On the Spectra of Bipartite Directed Subgraphs of K_4^*

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Abstract

The complete directed graph of order n, denoted K_n^*, is the directed graph on n vertices that contains the arcs (u, v) and (v, u) for every pair of distinct vertices u and v. For a given directed graph, D, the set of all n for which K_n^* admits a D-decomposition is called the spectrum of D. In this paper, we find the spectrum for each bipartite subgraph of K_4^* with 5 or fewer arcs.

1 Introduction

One of the primary questions in the field of graph decompositions is "Given a graph G, which complete graphs admit a G-decomposition?" Answering this question for all graphs of small order has been the topic of various papers. In particular, this question was answered for graphs and uniform hypergraphs of small order in [3] and [5], respectively. In this paper we follow this line of inquiry to directed graphs by classifying the spectra of bipartite directed graphs of order at most 4, with up to 5 arcs.

If a and b are integers, we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, then $[a, b] = \varnothing$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_m$ the group of integers modulo m. Throughout this paper, we refer to directed graphs

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as digraphs. For a digraph, D, let $V(D)$ and $E(D)$ denote the vertex set of D and the arc set of D, respectively. The *order* and the *size* of a digraph D are $|V(D)|$ and $|E(D)|$, respectively. Throughout this paper we will use the names for digraphs, displayed in Tables 1 and 2, found in *An Atlas of Graphs* [13] by Read and Wilson.

Let H and D be digraphs such that D is a subgraph of H. A *D-decomposition* of H is a set $\Delta = \{D_1, D_2, \ldots, D_r\}$ of pairwise edge-disjoint subgraphs of H each of which is isomorphic to D and such that $E(H) = \bigcup_{i=1}^{r} E(D_i)$. If a D-decomposition of H exists, then we say that D *decomposes* H. The *reverse orientation* of D is the digraph with vertex set $V(D)$ and arc set $\{(v,u)\colon (u,v) \in E(D)\}$. Let D' and H' denote the reverse orientations of D and H, respectively. We note that the existence of a D-decomposition of H necessarily implies the existence of a D'-decomposition of H'. A D-decomposition of K_n^* is also known as a *(K_n^*, D)-design*. The set of all n for which K_n^* admits a D-decomposition is called the *spectrum of D*. Since K_n^* is its own reverse orientation, we note that the spectrum of D is equivalent to the spectrum of D'.

The necessary conditions for a digraph D to decompose K_n^* include

(A) $|V(D)| \leq n$,
(B) $|E(D)|$ divides $n(n-1)$, and
(C) $\gcd\{\text{outdegree}(v)\colon v \in V(D)\}$ and $\gcd\{\text{indegree}(v)\colon v \in V(D)\}$ both divide $n-1$.

There are 51 bipartite subgraphs of K_4^* with no isolated vertices. It was shown in [6] that 37 of these decompose K_{mx+1}^* cyclically (defined in Section 2), where m is the number of arcs in the subgraph. Until now, the spectra for the majority of these graphs had not been classified. This paper extends those results by finding the spectra for all 42 bipartite subgraphs of K_4^* with 5 arcs or fewer. If D is a subgraph of K_4^* with $m \leq 5$ arcs, then by condition (B) the existence of a (K_n^*, D)-design necessitates that $n \equiv 0$ or $1 \pmod{m}$. Through a series of lemmas, we prove the following.

Theorem 1. *The spectra for the 42 bipartite subgraphs of K_4^* with 5 or fewer arcs are as displayed in Table 3.*

The spectra for certain subgraphs (both bipartite and non-bipartite) of K_4^* have already been studied. When D is a cyclic orientation of K_3, then a (K_n^*, D)-design is known as a Mendelsohn triple system. The spectrum for Mendelsohn triple systems was found independently by Mendelsohn [11] and Bermond [2].When D is a transitive orientation of K_3, then a (K_n^*, D)-design is known as a transitive triple system. The spectrum for transitive triple systems was found by Hung and Mendelsohn [9]. In [8], Hartman and Mendelsohn found the spectra for all remaining simple connected subgraphs of K_3^*, which include graphs D7, D10, D11, D13, D16, and D25 (see Table 1).

