\cdot\tfrac{r}{4}+27\cdot\tfrac{s}{4}-1, 27\cdot\tfrac{r}{4}+8s-3].
\end{aligned}
$$

Thus, $A_1 \le A_2 < B_2 < B_1 < A_3 \le A_4 < B_4 < B_3$. Note that $V(G_1) \cap V(G_2) = \{5\cdot\frac{r}{4}-2\}$ and $V(G_3) \cap V(G_4) = \{9\cdot\frac{r}{2}+23\cdot\frac{s}{4}-2\}$; otherwise, G_i and G_j are vertex-disjoint for $i \ne j$. Therefore, $G_1+G_2+(9\cdot\frac{r}{4}-2, 9\cdot\frac{r}{2}+9\cdot\frac{s}{2}-1)$ is a cycle of length $4r$ and $G_3+G_4+(9\cdot\frac{r}{2}+27\cdot\frac{s}{4}-2, 27\cdot\frac{r}{4}+9s-1)$ is a cycle of length $4s$.

Next, let E_i denote the set of edge labels in G_i for $1 \le i \le 4$. By (Q3) and (P3), we have edge labels

$$\begin{aligned}
E_1 &= \bigcup_{i=1}^{\frac{r}{4}-1} \left([9 \cdot \tfrac{r}{2} + 9 \cdot \tfrac{s}{2} - 9i + 1, 9 \cdot \tfrac{r}{2} + 9 \cdot \tfrac{s}{2} - 9i + 8]\right) \\
&\qquad \cup [9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{2} + 2, 9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{2} + 8] \\
&= [9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{2} + 2, 9 \cdot \tfrac{r}{2} + 9 \cdot \tfrac{s}{2} - 1] \\
&\qquad \setminus \{9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{2} + 9, 9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{2} + 18, \ldots, 9 \cdot \tfrac{r}{2} + 9 \cdot \tfrac{s}{2} - 9\}, \\
E_2 &= \bigcup_{i=1}^{\frac{r}{4}} \left([9 \cdot \tfrac{r}{4} - 9i + 1, 9 \cdot \tfrac{r}{4} - 9i + 8]\right) \\
&= [1, 9 \cdot \tfrac{r}{4} - 1] \setminus \{9, 18, \ldots, 9 \cdot \tfrac{r}{4} - 9\}, \\
E_3 &= \bigcup_{i=1}^{\frac{s}{4}-1} \left([9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{2} - 9i + 1, 9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{2} - 9i + 8]\right) \\
&\qquad \cup [9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{4} + 2, 9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{4} + 8] \\
&= [9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{4} + 2, 9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{2} - 1] \\
&\qquad \setminus \{9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{4} + 9, 9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{4} + 18, \ldots, 9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{2} - 9\}, \\
E_4 &= \bigcup_{i=1}^{\frac{s}{4}} \left([9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{4} - 9i + 1, 9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{4} - 9i + 8]\right) \\
&= [9 \cdot \tfrac{r}{4} + 1, 9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{4} - 1] \\
&\qquad \setminus \{9 \cdot \tfrac{r}{4} + 9, 9 \cdot \tfrac{r}{4} + 18, \ldots, 9 \cdot \tfrac{r}{4} + 9 \cdot \tfrac{s}{4} - 9\}.
\end{aligned}$$

Moreover, the path $(9 \cdot \frac{r}{4} - 2, 9 \cdot \frac{r}{2} + 9 \cdot \frac{s}{2} - 1)$ consists of an edge with label $9 \cdot \frac{r}{4} + 9 \cdot \frac{s}{2} + 1$, and the path $(9 \cdot \frac{r}{2} + 27 \cdot \frac{s}{4} - 2, 27 \cdot \frac{r}{4} + 9s - 1)$ consists of the edge with label $9 \cdot \frac{r}{4} + 9 \cdot \frac{s}{4} + 1$. Thus, the edge set of G has one edge of each label i, where $1 \leq i \leq 9 \cdot \frac{r}{2} + 9 \cdot \frac{s}{2} - 1$ except 9, 18, ..., $9 \cdot \frac{r}{2} + 9 \cdot \frac{s}{2} - 9$. That is, the set of edge labels is $[1, \lfloor cd/2 \rfloor] \setminus c\mathbb{Z}$. Therefore, we have an $(r+s)$-modular ρ-labeling of G.

Case 3.2: $r \equiv 0$ and $s \equiv 1 \pmod 4$.
If $s = 1$, let $C_{4s} = (27 \cdot \frac{r}{4} + 9, 9 \cdot \frac{r}{2} + 5, 27 \cdot \frac{r}{4} + 7, 9 \cdot \frac{r}{2} + 6, 27 \cdot \frac{r}{4} + 9)$. Otherwise, let $C_{4r} = G_1 + G_2 + (9 \cdot \frac{r}{4} - 1, 9 \cdot \frac{r}{2} + 9 \cdot \frac{s-1}{2} + 4)$ and $C_{4s} = G_3 + (9 \cdot \frac{r}{2} + 23 \cdot \frac{s-1}{4} + 5, 27 \cdot \frac{r}{4} + 8s - 1, 9 \cdot \frac{r}{2} + 23 \cdot \frac{s-1}{4} + 6) + G_4 + (9 \cdot \frac{r}{2} + 27 \cdot \frac{s-1}{4} + 6, 27 \cdot \frac{r}{4} + 9s)$ where

$$\begin{aligned}
G_1 &= Q(0, 9 \cdot \tfrac{r}{2} + 9 \cdot \tfrac{s-1}{2}, 4) + \textstyle\sum_{i=1}^{\frac{r}{4}-1} \left(Q(5i - 2, 9 \cdot \tfrac{r}{2} + 9 \cdot \tfrac{s-1}{2} - 4i - 2, 8)\right) \\
&\qquad + Q(5 \cdot \tfrac{r}{4} - 2, 7 \cdot \tfrac{r}{2} + 9 \cdot \tfrac{s-1}{2} + 3, 3), \\
G_2 &= \textstyle\sum_{i=1}^{\frac{r}{4}} \left(P(5 \cdot \tfrac{r}{4} + 4i - 5, 7 \cdot \tfrac{r}{2} - 5i - 5, 8)\right), \\
G_3 &= Q(9 \cdot \tfrac{r}{2} + 9 \cdot \tfrac{s-1}{2} + 5, 27 \cdot \tfrac{r}{4} + 9s - 4, 4) \\
&\qquad + \textstyle\sum_{i=1}^{\frac{s-1}{4}-1} \left(Q(9 \cdot \tfrac{r}{2} + 9 \cdot \tfrac{s-1}{2} + 5i + 3, 27 \cdot \tfrac{r}{4} + 9s - 4i - 6, 8)\right) \\
&\qquad + Q(9 \cdot \tfrac{r}{2} + 23 \cdot \tfrac{s-1}{4} + 3, 27 \cdot \tfrac{r}{4} + 8s - 2, 5), \\
G_4 &= \textstyle\sum_{i=1}^{\frac{s-1}{4}} \left(P(9 \cdot \tfrac{r}{2} + 23 \cdot \tfrac{s-1}{4} + 4i + 2, 27 \cdot \tfrac{r}{4} + 8s - 5i - 6, 8)\right).
\end{aligned}$$

If we continue as in the proof for Case 3.1, we can see that we have an $(r+s)$-modular ρ-labeling of G.

Case 3.3: $r \equiv 0$ and $s \equiv 2 \pmod 4$.
Let $C_{4r} = G_1 + G_2 + (9 \cdot \frac{r}{4} - 2, 9 \cdot \frac{r}{2} + 9 \cdot \frac{s-2}{2} + 8)$ and $C_{4s} = G_3 + G_4 + (9 \cdot \frac{r}{2} + 27 \cdot \frac{s-2}{4} + 12, 27 \cdot \frac{r}{4} + 9s - 1)$ where

$$\begin{aligned} G_1 &= \textstyle\sum_{i=1}^{\frac{r}{4}-1} \big(Q(5i-5, 9 \cdot \frac{r}{2} + 9 \cdot \frac{s-2}{2} - 4i + 4, 8)\big) \\ &\quad + Q(5 \cdot \frac{r}{4} - 5, 7 \cdot \frac{r}{2} + 9 \cdot \frac{s-2}{2} + 5, 7), \\ G_2 &= \textstyle\sum_{i=1}^{\frac{r}{4}} \big(P(5 \cdot \frac{r}{4} + 4i - 6, 7 \cdot \frac{r}{2} - 5i - 6, 8)\big), \\ G_3 &= \textstyle\sum_{i=1}^{\frac{s-2}{4}} \big(Q(9 \cdot \frac{r}{2} + 9 \cdot \frac{s-2}{2} + 5i + 4, 27\frac{r}{4} + 9s - 4i - 5, 8)\big) \\ &\quad + Q(9 \cdot \frac{r}{2} + 23 \cdot \frac{s-2}{4} + 9, 27 \cdot \frac{r}{4} + 8s - 2, 3), \\ G_4 &= P(9 \cdot \frac{r}{2} + 23 \cdot \frac{s-2}{4} + 10, 27 \cdot \frac{r}{4} + 8s - 6, 4) \\ &\quad + \textstyle\sum_{i=1}^{\frac{s-2}{4}} \big(P(9 \cdot \frac{r}{2} + 23 \cdot \frac{s-2}{4} + 4i + 8, 27 \cdot \frac{r}{4} + 8s - 5i - 8, 8)\big). \end{aligned}$$

If we continue as in the proof for Case 3.1, we can see that we have an $(r+s)$-modular ρ-labeling of G.