There are exactly four orientations of a 4-cycle. It was shown in [15] that if D is a cyclic orientation of a 4-cycle (i.e., D67), then a (K_n^*, D)-design exists if and only if $n \equiv 0$ or 1 (mod 4) and $n \neq 4$. The spectra for the remaining three orientations of a 4-cycle (i.e., D46, D59, D60) were found in [7].

A *directed path on* n *vertices*, denoted DP_n, is the directed graph whose underlying simple graph is a path on n vertices such that every vertex that is not a leaf in the underlying graph has both indegree and outdegree of one. In [12], necessary and sufficient conditions for a decomposition of the complete multi-digraph into directed paths of arbitrarily prescribed lengths were found. In particular, this characterizes the spectra of D25 and D38. Here, we state their result only for edge multiplicity one and fixed path length.

Theorem 2. *[12] Necessary and sufficient conditions for the existence of a* (K_n^*, DP_{m+1})*-design are*

$$m \leq n-1 \qquad \text{and} \qquad n(n-1) \equiv 0 \pmod{m},$$

unless (m, n) *is either* $(4, 5)$ *or* $(2, 3)$*.*

A digraph is called an *antidirected path* if its underlying graph is a path, and it does not admit a directed path of length 2 as a subgraph. Let D be an antidirected path. Necessary and sufficient conditions for the existence of a (K_n^*, D)-design were obtained in [16]. Notice that D7, D10, and D33 are antidirected paths.

Theorem 3. *[16] Let* AP_m *denote an antidirected path on* m *vertices. A* (K_n^*, AP_{m+1})*-design exists if and only if the following conditions are satisfied:*

(1) $m \leq n-1$,
(2) $n(n-1) \equiv 0 \pmod{m}$,
(3) n *or* m *is odd.*

Let G be a bipartite subgraph of the 2-fold undirected complete graph on 4 vertices. In [1] necessary and sufficient conditions were obtained for the existence of a G-decomposition of the 2-fold undirected complete graph on n vertices. Several of these decompositions directly translate to digraph decompositions of interest in this paper. In particular, the spectra of D3, D4, D16, D27, D37, and D66 are obtained from these results.

2 Graph labelings

Let $V(K_n^*) = [0, n-1]$. The *length* of an arc (i, j) is $j - i$ if $j > i$, and it is $n + j - i$ otherwise. Note that $E(K_n^*)$ consists of n arcs of length i for each $i \in [1, n-1]$. Let D be a subgraph of K_n^*. By *rotating* D, we mean applying the permutation $i \mapsto i+1$ to $V(D)$ where the addition is done modulo n. Moreover in this case, if $j \in \mathbb{N}$, then $D + j$ is the digraph obtained from D by successively rotating D a total of j times. Note that rotating an arc does not change its length. Also note that $D + j$ is isomorphic to D for every $j \in \mathbb{N}$. A (K_n^*, D)-design Δ is *cyclic* if rotating is an automorphism of Δ.

Graph labelings were introduced by Rosa in [14] and provided a means of obtaining cyclic designs for undirected graphs. In particular, the well-known *graceful labeling* was defined in [14]. In 1985, Bloom and Hsu [4] extended the concept of a graceful labeling to directed graphs. With the notation adapted to better suit this paper, we present the following definition from [4].

Let D be a directed graph with m arcs and at most $m + 1$ vertices. Let $f\colon V(D) \to [0, m]$ be an injective function, and define a function $\bar{f}\colon E(D) \to [1, m]$ as follows: $\bar{f}((a, b)) = f(b) - f(a)$, if $f(b) > f(a)$, and $\bar{f}((a, b)) = m + 1 + f(b) - f(a)$, otherwise. We call f a *directed ρ-labeling of D* if $\{\bar{f}((a, b))\colon (a, b) \in E(D)\} = [1, m]$. Thus, a directed ρ-labeling of D is an embedding of D in K_{m+1}^* such that there is exactly one arc in D of length i for each $i \in [1, m]$.