Case 3.4: $r \equiv 0$ and $s \equiv 3 \pmod 4$.
Let $C_{4r} = G_1 + G_2 + (9 \cdot \frac{r}{4} - 1, 9 \cdot \frac{r}{2} + 9 \cdot \frac{s-3}{2} + 13)$ and $C_{4s} = G_3 + (27 \cdot \frac{r}{4} + 8s + 1, 9 \cdot \frac{r}{2} + 23 \cdot \frac{s-3}{4} + 17) + G_4 + (9 \cdot \frac{r}{2} + 27 \cdot \frac{s-3}{4} + 20, 27 \cdot \frac{r}{4} + 9s)$ where

$$\begin{aligned} G_1 &= Q(0, 9 \cdot \frac{r}{2} + 9 \cdot \frac{s-3}{2} + 9, 4) \\ &\quad + \textstyle\sum_{i=1}^{\frac{r}{4}-1} \big(Q(5i-2, 9 \cdot \frac{r}{2} + 9 \cdot \frac{s-3}{2} - 4i + 7, 8)\big) \\ &\quad + Q(5 \cdot \frac{r}{4} - 2, 7 \cdot \frac{r}{2} + 9 \cdot \frac{s-3}{2} + 12, 3), \\ G_2 &= \textstyle\sum_{i=1}^{\frac{r}{4}} \big(P(5 \cdot \frac{r}{4} + 4i - 5, 7 \cdot \frac{r}{2} - 5i - 5, 8)\big), \\ G_3 &= Q(9 \cdot \frac{r}{2} + 9 \cdot \frac{s-3}{2} + 14, 27 \cdot \frac{r}{4} + 9s - 4, 4) \\ &\quad + \textstyle\sum_{i=1}^{\frac{s-3}{4}} \big(Q(9 \cdot \frac{r}{2} + 9 \cdot \frac{s-3}{2} + 5i + 12, 27 \cdot \frac{r}{4} + 9s - 4i - 6, 8)\big), \\ G_4 &= P(9 \cdot \frac{r}{2} + 23 \cdot \frac{s-3}{4} + 17, 27 \cdot \frac{r}{4} + 8s - 7, 6) \\ &\quad + \textstyle\sum_{i=1}^{\frac{s-3}{4}} \big(P(9 \cdot \frac{r}{2} + 23 \cdot \frac{s-3}{4} + 4i + 16, 27 \cdot \frac{r}{4} + 8s - 5i - 8, 8)\big). \end{aligned}$$

If we continue as in the proof for Case 3.1, we can see that we have an $(r+s)$-modular ρ-labeling of G.

Case 3.5: $r \equiv s \equiv 1 \pmod 4$.
If $s = 1$, let $C_{4s} = (27 \cdot \frac{r-1}{4} + 15, 9 \cdot \frac{r-1}{2} + 9, 27 \cdot \frac{r-1}{4} + 13, 9 \cdot \frac{r-1}{2} + 10, 27 \cdot \frac{r-1}{4} + 15)$. Otherwise, let $C_{4r} = G_1 + (7 \cdot \frac{r-1}{2} + 9 \cdot \frac{s-1}{2} + 8, 5 \cdot \frac{r-1}{4}, 14 \cdot \frac{r-1}{4} + 2, 5 \cdot \frac{r-1}{4} + 1) + G_2 + (9 \cdot \frac{r-1}{4} + 1, 9 \cdot \frac{r-1}{2} + 9 \cdot \frac{s-1}{2} + 8)$ and

$C_{4s} = G_3 + G_4 + (9 \cdot \frac{r-1}{2} + 27 \cdot \frac{s-1}{4} + 10, 27 \cdot \frac{r-1}{4} + 9s + 6)$ where

$$
\begin{aligned}
G_1 &= \textstyle\sum_{i=1}^{\frac{r-1}{4}} \big(Q(5i-5, 9 \cdot \frac{r-1}{2} + 9 \cdot \frac{s-1}{2} - 4i + 4, 8)\big),\\
G_2 &= \textstyle\sum_{i=1}^{\frac{r-1}{4}} \big(P(5 \cdot \frac{r-1}{4} + 4i - 3, 7 \cdot \frac{r-1}{2} - 5i - 3, 8)\big),\\
G_3 &= Q(9 \cdot \tfrac{r-1}{2} + 9 \cdot \tfrac{s-1}{2} + 9, 27 \cdot \tfrac{r-1}{4} + 9s, 6)\\
&\quad + \textstyle\sum_{i=1}^{\frac{s-1}{4}-1} \big(Q(9 \cdot \frac{r-1}{2} + 9 \cdot \frac{s-1}{2} + 5i + 8, 27 \cdot \frac{r-1}{4} + 9s - 4i - 1, 8)\big)\\
&\quad + Q(9 \cdot \tfrac{r-1}{2} + 23 \cdot \tfrac{s-1}{4} + 8, 27 \cdot \tfrac{r-1}{4} + 8s + 5, 3),\\
G_4 &= P(9 \cdot \tfrac{r-1}{2} + 23 \cdot \tfrac{s-1}{4} + 9, 27 \cdot \tfrac{r-1}{4} + 8s + 1, 4)\\
&\quad + \textstyle\sum_{i=1}^{\frac{s-1}{4}-1} \big(P(9 \cdot \frac{r-1}{2} + 23 \cdot \frac{s-1}{4} + 4i + 7, 27 \cdot \frac{r-1}{4} + 8s - 5i - 1, 8)\big)\\
&\quad + P(9 \cdot \tfrac{r-1}{2} + 27 \cdot \tfrac{s-1}{4} + 7, 27 \cdot \tfrac{r-1}{4} + 27 \cdot \tfrac{s-1}{4} + 9, 6).
\end{aligned}
$$

If we continue as in the proof for Case 3.1, we can see that we have an $(r+s)$-modular ρ-labeling of G.

Case 3.6: $r \equiv 1$ and $s \equiv 2 \pmod 4$.
If $r = 1$, let $C_{4r} = (9 \cdot \frac{s-2}{2} + 13, 0, 2, 1, 9 \cdot \frac{s-2}{2} + 13)$. If $s = 2$, let $C_{4s} = (27 \cdot \frac{r-1}{4} + 25, 9 \cdot \frac{r-1}{2} + 14, 27 \cdot \frac{r-1}{4} + 24, 9 \cdot \frac{r-1}{2} + 16, 27 \cdot \frac{r-1}{4} + 22, 9 \cdot \frac{r-1}{2} + 17, 27 \cdot \frac{r-1}{4} + 21, 9 \cdot \frac{r-1}{2} + 18, 27 \cdot \frac{r-1}{4} + 25)$. Otherwise, let $C_{4r} = G_1 + (5 \cdot \frac{r-1}{4}, 7 \cdot \frac{r-1}{2} + 2, 5 \cdot \frac{r-1}{4} + 1) + G_2 + (9 \cdot \frac{r-1}{4} + 1, 9 \cdot \frac{r-1}{2} + 9 \cdot \frac{s-2}{2} + 13)$ and $C_{4s} = G_3 + (27 \cdot \frac{r-1}{4} + 8s + 8, 9 \cdot \frac{r-1}{2} + 23 \cdot \frac{s-2}{4} + 16) + G_4 + (9 \cdot \frac{r-1}{2} + 27 \cdot \frac{s-2}{4} + 18, 27 \cdot \frac{r-1}{4} + 9s + 7, 9 \cdot \frac{r-1}{2} + 9 \cdot \frac{s-2}{2} + 14, 27 \cdot \frac{r-1}{4} + 9s + 6)$ where

$$
\begin{aligned}
G_1 &= Q(0, 9 \cdot \tfrac{r-1}{2} + 9 \cdot \tfrac{s-2}{2} + 9, 4)\\
&\quad + \textstyle\sum_{i=1}^{\frac{r-1}{4}-1} \big(Q(5i-2, 9 \cdot \frac{r-1}{2} + 9 \cdot \frac{s-2}{2} - 4i + 7, 8)\big)\\
&\quad + Q(5 \cdot \tfrac{r-1}{4} - 2, 7 \cdot \tfrac{r-1}{2} + 9 \cdot \tfrac{s-2}{2} + 10, 5),\\
G_2 &= \textstyle\sum_{i=1}^{\frac{r-1}{4}} \big(P(5 \cdot \frac{r-1}{4} + 4i - 3, 7 \cdot \frac{r-1}{2} - 5i - 3, 8)\big),\\
G_3 &= \textstyle\sum_{i=1}^{\frac{s-2}{4}} \big(Q(9 \cdot \frac{r-1}{2} + 9 \cdot \frac{s-2}{2} + 5i + 11, 27 \cdot \frac{r-1}{4} + 9s - 4i + 2, 8)\big),\\
G_4 &= P(9 \cdot \tfrac{r-1}{2} + 23 \cdot \tfrac{s-2}{4} + 16, 27 \cdot \tfrac{r-1}{4} + 8s, 6)\\
&\quad + \textstyle\sum_{i=1}^{\frac{s-2}{4}-1} \big(P(9 \cdot \frac{r-1}{2} + 23 \cdot \frac{s-2}{4} + 4i + 15, 27 \cdot \frac{r-1}{4} + 8s - 5i - 1, 8)\big)\\
&\quad + P(9 \cdot \tfrac{r-1}{2} + 27 \cdot \tfrac{s-2}{4} + 15, 27 \cdot \tfrac{r-1}{4} + 27 \cdot \tfrac{s-2}{4} + 17, 6).
\end{aligned}
$$

If we continue as in the proof for Case 3.1, we can see that we have an $(r+s)$-modular ρ-labeling of G.

Case 3.7: $r \equiv 1$ and $s \equiv 3 \pmod 4$.
If $s = 3$, let $C_{4s} = (27 \cdot \frac{r-1}{4} + 33, 9 \cdot \frac{r-1}{2} + 18, 27 \cdot \frac{r-1}{4} + 32, 9 \cdot \frac{r-1}{2} + 19, 27 \cdot \frac{r-1}{4} + 31, 9 \cdot \frac{r-1}{2} + 20, 27 \cdot \frac{r-1}{4} + 28, 9 \cdot \frac{r-1}{2} + 21, 27 \cdot \frac{r-1}{4} + 27, 9 \cdot \frac{r-1}{2} + 22, 27 \cdot \frac{r-1}{4} + 26, 9 \cdot \frac{r-1}{2} + 23, 27 \cdot \frac{r-1}{4} + 33)$. Otherwise, let $C_{4r} = G_1 + (7 \cdot \frac{r-1}{2} + 9 \cdot$

$\frac{s-3}{2}+17, 5\cdot\frac{r-1}{4}, 7\cdot\frac{r-1}{2}+2, 5\cdot\frac{r-1}{4}+1)+G_2+(9\cdot\frac{r-1}{4}+1, 9\cdot\frac{r-1}{2}+9\cdot\frac{s-3}{2}+17)$ and $C_{4s} = G_3 + G_4 + (9\cdot\frac{r-1}{2} + 27\cdot\frac{s-3}{4} + 23, 27\cdot\frac{r-1}{4} + 9s + 6)$ where

$$G_1 = \sum_{i=1}^{\frac{r-1}{4}} \left(Q(5i-5, 9\cdot\tfrac{r-1}{2} + 9\cdot\tfrac{s-3}{2} - 4i + 13, 8)\right),$$
$$G_2 = \sum_{i=1}^{\frac{r-1}{4}} \left(P(5\cdot\tfrac{r-1}{4} + 4i - 3, 7\cdot\tfrac{r-1}{2} - 5i - 3, 8)\right),$$
$$\begin{aligned} G_3 = {} & Q(9\cdot\tfrac{r-1}{2} + 9\cdot\tfrac{s-3}{2} + 18, 27\cdot\tfrac{r-1}{4} + 9s, 6) \\ & + \sum_{i=1}^{\frac{s-3}{4}-1} \left(Q(9\cdot\tfrac{r-1}{2} + 9\cdot\tfrac{s-3}{2} + 5i + 17, 27\cdot\tfrac{r-1}{4} + 9s - 4i - 1, 8)\right) \\ & + Q(9\cdot\tfrac{r-1}{2} + 23\cdot\tfrac{s-3}{4} + 17, 27\cdot\tfrac{r-1}{4} + 8s + 3, 7), \end{aligned}$$
$$\begin{aligned} G_4 = {} & \sum_{i=1}^{\frac{s-3}{4}} \left(P(9\cdot\tfrac{r-1}{2} + 23\cdot\tfrac{s-3}{4} + 4i + 16, 27\cdot\tfrac{r-1}{4} + 8s - 5i + 1, 8)\right) \\ & + P(9\cdot\tfrac{r-1}{2} + 27\cdot\tfrac{s-3}{4} + 20, 27\cdot\tfrac{r-1}{2} + 27\cdot\tfrac{s-3}{4} + 22, 6). \end{aligned}$$

If we continue as in the proof for Case 3.1, we can see that we have an $(r+s)$-modular ρ-labeling of G.

Case 3.8: $r \equiv s \equiv 2 \pmod 4$.
If $s = 2$, let $C_{4s} = (27\cdot\frac{r-2}{4} + 31, 9\cdot\frac{r-2}{2} + 18, 27\cdot\frac{r-2}{4} + 30, 9\cdot\frac{r-2}{2} + 19, 27\cdot\frac{r-2}{4} + 27, 9\cdot\frac{r-2}{2} + 20, 27\cdot\frac{r-2}{4} + 26, 9\cdot\frac{r-2}{2} + 21, 27\cdot\frac{r-2}{4} + 31)$. Otherwise, let $C_{4r} = G_1 + G_2 + (9\cdot\frac{r-2}{4} + 3, 9\cdot\frac{r-2}{2} + 9\cdot\frac{s-2}{2} + 17)$ and $C_{4s} = G_3 + G_4 + (9\cdot\frac{r-2}{2} + 27\cdot\frac{s-2}{4} + 21, 27\cdot\frac{r-2}{4} + 9s + 13)$ where

$$\begin{aligned} G_1 = {} & \sum_{i=1}^{\frac{r-2}{4}} \left(Q(5i-5, 9\cdot\tfrac{r-2}{2} + 9\cdot\tfrac{s-2}{2} - 4i + 13, 8)\right) \\ & + Q(5\cdot\tfrac{r-2}{4}, 7\cdot\tfrac{r-2}{2} + 9\cdot\tfrac{s-2}{2} + 14, 3), \end{aligned}$$
$$\begin{aligned} G_2 = {} & P(5\cdot\tfrac{r-2}{4} + 1, 7\cdot\tfrac{r-2}{2} + 1, 4) \\ & + \sum_{i=1}^{\frac{r-2}{4}} \left(P(5\cdot\tfrac{r-2}{4} + 4i - 1, 7\cdot\tfrac{r-2}{2} - 5i - 1, 8)\right), \end{aligned}$$
$$\begin{aligned} G_3 = {} & Q(9\cdot\tfrac{r-2}{2} + 9\cdot\tfrac{s-2}{2} + 18, 27\cdot\tfrac{r-2}{4} + 9s + 9, 4) \\ & + \sum_{i=1}^{\frac{s-2}{4}-1} \left(Q(9\cdot\tfrac{r-2}{2} + 9\cdot\tfrac{s-2}{2} + 5i + 16, 27\cdot\tfrac{r-2}{4} + 9s - 4i + 7, 8)\right) \\ & + Q(9\cdot\tfrac{r-2}{2} + 23\cdot\tfrac{s-2}{4} + 16, 27\cdot\tfrac{r-2}{4} + 8s + 10, 7), \end{aligned}$$
$$\begin{aligned} G_4 = {} & \sum_{i=1}^{\frac{s-2}{4}} \left(P(9\cdot\tfrac{r-2}{2} + 23\cdot\tfrac{s-2}{4} + 4i + 15, 27\cdot\tfrac{r-2}{4} + 8s - 5i + 8, 8)\right) \\ & + P(9\tfrac{r-2}{2} + 27\cdot\tfrac{s-2}{4} + 15, 27\cdot\tfrac{r-2}{4} + 27\cdot\tfrac{r-2}{4} + 27, 4). \end{aligned}$$

If we continue as in the proof for Case 3.1, we can see that we have an $(r+s)$-modular ρ-labeling of G.

Case 3.9: $r \equiv 2$ and $s \equiv 3 \pmod 4$.
If $r = 2$, let $C_{4r} = (9\cdot\frac{s-3}{2} + 22, 0, 9\cdot\frac{s-3}{2} + 21, 1, 5, 2, 4, 3, 9\cdot\frac{s-3}{2} + 22)$. Otherwise, let $C_{4r} = G_1 + G_2 + (9\cdot\frac{r-2}{4} + 3, 9\cdot\frac{r-2}{2} + 9\cdot\frac{s-3}{2} + 22)$ and $C_{4s} = G_3 + (9\cdot\frac{r-2}{2} + 23\cdot\frac{s-3}{4} + 25, 27\cdot\frac{r-2}{4} + 8s + 12, 9\cdot\frac{r-2}{2} + 23\cdot\frac{s-3}{4} +$

$26) + G_4 + (9 \cdot \frac{r-2}{2} + 27 \cdot \frac{s-3}{4} + 28, 27 \cdot \frac{r-2}{4} + 9s + 13)$ where

$$\begin{aligned} G_1 = {} & Q(0, 9 \cdot \tfrac{r-2}{2} + 9 \cdot \tfrac{s-3}{2} + 18, 4) \\ & + \textstyle\sum_{i=1}^{\frac{r-2}{4}-1} \big(Q(5i-2, 9 \cdot \tfrac{r-2}{2} + 9 \cdot \tfrac{s-3}{2} - 4i + 16, 8)\big) \\ & + Q(5 \cdot \tfrac{r-2}{4} - 2, 7 \cdot \tfrac{r-2}{2} + 9 \cdot \tfrac{s-3}{2} + 17, 7), \\ G_2 = {} & P(5 \cdot \tfrac{r-2}{4} + 1, 7 \cdot \tfrac{r-2}{2} + 1, 4) \\ & + \textstyle\sum_{i=1}^{\frac{r-2}{4}} \big(P(5 \cdot \tfrac{r-2}{4} + 4i - 1, 7 \cdot \tfrac{r-2}{2} - 5i - 1, 8)\big), \\ G_3 = {} & \textstyle\sum_{i=1}^{\frac{s-3}{4}} \big(Q(9 \cdot \tfrac{r-2}{2} + 9 \cdot \tfrac{s-3}{2} + 5i + 18, 27 \cdot \tfrac{r-2}{4} + 9s - 4i + 9, 8)\big) \\ & + Q(9 \cdot \tfrac{r-2}{2} + 23 \cdot \tfrac{s-3}{4} + 23, 27 \cdot \tfrac{r-2}{4} + 8s + 11, 5), \\ G_4 = {} & \textstyle\sum_{i=1}^{\frac{s-3}{4}} \big(P(9 \cdot \tfrac{r-2}{2} + 23 \cdot \tfrac{s-3}{4} + 4i + 22, 27 \cdot \tfrac{r-2}{4} + 8s - 5i + 7, 8)\big) \\ & + P(9 \cdot \tfrac{r-2}{2} + 27 \cdot \tfrac{s-3}{4} + 26, 27 \cdot \tfrac{r-2}{4} + 27 \cdot \tfrac{s-3}{4} + 30, 4). \end{aligned}$$

If we continue as in the proof for Case 3.1, we can see that we have an $(r+s)$-modular ρ-labeling of G.

Case 3.10: $r \equiv s \equiv 3 \pmod 4$.
Let $C_{4r} = G_1 + G_2 + (9 \cdot \frac{r-3}{4} + 5, 9 \cdot \frac{r-3}{2} + 9 \cdot \frac{s-3}{2} + 26)$ and $C_{4s} = G_3 + G_4 + (9 \cdot \frac{r-3}{2} + 27 \cdot \frac{s-3}{4} + 32, 27 \cdot \frac{r-3}{4} + 27 \cdot \frac{s-3}{4} + 40, 9 \cdot \frac{r-3}{2} + 27 \cdot \frac{s-3}{4} + 33, 27 \cdot \frac{r-3}{4} - 9s + 20, 9 \cdot \frac{r-3}{2} + 9 \cdot \frac{s-3}{2} + 27, 27 \cdot \frac{r-3}{4} + 9s + 19)$ where

$$\begin{aligned} G_1 = {} & \textstyle\sum_{i=1}^{\frac{r-3}{4}} \big(Q(5i-5, 9 \cdot \tfrac{r-3}{2} + 9 \cdot \tfrac{s-3}{2} - 4i + 22, 8)\big) \\ & + Q(5 \cdot \tfrac{r-3}{4}, 7 \cdot \tfrac{r-3}{2} + 9 \cdot \tfrac{s-3}{2} + 21, 5), \\ G_2 = {} & P(5 \cdot \tfrac{r-3}{4} + 2, 7 \cdot \tfrac{r-3}{2} + 2, 6) \\ & + \textstyle\sum_{i=1}^{\frac{r-3}{4}} \big(P(5 \cdot \tfrac{r-3}{4} + 4i + 1, 7 \cdot \tfrac{r-3}{2} - 5i + 1, 8)\big), \\ G_3 = {} & \textstyle\sum_{i=1}^{\frac{s-3}{4}} \big(Q(9 \cdot \tfrac{r-3}{2} + 9 \cdot \tfrac{s-3}{2} + 5i + 24, 27 \cdot \tfrac{r-3}{4} + 9s - 4i + 15, 8)\big) \\ & + Q(9 \cdot \tfrac{r-3}{2} + 23 \cdot \tfrac{s-3}{4} + 29, 27 \cdot \tfrac{r-3}{4} + 8s + 19, 3), \\ G_4 = {} & P(9 \cdot \tfrac{r-3}{2} + 23 \cdot \tfrac{s-3}{4} + 30, 27 \cdot \tfrac{r-3}{4} + 8s + 15, 4) \\ & + \textstyle\sum_{i=1}^{\frac{s-3}{4}} \big(P(9 \cdot \tfrac{r-3}{2} + 23 \cdot \tfrac{s-3}{4} + 4i + 28, 27 \cdot \tfrac{r-3}{4} + 8s - 5i + 13, 8)\big). \end{aligned}$$

If we continue as in the proof for Case 3.1, we can see that we have an $(r+s)$-modular ρ-labeling of G.