It was shown in [10] that for a digraph D with size m and order at most $m + 1$, a cyclic (K_{m+1}^*, D)-design exists if and only if D admits a directed ρ-labeling. In order to obtain an infinite family of cyclic designs, it is necessary to extend the notion of a directed ρ-labeling. In our case this is accomplished by restricting ourselves to bipartite digraphs and imposing an order on the labeling.

Let D be a bipartite directed graph with m arcs and at most $m + 1$ vertices. Let $\{A, B\}$ be a vertex bipartition of $V(D)$. A directed ρ-labeling f of D is *ordered* if $f(a) < f(b)$ for each arc with end vertices $a \in A$ and $b \in B$. An ordered directed ρ-labeling is also called a *directed ρ^+-labeling*. The connection between directed ρ^+-labelings and cyclic digraph designs is found in a result from [6].

Theorem 4. *If D is a bipartite directed graph with m arcs that admits a directed ρ^+-labeling, then there exists a cyclic (K_{mx+1}^*, D)-design for all $x \in \mathbb{Z}^+$.*

Next we consider an analogue of 1-rotational designs for directed graphs. If we let $V(K_n^*) = [0, n-2] \cup \{\infty\}$, then the length of an arc (i, j), where $\infty \notin \{i, j\}$, is $j - i$ if $j > i$, and it is $n - 1 + j - i$ otherwise. Furthermore, we say that the length of an arc of the form (i, ∞) is $+\infty$, and the length of an arc of the form (∞, j) is $-\infty$. Let D be a subgraph of K_n^*. A (K_n^*, D)-design Δ is called a *1-rotational directed design* if applying the permutation $(0, 1, 2, \ldots, n-2)(\infty)$ is an automorphism of Δ.

Let D be a directed graph with m arcs and at most m vertices. Let $f\colon V(D) \to [0, m-1] \cup \{\infty\}$ be an injective function and define a function $\bar{f}\colon E(D) \to [1, m-2] \cup \{+\infty, -\infty\}$ as follows:

$$\bar{f}((a,b)) = \begin{cases} f(b) - f(a) & \text{if } f(b) > f(a), \\ m - 1 + f(b) - f(a) & \text{if } f(b) < f(a), \\ +\infty & \text{if } f(b) = \infty, \\ -\infty & \text{if } f(a) = \infty. \end{cases}$$

We call f a *1-rotational directed ρ-labeling* of D if $\{\bar{f}((a,b))\colon (a,b) \in E(D)\} = [1, m-2] \cup \{+\infty, -\infty\}$. Thus, a 1-rotational directed ρ-labeling of D is an embedding of D in K_m^* such that there is exactly one arc in D of length i for each $i \in [1, m-2] \cup \{+\infty, -\infty\}$. The following theorem formalizes this observation.

Theorem 5. *Let D be a bipartite directed graph with m arcs. There exists a 1-rotational (K_m^*, D)-design if and only if D admits a 1-rotational directed ρ-labeling.*

3 Main Results

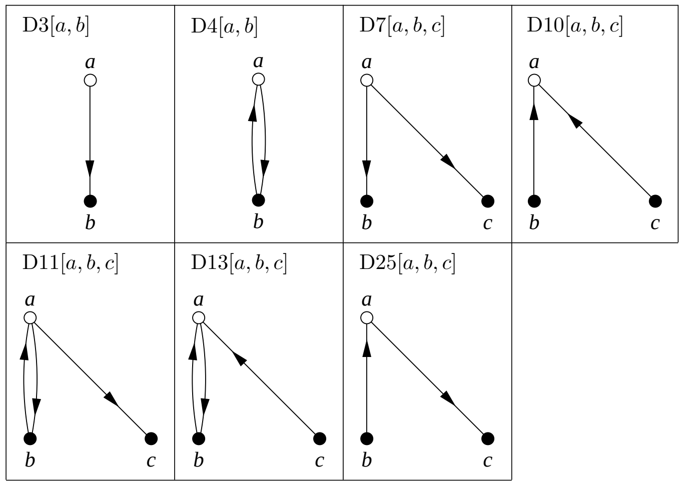
The 42 non-isomorphic bipartite subgraphs of K_4^* of size at most 5 are shown in Tables 1 and 2. These tables also give a key that denotes a labeled copy for each bipartite directed graph. For example, D11$[a, b, c]$ refers to the graph with three vertices labeled a, b, and c with two arcs between a and b and a single arc directed from a to c. Our general method for classifying the spectrum for such a graph of size m is to break into two cases: complete graphs with $mx + 1$ vertices and complete graphs with mx vertices. However, we first note the following negative results for the sufficiency of the necessary conditions, which can be easily verified.