Case 4: $d = 2(r+s)$.
Let $c = 2(4r+4s)/(2r+2s)+1$, so the complete multipartite graph we are working in is $K_{c \times d} = K_{5 \times (2r+2s)}$.

Case 4.1: r is odd, s is odd.
If $s = 1$, let $C_{4s} = (15 \cdot \frac{r-1}{2} + 17, 5r + 5, 15 \cdot \frac{r-1}{2} + 14, 5r + 6, 15 \cdot \frac{r-1}{2} + 17)$.

Otherwise, let $C_{4r} = G_1 + (4r+5s, 3\cdot\frac{r-1}{2}, 4r-2, 3\cdot\frac{r-1}{2}+1) + G_2 + (5\cdot\frac{r-1}{2}+1, 5r+5s-1)$ and $C_{4s} = G_3 + (15\cdot\frac{r-1}{2}+9s+9, 5r+13\cdot\frac{s-1}{2}+4, 15\cdot\frac{r-1}{2}+9s+8, 5r+13\cdot\frac{s-1}{2}+5) + G_4 + (5r+15\cdot\frac{s-1}{2}+5, 15\cdot\frac{r-1}{2}+15\cdot\frac{s-1}{2}+14, 5r+15\cdot\frac{s-1}{2}+6, 15\cdot\frac{r-1}{2}+10s+7, 5r+5s, 15\cdot\frac{r-1}{2}+10s+6)$ where

$$\begin{aligned}
G_1 &= \sum_{i=1}^{\frac{r-1}{2}} Q(3i-3, 5r+5s-2i-3, 4),\\
G_2 &= \sum_{i=1}^{\frac{r-1}{2}} P(3\cdot\tfrac{r-1}{2}+2i-1, 4r-3i-5, 4),\\
G_3 &= \sum_{i=1}^{\frac{s-1}{2}-1} Q(5r+5s+3i-1, 15\cdot\tfrac{r-1}{2}+10s-2i+4, 4),\\
G_4 &= \sum_{i=1}^{\frac{s-1}{2}} P(5r+13\cdot\tfrac{s-1}{2}+2i+3, 15\cdot\tfrac{r-1}{2}+9s-3i+4, 4).
\end{aligned}$$

If we continue as in the proof for Case 3.1, we can see that we have a $(2r+2s)$-modular ρ-labeling of G.

Case 4.2: r is odd, s is even.

Let $C_{4r} = G_1 + (4r+5s, 3\cdot\frac{r-1}{2}, 4r-2, 3\cdot\frac{r-1}{2}+1) + G_2 + (5\cdot\frac{r-1}{2}+1, 5r+5s-1)$ and $C_{4s} = G_3 + (15\cdot\frac{r-1}{2}+9s+8, 5r+13\cdot\frac{s}{2}-1, 15\cdot\frac{r-1}{2}+9s+6, 5r+13\cdot\frac{s}{2}) + G_4 + (5r+15\cdot\frac{s}{2}-2, 15\cdot\frac{r-1}{2}+15\cdot\frac{s}{2}+7, 5r+15\cdot\frac{s}{2}-1, 15\cdot\frac{r-1}{2}+10s+7, 5r+5s, 15\cdot\frac{r-1}{2}+10s+6)$ where

$$\begin{aligned}
G_1 &= \sum_{i=1}^{\frac{r-1}{2}} Q(3i-3, 5r+5s-2i-3, 4),\\
G_2 &= \sum_{i=1}^{\frac{r-1}{2}} P(3\cdot\tfrac{r-1}{2}+2i-1, 4r-3i-5, 4),\\
G_3 &= \sum_{i=1}^{\frac{s}{2}-1} Q(5r+5s+3i-1, 15\cdot\tfrac{r-1}{2}+10s-2i+4, 4),\\
G_4 &= \sum_{i=1}^{\frac{s}{2}-1} P(5r+13\cdot\tfrac{s}{2}+2i-2, 15\cdot\tfrac{r-1}{2}+9s-3i+3, 4).
\end{aligned}$$

If we continue as in the proof for Case 3.1, we can see that we have a $(2r+2s)$-modular ρ-labeling of G.

Case 4.3: r is even, s is odd.

Let $C_{4r} = G_1 + (4r+5s+1, 3\cdot\frac{r}{2}-3, 4r+5s, 3\cdot\frac{r}{2}-2) + G_2 + (5\cdot\frac{r}{2}-2, 5r+5s-1)$ and $C_{4s} = G_3 + (15\cdot\frac{r}{2}+9s, 5r+13\cdot\frac{s-1}{2}+5, 15\cdot\frac{r}{2}+9s-2, 5r+13\cdot\frac{s-1}{2}+6) + G_4 + (5r+15\cdot\frac{s-1}{2}+6, 15\cdot\frac{r}{2}+10s-1)$ where

$$\begin{aligned}
G_1 &= \sum_{i=1}^{\frac{r}{2}-1} Q(3i-3, 5r+5s-2i-3, 4),\\
G_2 &= \sum_{i=1}^{\frac{r}{2}} P(3\cdot\tfrac{r}{2}+2i-4, 4r-3i-4, 4),\\
G_3 &= \sum_{i=1}^{\frac{s-1}{2}} Q(5r+5s+3i-3, 15\cdot\tfrac{r}{2}+10s-2i-3, 4),\\
G_4 &= \sum_{i=1}^{\frac{s-1}{2}} P(5r+13\cdot\tfrac{s-1}{2}+2i+4, 15\cdot\tfrac{r}{2}+9s-3i-5, 4).
\end{aligned}$$

If we continue as in the proof for Case 3.1, we can see that we have a $(2r+2s)$-modular ρ-labeling of G.

Case 4.4: r is even, s is even.
Let $C_{4r} = G_1 + (4r+5s+1, 3\cdot\frac{r}{2}-3, 4r+5s, 3\cdot\frac{r}{2}-2) + G_2 + (5\cdot\frac{r}{2}-2, 5r+5s-1)$ and $C_{4s} = G_3 + (15\cdot\frac{r}{2}+9s+1, 5r+13\cdot\frac{s}{2}-3, 15\cdot\frac{r}{2}+9s, 5r+13\cdot\frac{r}{2}-2) + G_4 + (5r+15\cdot\frac{s}{2}-2, 15\cdot\frac{r}{2}+10s-1)$ where

$$G_1 = \sum_{i=1}^{\frac{r}{2}-1} Q(3i-3, 5r+5s-2i-3, 4),$$
$$G_2 = \sum_{i=1}^{\frac{r}{2}} P(3\cdot\tfrac{r}{2}+2i-4, 4r-3i-4, 4),$$
$$G_3 = \sum_{i=1}^{\frac{s}{2}-1} Q(5r+5s+3i-3, 15\cdot\tfrac{r}{2}+10s-2i-3, 4),$$
$$G_4 = \sum_{i=1}^{\frac{s}{2}} P(5r+13\cdot\tfrac{s}{2}+2i-4, 15\cdot\tfrac{r}{2}+9s-3i-4, 4).$$

If we continue as in the proof for Case 3.1, we can see that we have a $(2r+2s)$-modular ρ-labeling of G.

Case 5: $d = 4(r+s)$.
Let $c = 2(4r+4s)/(4r+4s)+1$, so the complete multipartite graph we are working in is $K_{c\times d} = K_{3\times(4r+4s)}$. Let $C_{4r} = G_1 + (5r+6s, 2r-2) + G_2 + (3r-2, 6r+6s-1)$ and $C_{4s} = G_3 + (9r+11s-1, 6r+8s+1) + G_4 + (6r+9s, 9r+12s-1)$ where

$$G_1 = \sum_{i=1}^{r-1} Q(2i-2, 6r+6s-i-2, 2),$$
$$G_2 = \sum_{i=1}^{r} P(2r-3+i, 5r-3-2i, 2),$$
$$G_3 = \sum_{i=1}^{s} Q(6r+6s+2i-2, 9r+12s-i-2, 2),$$
$$G_4 = \sum_{i=1}^{s-1} P(6r+8s+i, 9r+11s-2i-3, 2).$$

(In the case when $r=1$, the path G_1 is empty, and when $s=1$, the path G_4 is empty. However, this does not change the proof in any way.) If we continue as in the proof for Case 3.1, we can see that we have a $(4r+4s)$-modular ρ-labeling of G. ■

Theorem 9. *Let $G = C_{4r} \cup C_{4s}$ and let $n = 4r+4s$. Then there exists a cyclic G-decomposition of $K_{(2n+1)\times t}$, $K_{(n+1)\times 2t}$, $K_{(n/2+1)\times 4t}$, $K_{(n/4+1)\times 8t}$, $K_{9\times(n/4)t}$, $K_{5\times(n/2)t}$, $K_{3\times nt}$, and of $K_{2\times 2nt}$ for every positive integer t.*

Lemma 10. *A d-modular ρ-labeling of $C_{4r} \cup C_{4s+2}$ exists for $r, s \geq 1$ and $d \in \{1, 4, 2r+2s+1, 4(2r+2s+1)\}$.*

Proof. Let $G = C_{4r} \cup C_{4s+2}$ where $r, s \geq 1$. The cases $d=1$ and $d = 4(2r+2s+1)$ can be obtained from the fact that such a G necessarily admits a ρ^+-labeling (see [4]).

Case 1: $d = 4$.
Let $c = 2(4r+4s+2)/4+1$, so that the complete multipartite graph we are working in is $K_{c\times d} = K_{(2r+2s+2)\times 4}$.

Case 1.1: $r \leq s$.
Let $C_{4r} = G_1 + G_2 + (2r-1, 4r)$ and $C_{4s+2} = G_3 + G_4 + G_5 + (4r+2s+2, 8r+4s+4)$ where

$$G_1 = Q(0, 2r+1, 2r-1),$$
$$G_2 = P(r-1, r-1, 2r),$$
$$G_3 = Q(4r+1, 8r+2s+3, 2s+1),$$
$$G_4 = P(4r+s+1, 6r+3s+3, 2r-1),$$
$$G_5 = Q(5r+s+2, 9r+s+2, 2s-2r+1).$$

If we continue as in the proof for Case 1 in Lemma 1, we can see that we have a 4-modular ρ-labeling of G.

Case 1.2: $r > s$.
Let $C_{4r} = G_1 + G_2 + G_3 + (2r-1, 4r+2, 0)$ and $C_{4s+2} = G_4 + G_5 + (8r+2s+5, 4r+2s+4, 8r+4s+6)$ where

$$G_1 = P(0, 2r+2s+2, 2r-2s-2),$$
$$G_2 = P(r-s-1, 3r-s+2, 2s-2),$$
$$G_3 = P(r-2, r-2, 2r+2),$$
$$G_4 = Q(4r+3, 8r+2s+5, 2s+1),$$
$$G_5 = P(4r+s+3, 8r+s+5, 2s-1).$$

If we continue as in the proof for Case 1 in Lemma 1, we can see that we have a 4-modular ρ-labeling of G.

Case 2: $d = 2r+2s+1$.
Let $c = 2(4r+4s+2)/(2r+2s+1)+1$, so the complete multipartite graph we are working in is $K_{c\times d} = K_{5\times(2r+2s+1)}$. In order to show that G admits a d-modular ρ-labeling, we examine when r is odd or even and when s is odd or even and show that any of the four possible combinations will satisfy the necessary conditions for the desired labeling.

Case 2.1: r is odd.
Let $C_{4r} = G_1 + (9r+5s+4, 13\cdot\frac{r-1}{2}+5s+9, 9r+5s+2, 13\cdot\frac{r-1}{2}+5s+10) + G_2 + (15\cdot\frac{r-1}{2}+5s+10, 10r+5s+3)$ where

$$G_1 = \sum_{i=1}^{\frac{r-1}{2}} Q(5r+5s+3i+1, 10r+5s-2i+1, 4),$$
$$G_2 = \sum_{i=1}^{\frac{r-1}{2}} P(13\cdot\tfrac{r-1}{2}+5s+2i+8, 9r+5s-3i-1, 4).$$

If we continue as in Case 3.1 in Lemma 1, we can see that the set of edge labels is $[1, 5r-1] \setminus c\mathbb{Z}$ with $5r+5s+4 \leq V(C_{4r}) \leq 10r+5s+3$.

Case 2.2: r is even.
Let $C_{4r} = G_1 + (9r + 5s + 5, 13 \cdot \frac{r}{2} + 5s + 1, 9r + 5s + 4, 13 \cdot \frac{r}{2} + 5s + 2) + G_2 + (15 \cdot \frac{r}{2} + 5s + 2, 10r + 5s + 3)$ where

$$G_1 = \sum_{i=1}^{\frac{r}{2}-1} Q(5r + 5s + 3i + 1, 10r + 5s - 2i + 1, 4),$$
$$G_2 = \sum_{i=1}^{\frac{r}{2}} P(13 \cdot \tfrac{r}{2} + 5s + 2i, 9r + 5s - 3i, 4).$$

If we continue as in Case 3.1 in Lemma 1, we can see that the set of edge labels is $[1, 5r - 1] \setminus c\mathbb{Z}$ with $5r + 5s + 4 \leq V(C_{4r}) \leq 10r + 5s + 3$.

Case 2.3: s is odd.
Let $C_{4s+2} = G_3 + (5r + 4s + 2, 3 \cdot \frac{s-1}{2} + 3, 5r + 4s + 1, 3 \cdot \frac{s-1}{2} + 4) + G_4 + (5 \cdot \frac{s-1}{2} + 4, 5r + 5s + 3, 0, 5r + 5s + 1)$ where

$$G_3 = \sum_{i=1}^{\frac{s-1}{2}} Q(3i - 1, 5r + 5s - 2i - 1, 4),$$
$$G_4 = \sum_{i=1}^{\frac{s-1}{2}} P(3 \cdot \tfrac{s-1}{2} + 2i + 2, 5r + 4s - 3i - 2, 4).$$

If we continue as in Case 3.1 in Lemma 1, we can see that the set of edge labels is $[5r + 1, \lfloor (cd - 1)/2 \rfloor] \setminus c\mathbb{Z}$ with $0 \leq V(C_{4s}) \leq 5r + 5s + 3$.

Case 2.4: s is even.
Let $C_{4s+2} = G_3 + (5r + 4s + 3, 3 \cdot \frac{s}{2} - 1, 5r + 4s + 1, 3 \cdot \frac{s}{2}) + G_4 + (5 \cdot \frac{s}{2}, 5r + 5s + 3, 0, 5r + 5s + 1)$ where

$$G_3 = \sum_{i=1}^{\frac{s}{2}-1} Q(3i - 1, 5r + 5s - 2i - 1, 4),$$
$$G_4 = \sum_{i=1}^{\frac{s}{2}} P(3 \cdot \tfrac{s}{2} + 2i - 2, 5r + 4s - 3i - 2, 4).$$

If we continue as in Case 3.1 in Lemma 1, we can see that the set of edge labels is $[5r + 1, \lfloor (cd - 1)/2 \rfloor] \setminus c\mathbb{Z}$ with $0 \leq V(C_{4s}) \leq 5r + 5s + 3$.

Since a labeling of C_{4r} from either of the first two subcases will be vertex disjoint from a labeling of C_{4s+2} from either of the last two subcases, we have a labeling of $G = C_{4r} \cup C_{4s+2}$ where the set of edge labels is $[1, \lfloor cd/2 \rfloor] \setminus c\mathbb{Z}$. Therefore, we have a $(2r + 2s + 1)$-modular ρ-labeling of G. ■

Theorem 11. *Let $G = C_{4r} \cup C_{4s+2}$ where r and s are positive integers and and let $n = 4r + 4s + 2$. Then there exists a cyclic G-decomposition of $K_{(2n+1)\times t}$, $K_{(n/2+1)\times 4t}$, $K_{5\times(n/2)t}$, and of $K_{2\times 2nt}$ for every positive integer t.*

Before proceeding to our final case, we note that the parity condition (i.e., Lemma 5) rules out the existence of a d-modular ρ-labelings of G in Lemma 10 for $d = 2$ and for $d = 4r + 4s + 2$.

Lemma 12. *A d-modular ρ-labeling of* $C_{4r+2} \cup C_{4s+2}$ *exists for* $r, s \geq 1$ *and* $d \in \{1, 2, 4, 8, r+s+1, 2(r+s+1), 4(r+s+1), 8(r+s+1)\}$.

Proof. Let $G = C_{4r+2} \cup C_{4s+2}$ where $1 \leq r \leq s$. The cases $d = 1$, $d = 2$, and $d = 8(r+s)$ can be obtained from the fact that such a G necessarily admits an α-labeling (see [1]).

Case 1: $d = 4$.

Let $c = 2(4r + 4s + 4)/4 + 1$, so that the complete multipartite graph we are working in is $K_{c \times d} = K_{(2r+2s+3)\times 4}$.

Case 1.1: $r = s$.

If $r = s = 1$, let $C_{4r+2} = (0,\ 3,\ 2,\ 6,\ 4,\ 9,\ 0)$ and $C_{4s+2} = (10,\ 22,\ 11,\ 19,\ 13,\ 23,\ 10)$. We leave it to the reader to check that this yields a 4-modular ρ-labeling of G.

If $r = s > 1$, let $C_{4r+2} = G_1 + G_2 + (2r+1, 4r+5, 0)$ and $C_{4s+2} = G_3 + G_4 + (6s+5, 10s+9, 6s+7, 12s+11)$ where

$$G_1 = P(0, 2r+4, 2r-3), \qquad G_2 = Q(r, r, 2r+3),$$
$$G_3 = Q(4s+6, 10s+10, 2s+1), \qquad G_4 = P(5s+6, 9s+11, 2s-2).$$

If we continue as in the proof for Case 1 in Lemma 1, we can see that we have a 4-modular ρ-labeling of G.

Case 1.2: $r < s$.

Let $C_{4r+2} = G_1 + G_2 + (2r+1, 4r+3, 0)$ and $C_{4s+2} = G_3 + G_4 + G_5 + (8r+2s+7, 4r+2s+5, 8r+4s+9)$ where

$$G_1 = P(0, 2r+2, 2r-1),$$
$$G_2 = Q(r+1, r+1, 2r+1),$$
$$G_3 = Q(4r+4, 8r+2s+8, 2s+1),$$
$$G_4 = P(4r+s+4, 6r+3s+7, 2r),$$
$$G_5 = P(5r+s+4, 9r+s+7, 2s-2r-1).$$

If we continue as in the proof for Case 1 in Lemma 1, we can see that we have a 4-modular ρ-labeling of G.

Case 2: $d = 8$.

Let $c = 2(4r + 4s + 4)/8 + 1$, so that the complete multipartite graph we are working in is $K_{c \times d} = K_{(r+s+2)\times 8}$.

Case 2.1: $r = s$.

Let $C_{4r+2} = G_1 + G_2 + (6r+5, 2r+2, 8r+7)$ and $C_{4s+2} = G_3 + (9r+7, 11r+10) + G_4 + (10r+9, 12r+13, 8r+8)$ where

$$G_1 = Q(0, 6r+6, 2r+1), \qquad G_2 = P(r, 5r+5, 2r-1),$$
$$G_3 = P(8r+8, 10r+12, 2r-2), \qquad G_4 = Q(9r+9, 9r+9, 2r+1).$$

If we continue as in the proof for Case 1 in Lemma 1, we can see that we have an 8-modular ρ-labeling of G.

Case 2.2: $r < s < 3r+1$ and $r+s$ is odd.
Let $C_{4r+2} = G_1 + G_2 + G_3 + (2r+4s+5, 2r+1, 4r+4s+7)$ and $C_{4s+2} = G_4 + G_5 + G_6 + G_7 + (4r+6s+9, 4r+8s+13, 4r+4s+8)$ where

$$\begin{aligned}
G_1 &= Q(0, 2r+4s+6, 2r+1),\\
G_2 &= P(r, 4r+3s+6, s-r-1),\\
G_3 &= P(\tfrac{r+s-1}{2}, \tfrac{r+s-1}{2}+4s+5, 3r-s),\\
G_4 &= P(4r+4s+8, 6r+6s+12, 2s-2r-1),\\
G_5 &= Q(3r+5s+9, 3r+7s+13, 2r-1),\\
G_6 &= P(4r+5s+8, 5r+6s+10, s-r+1),\\
G_7 &= P(7\cdot\tfrac{r+s-1}{2}+2s+12, 7\cdot\tfrac{r+s-1}{2}+2s+12, r+s+1).
\end{aligned}$$

If we continue as in the proof for Case 1 in Lemma 1, we can see that we have an 8-modular ρ-labeling of G.