Lemma 6. *There does not exist an (H, D)-design for $(H, D) \in \{(K_4^*, \text{D50}), (K_5^*, \text{D50}), (K_4^*, \text{D51}), (K_6^*, \text{D99})\}$.*

Let G and H be vertex-disjoint directed graphs. The *join of G and H*, denoted $G \vee H$, is defined to be the directed graph with vertex set $V(G) \cup V(H)$ and arc set $E(G) \cup E(H) \cup \{(u, v), (v, u)\colon u \in V(G), v \in V(H)\}$. We use the shorthand notation $\bigvee_{i=1}^{t} G_i$ to denote $G_1 \vee G_2 \vee \cdots \vee G_t$, and when $G_i \cong G_j \cong G$ for all $1 \le i < j \le t$, we use the notation $\bigvee_{i=1}^{t} G$. For example, $K_{12}^* \cong \bigvee_{i=1}^{4} K_3^*$.

Table 1: Bipartite subgraphs of K_4^* with 3 or fewer arcs.

2–3 vertices



4 vertices

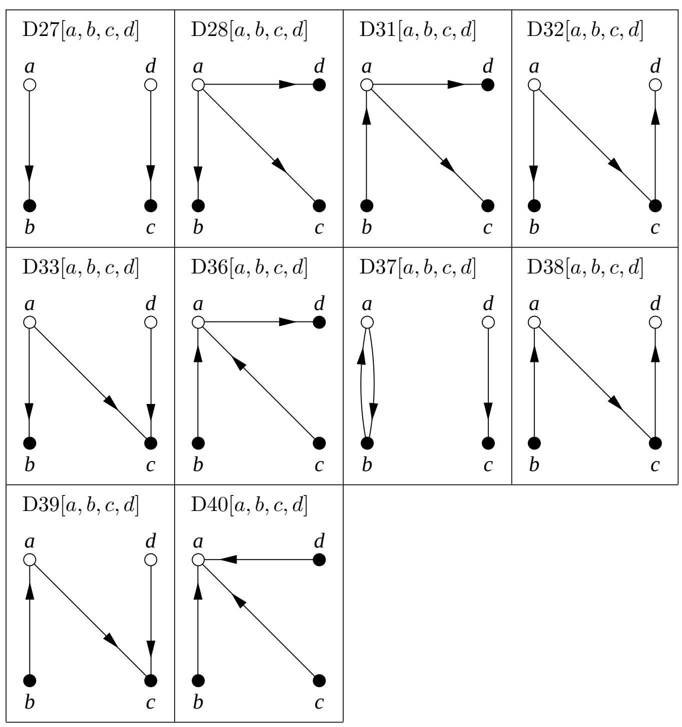


Table 2: Bipartite subgraphs of K_4^* with 4 or 5 arcs.