Case 2.3: $r < s < 3r+1$ and $r+s$ is even.
Let $C_{4r+2} = G_1 + G_2 + G_3 + (2r+1, 4r+4s+7)$ and $C_{4s+2} = G_4 + G_5 + G_6 + G_7 + (4r+6s+10, 4r+8s+12)$ where

$$\begin{aligned}
G_1 &= Q(0, 2r+4s+6, 2r+1),\\
G_2 &= P(r, 4r+3s+6, s-r-1),\\
G_3 &= Q(\tfrac{r+s}{2}+1, \tfrac{r+s}{2}+4s+5, 3r-s+1),\\
G_4 &= Q(4r+4s+8, 6r+6s+12, 2s-2r),\\
G_5 &= Q(3r+5s+9, 3r+7s+11, 2r+1),\\
G_6 &= P(4r+5s+9, 5r+6s+11, s-r-1),\\
G_7 &= Q(7\cdot\tfrac{r+s}{2}+2s+10, 7\cdot\tfrac{r+s}{2}+2s+10, r+s+1).
\end{aligned}$$

If we continue as in the proof for Case 1 in Lemma 1, we can see that we have an 8-modular ρ-labeling of G.

Case 2.4: $s = 3r+1$.
Let $C_{4r+2} = G_1 + G_2 + (2r-1, 14r+9, 2r+1, 16r+11)$ and $C_{4s+2} = G_3 + G_4 + G_5 + G_6 + (22r+16, 28r+23, 16r+12)$ where

$$\begin{aligned}
G_1 &= Q(0, 14r+10, 2r+1), & G_2 &= P(r, 13r+11, 2r-2),\\
G_3 &= P(16r+12, 24r+18, 4r+1), & G_4 &= Q(18r+14, 24r+21, 2r-2),\\
G_5 &= Q(19r+14, 23r+17, 2r+3), & G_6 &= P(20r+15, 20r+15, 4r+2).
\end{aligned}$$

If we continue as in the proof for Case 1 in Lemma 1, we can see that we have an 8-modular ρ-labeling of G.

Case 2.5: $s > 3r + 1$ and $r + s$ is odd.
Let $C_{4r+2} = G_1 + G_2 + (2r + 4s + 6, 2r + 1, 4r + 4s + 7)$ and $C_{4s+2} = G_3 + G_4 + G_5 + G_6 + G_7 + (4r + 6s + 10, 4r + 8s + 14, 4r + 4s + 8)$ where

$$
\begin{aligned}
G_1 &= Q(0, 2r + 4s + 6, 2r + 1), \\
G_2 &= P(r, r + 4s + 6, 2r - 1), \\
G_3 &= P(4r + 4s + 8, 7r + 7s + 14, s - 3r - 2), \\
G_4 &= Q(5 \cdot \tfrac{r+s-1}{2} + 2s + 11, p \cdot \tfrac{r+s-1}{2} + 2s + 17, r + s + 1), \\
G_5 &= Q(3r + 5s + 10, 3r + 7s + 14, 2r - 1), \\
G_6 &= P(4r + 5s + 9, 5r + 6s + 11, s - r + 1), \\
G_7 &= P(7 \cdot \tfrac{r+s-1}{2} + 2s + 13, 7 \cdot \tfrac{r+s-1}{2} + 2s + 13, r + s + 1).
\end{aligned}
$$

If we continue as in the proof for Case 1 in Lemma 1, we can see that we have an 8-modular ρ-labeling of G.

Case 2.6: $s > 3r + 1$ and $r + s$ is even.
Let $C_{4r+2} = G_1 + G_2 + (2r + 4s + 6, 2r + 1, 4r + 4s + 7)$ and $C_{4s+2} = G_3 + G_4 + G_5 + G_6 + G_7 + (4r + 6s + 10, 4r + 8s + 14, 4r + 4s + 8)$ where

$$
\begin{aligned}
G_1 &= Q(0, 2r + 4s + 6, 2r + 1), \\
G_2 &= P(r, r + 4s + 6, 2r - 1), \\
G_3 &= P(4r + 4s + 8, 7r + 7s + 14, s - 3r - 2), \\
G_4 &= P(5 \cdot \tfrac{r+s}{2} + 2s + 7, 9 \cdot \tfrac{r+s}{2} + 2s + 11, r + s + 1), \\
G_5 &= Q(3r + 5s + 9, 3r + 7s + 13, 2r - 1), \\
G_6 &= P(4r + 5s + 8, 5r + 6s + 10, s - r + 1), \\
G_7 &= Q(7 \cdot \tfrac{r+s}{2} + 2s + 10, 7 \cdot \tfrac{r+s}{2} + 2s + 10, r + s + 1).
\end{aligned}
$$

If we continue as in the proof for Case 1 in Lemma 1, we can see that we have an 8-modular ρ-labeling of G.

Case 3: $d = r + s + 1$.
Let $c = 2(4r + 4s + 4)/(r + s + 2) + 1$, so that the complete multipartite graph we are working in is $K_{c \times d} = K_{9 \times (r+s+1)}$.

Case 3.1: r and s are both odd.
In order to show that G admits a d-modular ρ-labeling, we examine when $r \equiv 1, 3 \pmod 4$ and when $s \equiv 1, 3 \pmod 4$ and show that any of the four possible combinations will satisfy the necessary conditions for the desired labeling.

Case 3.1.1: $r \equiv 1 \pmod 4$.
If $r = 1$, let $C_{4r+2} = (0, 9 \cdot \frac{s-1}{2} + 12, 1, 9 \cdot \frac{s-1}{2} + 9, 3, 9 \cdot \frac{s-1}{2} + 13, 0)$. We leave it to the reader to check that this yields an $(r + s + 1)$-modular ρ-labeling of G.

If $r > 1$, let $C_{4r+2} = G_1 + G_2 + (9 \cdot \frac{r-1}{4} + 1, 9 \cdot \frac{r+s}{2} - 9 \cdot \frac{r-1}{4}, 9 \cdot \frac{r-1}{4} + 3, 9 \cdot \frac{r+s}{2} + 4)$ where

$$\begin{aligned} G_1 &= Q(0, 9 \cdot \tfrac{r+s}{2}, 4) + \textstyle\sum_{i=1}^{\frac{r-1}{4}-1} \big(Q(5i-2, 9 \cdot \tfrac{r+s}{2} - 4i - 2, 8)\big) \\ &\quad + Q(5 \cdot \tfrac{r-1}{4} - 2, 7 \cdot \tfrac{r+s}{2} + s, 7), \\ G_2 &= \textstyle\sum_{i=1}^{\frac{r-1}{4}} \big(P(5 \cdot \tfrac{r-1}{4} + 4i - 3, 7 \cdot \tfrac{r+s}{2} + s - 5i - 2, 8)\big). \end{aligned}$$

Case 3.1.2: $r \equiv 3 \pmod 4$.
Let $C_{4r+2} = G_1 + G_2 + (9 \cdot \frac{r-3}{4} + 6, 9 \cdot \frac{r+s}{2} - 9 \cdot \frac{r-3}{4} - 4, 9 \cdot \frac{r-3}{4} + 8, 9 \cdot \frac{r+s}{2} + 4)$ where

$$\begin{aligned} G_1 &= Q(0, 9 \cdot \tfrac{r+s}{2}, 4) + \textstyle\sum_{i=1}^{\frac{r-3}{4}} \big(Q(5i-2, 9 \cdot \tfrac{r+s}{2} - 4i - 2, 8)\big) \\ &\quad + Q(5 \cdot \tfrac{r-3}{4} + 3, 7 \cdot \tfrac{r+s}{2} + s + 2, 3), \\ G_2 &= P(5 \cdot \tfrac{r-3}{4} + 4, 7 \cdot \tfrac{r+s}{2} + s - 2, 4) \\ &\quad + \textstyle\sum_{i=1}^{\frac{r-3}{4}} \big(P(5 \cdot \tfrac{r-3}{4} + 4i + 2, 7 \cdot \tfrac{r+s}{2} + s - 5i - 4, 8)\big). \end{aligned}$$

Case 3.1.3: $s \equiv 1 \pmod 4$.
If $s = 1$, let $C_{4s+2} = (9 \cdot \frac{r-1}{2} + 14, 9 \cdot \frac{r-1}{2} + 19, 9 \cdot \frac{r-1}{2} + 16, 9 \cdot \frac{r-1}{2} + 18, 9 \cdot \frac{r-1}{2} + 17, 9 \cdot \frac{r-1}{2} + 21, 9 \cdot \frac{r-1}{2} + 14)$. We leave it to the reader to check that this yields an $(r+s+1)$-modular ρ-labeling of G.

If $s > 1$, let $C_{4s+2} = G_3 + G_4 + (9 \cdot \frac{r+s}{2} + 9 \cdot \frac{s-1}{4} + 8, 9 \cdot \frac{r+s}{2} + 9 \cdot \frac{s-1}{2} + 12, 9 \cdot \frac{r+s}{2} + 5)$ where

$$\begin{aligned} G_3 &= P(9 \cdot \tfrac{r+s}{2} + 5, 9 \cdot \tfrac{r+s}{2} + 9 \cdot \tfrac{s-1}{2} + 5, 5) \\ &\quad + \textstyle\sum_{i=1}^{\frac{s-1}{4}-1} \big(Q(9 \cdot \tfrac{r+s}{2} + 5i + 4, 9 \cdot \tfrac{r+s}{2} + 9 \cdot \tfrac{s-1}{2} - 4i + 4, 8)\big) \\ &\quad + Q(9 \cdot \tfrac{r+s}{2} + 5 \cdot \tfrac{s-1}{4} + 4, 9 \cdot \tfrac{r+s}{2} + 7 \cdot \tfrac{s-1}{2} + 8, 4), \\ G_4 &= Q(9 \cdot \tfrac{r+s}{2} + 5 \cdot \tfrac{s-1}{4} + 7, 9 \cdot \tfrac{r+s}{2} + 7 \cdot \tfrac{s-1}{2} + 7, 3) \\ &\quad + \textstyle\sum_{i=1}^{\frac{s-1}{4}} \big(P(9 \cdot \tfrac{r+s}{2} + 5 \cdot \tfrac{s-1}{4} + 4i + 4, 9 \cdot \tfrac{r+s}{2} + 7 \cdot \tfrac{s-1}{2} - 5i + 4, 8)\big). \end{aligned}$$

Case 3.1.4: $s \equiv 3 \pmod 4$.
Let $C_{4s+2} = G_3 + G_4 + (9 \cdot \frac{r+s}{2} + 9 \cdot \frac{s-3}{4} + 13, 9 \cdot \frac{r+s}{2} + 9 \cdot \frac{s-3}{2} + 21, 9 \cdot \frac{r+s}{2} + 5)$ where

$$\begin{aligned} G_3 &= P(9 \cdot \tfrac{r+s}{2} + 5, 9 \cdot \tfrac{r+s}{2} + 9 \cdot \tfrac{s-3}{2} + 14, 5) \\ &\quad + \textstyle\sum_{i=1}^{\frac{s-3}{4}} \big(Q(9 \cdot \tfrac{r+s}{2} + 5i + 4, 9 \cdot \tfrac{r+s}{2} + 9 \cdot \tfrac{s-3}{2} - 4i + 13, 8)\big), \\ G_4 &= Q(9 \cdot \tfrac{r+s}{2} + 5 \cdot \tfrac{s-3}{4} + 10, 9 \cdot \tfrac{r+s}{2} + 7 \cdot \tfrac{s-3}{2} + 10, 7) \\ &\quad + \textstyle\sum_{i=1}^{\frac{s-3}{4}} \big(P(9 \cdot \tfrac{r+s}{2} + 5 \cdot \tfrac{s-3}{4} + 4i + 9, 9 \cdot \tfrac{r+s}{2} + 7 \cdot \tfrac{s-3}{2} - 5i + 9, 8)\big). \end{aligned}$$

If we continue as in the proof for Case 2 in Lemma 2, we can see that we have an $(r+s+1)$-modular ρ-labeling of G.

Case 3.2: r and s are both even.
In order to show that G admits a d-modular ρ-labeling, we examine when $r \equiv 0,2 \pmod 4$ and when $s \equiv 0,2 \pmod 4$ and show that any of the four possible combinations will satisfy the necessary conditions for the desired labeling.

Case 3.2.1: $r \equiv 0 \pmod 4$.
Let $C_{4r+2} = G_1 + G_2 + (9\cdot\frac{r+s}{2} - 9\cdot\frac{r}{4} + 3, 9\cdot\frac{r}{4} + 1, 9\cdot\frac{r+s}{2} + 4)$ where

$$\begin{aligned} G_1 &= Q(0, 9\cdot\tfrac{r+s}{2}, 4) + \textstyle\sum_{i=1}^{\frac{r}{4}-1}\big(Q(5i-2, 9\cdot\tfrac{r+s}{2} - 4i - 2, 8)\big)\\ &\quad + Q(5\cdot\tfrac{r}{4} - 2, 7\cdot\tfrac{r+s}{2} + s + 1, 5),\\ G_2 &= P(5\cdot\tfrac{r}{4}, 7\cdot\tfrac{r+s}{2} + s, 2)\\ &\quad + \textstyle\sum_{i=1}^{\frac{r}{4}-1}\big(P(5\cdot\tfrac{r}{4} + 4i - 3, 7\cdot\tfrac{r+s}{2} + s - 5i - 3, 8)\big)\\ &\quad + P(9\cdot\tfrac{r}{4} - 3, 9\cdot\tfrac{r+s}{2} - 9\cdot\tfrac{r}{4}, 5). \end{aligned}$$

Case 3.2.2: $r \equiv 2 \pmod 4$.
If $r = 2$, let $C_{4r+2} = (0, 9\cdot\frac{s}{2}+12, 1, 9\cdot\frac{s}{2}+11, 3, 9\cdot\frac{s}{2}+9, 4, 9\cdot\frac{s}{2}+8, 6, 9\cdot\frac{s}{2}+13, 0)$. We leave it to the reader to check that this yields an $(r+s+1)$-modular ρ-labeling of G.

If $r > 2$, let $C_{4r+2} = G_1 + (7\cdot\frac{r+s}{2} + s + 4, 5\cdot\frac{r-2}{4} + 3) + G_2 + (9\cdot\frac{r+s}{2} - 9\cdot\frac{r-2}{4} - 1, 9\cdot\frac{r-2}{4} + 6, 9\cdot\frac{r+s}{2} + 4)$ where

$$\begin{aligned} G_1 &= Q(0, 9\cdot\tfrac{r+s}{2}, 4) + \textstyle\sum_{i=1}^{\frac{r-2}{4}}\big(Q(5i-2, 9\cdot\tfrac{r+s}{2} - 4i - 2, 8)\big),\\ G_2 &= P(5\cdot\tfrac{r-2}{4} + 3, 7\cdot\tfrac{r+s}{2} + s - 4, 6)\\ &\quad + \textstyle\sum_{i=1}^{\frac{r-2}{4}-1}\big(P(5\cdot\tfrac{r-2}{4} + 4i + 2, 7\cdot\tfrac{r+s}{2} + s - 5i - 5, 8)\big)\\ &\quad + P(9\cdot\tfrac{r-2}{4} + 2, 9\cdot\tfrac{r+s}{2} - 9\cdot\tfrac{r-2}{4} - 4, 5). \end{aligned}$$

Case 3.2.3: $s \equiv 0 \pmod 4$.
Let $C_{4s+2} = G_3 + (9\cdot\frac{r+s}{2} + 7\cdot\frac{s}{2} + 7, 9\cdot\frac{r+s}{2} + 5\cdot\frac{s}{4} + 6) + G_4 + (9\cdot\frac{r+s}{2} + 9\cdot\frac{s}{4} + 6, 9\cdot\frac{r+s}{2} + 9\cdot\frac{s}{2} + 8, 9\cdot\frac{r+s}{2} + 5, 9\cdot\frac{r+s}{2} + 9\cdot\frac{s}{2} + 6)$ where

$$\begin{aligned} G_3 &= \textstyle\sum_{i=1}^{\frac{s}{4}-1}\big(Q(9\cdot\tfrac{r+s}{2} + 5i + 2, 9\cdot\tfrac{r+s}{2} + 9\cdot\tfrac{s}{2} - 4i + 2, 8)\big)\\ &\quad + Q(9\cdot\tfrac{r+s}{2} + 5\cdot\tfrac{s}{4} + 2, 9\cdot\tfrac{r+s}{2} + 7\cdot\tfrac{s}{2} + 4, 6),\\ G_4 &= \textstyle\sum_{i=1}^{\frac{s}{4}}\big(P(9\cdot\tfrac{r+s}{2} + 5\cdot\tfrac{s}{4} + 4i + 2, 9\cdot\tfrac{r+s}{2} + 7\cdot\tfrac{s}{2} - 5i + 2, 8)\big). \end{aligned}$$

Case 3.2.4: $s \equiv 2 \pmod 4$.
Let $C_{4s+2} = G_3 + (9\cdot\frac{r+s}{2} + 7\cdot\frac{s-2}{2} + 15, 9\cdot\frac{r+s}{2} + 5\cdot\frac{s-2}{4} + 7, 9\cdot\frac{r+s}{2} + 7\cdot\frac{s-2}{2} + 14) +$

$G_4 + (9 \cdot \frac{r+s}{2} + 9 \cdot \frac{s-2}{4} + 11, 9 \cdot \frac{r+s}{2} + 9 \cdot \frac{s-2}{2} + 17, 9 \cdot \frac{r+s}{2} + 5, 9 \cdot \frac{r+s}{2} + 9 \cdot \frac{s-2}{2} + 15)$ where

$$G_3 = \sum_{i=1}^{\frac{s-2}{4}} \big(Q(9 \cdot \tfrac{r+s}{2} + 5i + 2, 9 \cdot \tfrac{r+s}{2} + 9 \cdot \tfrac{s-2}{2} - 4i + 11, 8)\big),$$
$$G_4 = Q(9 \cdot \tfrac{r+s}{2} + 5 \cdot \tfrac{s-2}{4} + 9, 9 \cdot \tfrac{r+s}{2} + 7 \cdot \tfrac{s-2}{2} + 9, 5)$$
$$+ \sum_{i=1}^{\frac{s-2}{4}} \big(P(9 \cdot \tfrac{r+s}{2} + 5 \cdot \tfrac{s-2}{4} + 4i + 7, 9 \cdot \tfrac{r+s}{2} + 7 \cdot \tfrac{s-2}{2} - 5i + 7, 8)\big).$$

If we continue as in the proof for Case 2 in Lemma 2, we can see that we have an $(r+s+1)$-modular ρ-labeling of G.

Case 3.3: $r+s$ is odd.
For this case, we relax the condition that $r \leq s$. Then without loss of generality, we need only consider when r is odd and s is even. In order to show that G admits a d-modular ρ-labeling, we examine when $r \equiv 1, 3 \pmod 4$ and when $s \equiv 0, 2 \pmod 4$ and show that any of the four possible combinations will satisfy the necessary conditions for the desired labeling.

Case 3.3.1: $r \equiv 1 \pmod 4$.
If $r = 1$, let $C_{4r+2} = (0, 9 \cdot \frac{s}{2} + 7, 1, 9 \cdot \frac{s}{2} + 5, 3, 9 \cdot \frac{s}{2} + 8, 0)$. We leave it to the reader to check that this yields an $(r+s+1)$-modular ρ-labeling of G.

If $r > 1$, let $C_{4r+2} = G_1 + G_2 + (9 \cdot \frac{r+s-1}{2} - 9 \cdot \frac{r-1}{4} + 5, 9 \cdot \frac{r-1}{4} + 3, 9 \cdot \frac{r+s-1}{2} + 8)$ where

$$G_1 = \sum_{i=1}^{\frac{r-1}{4}} \big(Q(5i - 5, 9 \cdot \tfrac{r+s-1}{2} - 4i + 4, 8)\big)$$
$$+ Q(5 \cdot \tfrac{r-1}{4}, 7 \cdot \tfrac{r+s-1}{2} + s + 5, 3),$$
$$G_2 = P(5 \cdot \tfrac{r-1}{4} + 1, 7 \cdot \tfrac{r+s-1}{2} + s, 4)$$
$$+ \sum_{i=1}^{\frac{r-1}{4}-1} \big(P(5 \cdot \tfrac{r-1}{4} + 4i - 1, 7 \cdot \tfrac{r+s-1}{2} + s - 5i - 1, 8)\big)$$
$$+ P(9 \cdot \tfrac{r-1}{4} - 1, 9 \cdot \tfrac{r+s-1}{2} - 9 \cdot \tfrac{r-1}{4} + 2, 5).$$

Case 3.3.2: $r \equiv 3 \pmod 4$.
Let $C_{4r+2} = G_1 + G_2 + (9 \cdot \frac{r+s-1}{2} - 9 \cdot \frac{r-3}{4}, 9 \cdot \frac{r-3}{4} + 7, 9 \cdot \frac{r+s-1}{2} + 8)$ where

$$G_1 = \sum_{i=1}^{\frac{r-3}{4}} \big(Q(5i - 5, 9 \cdot \tfrac{r+s-1}{2} - 4i + 4, 8)\big)$$
$$+ Q(5 \cdot \tfrac{r-3}{4}, 7 \cdot \tfrac{r+s-1}{2} + s + 3, 7),$$
$$G_2 = \sum_{i=1}^{\frac{r-3}{4}} \big(P(5 \cdot \tfrac{r-3}{4} + 4i - 1, 7 \cdot \tfrac{r+s-1}{2} + s - 5i + 1, 8)\big)$$
$$+ P(9 \cdot \tfrac{r-3}{4} + 3, 9 \cdot \tfrac{r+s-1}{2} - 9 \cdot \tfrac{r-3}{4} - 3, 5).$$

Case 3.3.3: $s \equiv 0 \pmod 4$.
Let $C_{4s+2} = G_3 + (9 \cdot \frac{r+s-1}{2} + 7 \cdot \frac{s}{2} + 11, 9 \cdot \frac{r+s-1}{2} + 5 \cdot \frac{s}{4} + 10) + G_4 + (9 \cdot$

$\frac{r+s-1}{2}+9\cdot\frac{s}{4}+10, 9\cdot\frac{r+s-1}{2}+9\cdot\frac{s}{2}+12, 9\cdot\frac{r+s-1}{2}+9, 9\cdot\frac{r+s-1}{2}+9\cdot\frac{s}{2}+10)$ where

$$\begin{aligned} G_3 &= \textstyle\sum_{i=1}^{\frac{s}{4}-1}\big(Q(9\cdot\frac{r+s-1}{2}+5i+6, 9\cdot\frac{r+s-1}{2}+9\cdot\frac{s}{2}-4i+6, 8)\big)\\ &\quad + Q(9\cdot\frac{r+s-1}{2}+5\cdot\frac{s}{4}+6, 9\cdot\frac{r+s-1}{2}+7\cdot\frac{s}{2}+8, 6),\\ G_4 &= \textstyle\sum_{i=1}^{\frac{s}{4}}\big(P(9\cdot\frac{r+s-1}{2}+5\cdot\frac{s}{4}+4i+6, 9\cdot\frac{r+s-1}{2}+7\cdot\frac{s}{2}-5i+6, 8)\big). \end{aligned}$$

Case 3.3.4: $s \equiv 2 \pmod 4$.
Let $C_{4s+2} = G_3 + (9\cdot\frac{r+s-1}{2}+7\cdot\frac{s-2}{2}+19, 9\cdot\frac{r+s-1}{2}+5\cdot\frac{s-2}{4}+11, 9\cdot\frac{r+s-1}{2}+7\cdot\frac{s-2}{2}+18) + G_4 + (9\cdot\frac{r+s-1}{2}+9\cdot\frac{s-2}{4}+15, 9\cdot\frac{r+s-1}{2}+9\cdot\frac{s-2}{2}+21, 9\cdot\frac{r+s-1}{2}+9, 9\cdot\frac{r+s-1}{2}+9\cdot\frac{s-2}{2}+19)$ where

$$\begin{aligned} G_3 &= \textstyle\sum_{i=1}^{\frac{s-2}{4}}\big(Q(9\cdot\frac{r+s-1}{2}+5i+6, 9\cdot\frac{r+s-1}{2}+9\cdot\frac{s-2}{2}-4i+15, 8)\big),\\ G_4 &= Q(9\cdot\frac{r+s-1}{2}+5\cdot\frac{s-2}{4}+13, 9\cdot\frac{r+s-1}{2}+7\cdot\frac{s-2}{2}+13, 5)\\ &\quad + \textstyle\sum_{i=1}^{\frac{s-2}{4}}\big(P(9\cdot\frac{r+s-1}{2}+5\cdot\frac{s-2}{4}+4i+11, 9\cdot\frac{r+s-1}{2}+7\cdot\frac{s-2}{2}-5i+11, 8)\big). \end{aligned}$$

If we continue as in the proof for Case 2 in Lemma 2, we can see that we have an $(r+s+1)$-modular ρ-labeling of G.

Case 4: $d = 2(r+s+1)$.
Let $c = 2(4r+4s+4)/(2r+2s+2)+1$, so that the complete multipartite graph we are working in is $K_{c\times d} = K_{5\times 2(r+s+1)}$. In order to show that G admits a d-modular ρ-labeling, we examine when r is even or odd and when s is even or odd and show that any of the four possible combinations will satisfy the necessary conditions for the desired labeling.

Case 4.1: r is odd.
Let $C_{4r+2} = G_1 + G_2 + (5\cdot\frac{r-1}{2}+1, 5\cdot\frac{r-1}{2}+5s+5, 5\cdot\frac{r-1}{2}+3, 5r+5s+4)$ where

$$\begin{aligned} G_1 &= \textstyle\sum_{i=1}^{\frac{r-1}{2}}\big(Q(3i-3, 5r+5s-2i+2, 4)\big) + Q(3\cdot\frac{r-1}{2}, 4r+5s+2, 3),\\ G_2 &= \textstyle\sum_{i=1}^{\frac{r-1}{2}}\big(P(3\cdot\frac{r-1}{2}+2i-1, 4r+5s-3i, 4)\big). \end{aligned}$$

Case 4.2: r is even.
Let $C_{4r+2} = G_1 + (4r+5s+4, 3\cdot\frac{r}{2}, 4r+5s+2, 3\cdot\frac{r}{2}+1) + G_2 + (5\cdot\frac{r}{2}-1, 5\cdot\frac{r}{2}+5s+3, 5\cdot\frac{r}{2}+1, 5r+5s+4)$ where

$$\begin{aligned} G_1 &= \textstyle\sum_{i=1}^{\frac{r}{2}}\big(Q(3i-3, 5r+5s-2i+2, 4)\big),\\ G_2 &= \textstyle\sum_{i=1}^{\frac{r}{2}-1}\big(P(3\cdot\frac{r}{2}+2i-1, 4r+5s-3i-1, 4)\big). \end{aligned}$$

Case 4.3: s is odd.
Let $C_{4s+2} = G_3 + G_4 + (15\cdot\frac{s+1}{2}+5r-1, 5r+10s+8, 5r+5s+5, 5r+10s+6)$

where

$$
\begin{aligned}
G_3 &= \textstyle\sum_{i=1}^{\frac{s-1}{2}} \left(Q(5r+5s+3i+4, 5r+10s-2i+4, 4)\right), \\
G_4 &= Q(13 \cdot \tfrac{s+1}{2} + 5r, 5r+9s+4, 3) \\
&\quad + \textstyle\sum_{i=1}^{\frac{s-1}{2}} \big(P(13 \cdot \tfrac{s+1}{2} + 5r + 2i - 1, 5r+9s-3i+3, 4)\big).
\end{aligned}
$$

Case 4.4: s is even.
Let $C_{4s+2} = G_3 + (5r+9s+8, 13 \cdot \frac{s}{2} + 5r+4, 5r+9s+7, 13 \cdot \frac{s}{2} + 5r + 6) + G_4 + (15 \cdot \frac{s}{2} + 5r + 6, 5r+10s+8, 5r+5s+5, 5r+10s+6)$ where

$$
\begin{aligned}
G_3 &= \textstyle\sum_{i=1}^{\frac{s}{2}-1} \left(Q(5r+5s+3i+4, 5r+10s-2i+4, 4)\right), \\
G_4 &= \textstyle\sum_{i=1}^{\frac{s}{2}} \big(P(13 \cdot \tfrac{s}{2} + 5r + 2i + 4, 5r+9s-3i+4, 4)\big).
\end{aligned}
$$

If we continue as in the proof for Case 2 in Lemma 2, we can see that we have a $(2r+2s+2)$-modular ρ-labeling of G.

Case 5: $d = 4(r+s+1)$.
Let $c = 2(4r+4s+4)/(4r+4s+4)+1$, so that the complete multipartite graph we are working in is $K_{c \times d} = K_{3 \times (4r+4s+4)}$. If $s = 1$, let $C_{4r+2} = (0, 16, 2, 12, 4, 17, 0)$ and $C_{4s+2} = (18, 20, 19, 26, 22, 29, 18)$. We leave it to the reader to check that this yields a $(4r+4s+4)$-modular ρ-labeling of G.

If $s > 1$, let $C_{4r+2} = G_1 + (5r+6s+5, 2r) + G_2 + (3r-1, 3r+6s+3, 3r+1, 6r+6s+5)$ and $C_{4s+2} = G_3 + (6r+11s+9, 6r+8s+5) + G_4 + (6r+9s+6, 6r+12s+11, 6r+6s+6, 6r+12s+7)$ where

$$
\begin{aligned}
G_1 &= \textstyle\sum_{i=1}^{r} Q(2i-2, 6r+6s-i+4, 2), \\
G_2 &= \textstyle\sum_{i=1}^{r-1} P(2r+i-1, 5r+6s-2i+2, 2), \\
G_3 &= \textstyle\sum_{i=1}^{s-2} Q(6r+6s+2i+6, 6r+12s-i+6, 2), \\
G_4 &= \textstyle\sum_{i=1}^{s+1} P(6r+8s+i+4, 6r+11s-2i+7, 2).
\end{aligned}
$$

If we continue as in the proof for Case 3.1 in Lemma 1, we can see that we have a $(4r+4s+4)$-modular ρ-labeling of G. ■

Theorem 13. *Let $G = C_{4r+2} \cup C_{4s+2}$ where r and s are positive integers and let $n = 4r+4s+4$. Then there exists a cyclic G-decomposition of $K_{(2n+1)\times t}$, $K_{(n+1)\times 2t}$, $K_{(n/2+1)\times 4t}$, $K_{(n/4+1)\times 8t}$, $K_{9\times(n/4)t}$, $K_{5\times(n/2)t}$, $K_{3\times nt}$, and of $K_{2\times 2nt}$ for every positive integer t.*

5 Acknowledgement and Final Note

This work was done under the supervision of the third and fourth authors as part of *REU Site: Mathematics Research Experience for Pre-service and*

for In-service Teachers at Illinois State University. The affiliations of the remaining authors at the time were as follows: A. Su: University High School (Normal, IL); J. Buchanan and E. Sparks: Illinois State University; E. Pelttari: Northen Illinois University; G. Rasmuson: Roanoke-Benson High School (Roanoke, IL); S. Tagaris: Illinois Wesleyan University.

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