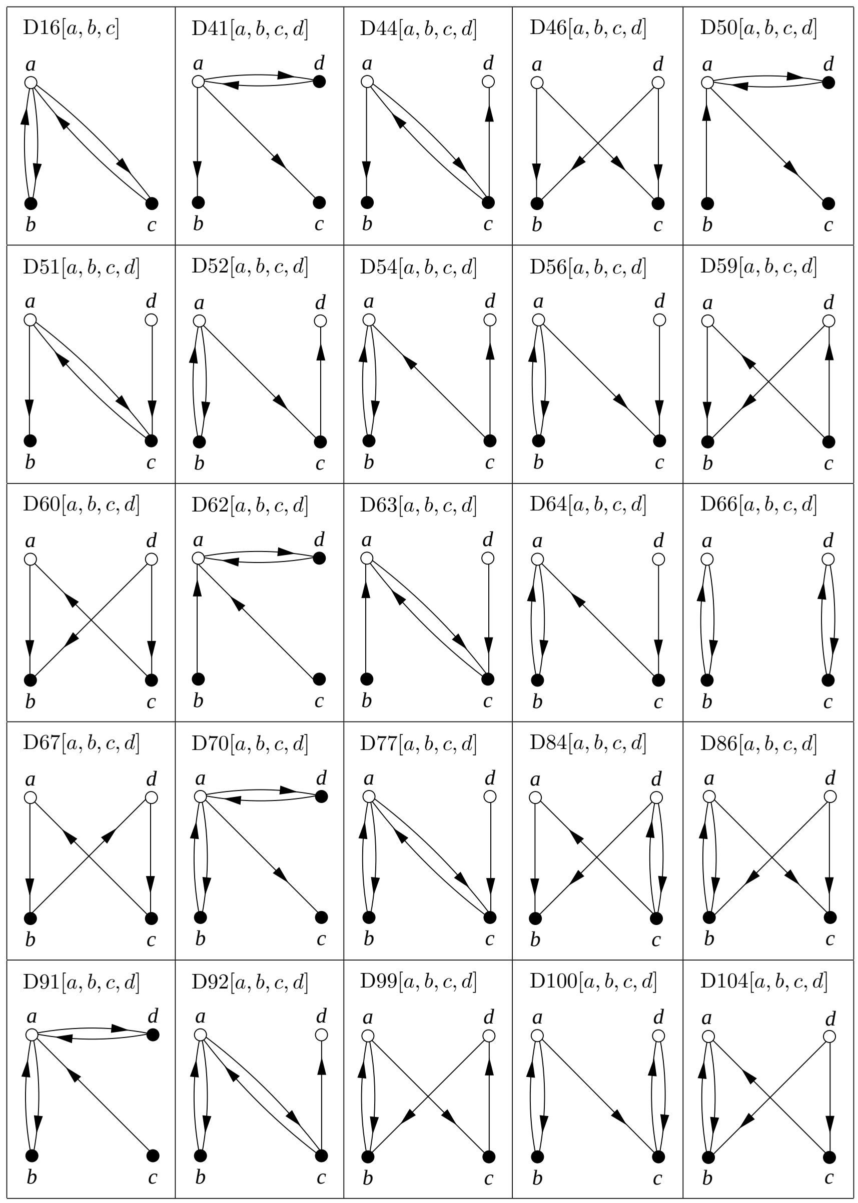


Table 3: The necessary and sufficient conditions for the given digraphs to decompose K_n^*.

Diraph	Conditions	References
D3	$n \geq 1$	[1]
D4	$n \geq 1$	[1]
D7	$n \equiv 1 \pmod 2$	[8], [16], [6]
D10	$n \equiv 1 \pmod 2$	[8], [16], [6]
D11	$n \equiv 0$ or $1 \pmod 3$	[8], [6]
D13	$n \equiv 0$ or $1 \pmod 3$	[8], [6]
D16	$n \equiv 0$ or $1 \pmod 4$	[8], [6]
D25	$n = 1$ or $n \geq 4$	[8], [12]
D27	$n = 1$ or $n \geq 4$	[1]
D28	$n \equiv 1 \pmod 3$	[6]
D31	$n \equiv 0$ or $1 \pmod 3$, $n \neq 3$	[6], Lemma 12
D32	$n \equiv 0$ or $1 \pmod 3$, $n \neq 3$	Lemmas 8 and 13
D33	$n \equiv 0$ or $1 \pmod 3$, $n \neq 3$	[16], [6]
D36	$n \equiv 0$ or $1 \pmod 3$, $n \neq 3$	[6], Lemma 12
D37	$n \equiv 0$ or $1 \pmod 3$, $n = 1$ or $n \geq 6$	[1]
D38	$n \equiv 0$ or $1 \pmod 3$, $n \neq 3$	[12], [6]
D39	$n \equiv 0$ or $1 \pmod 3$, $n \neq 3$	Lemmas 8 and 13
D40	$n \equiv 1 \pmod 3$	[6]
D41	$n \equiv 0$ or $1 \pmod 4$	[6], Lemma 14
D44	$n \equiv 1 \pmod 4$	[6]
D46	$n \equiv 1 \pmod 4$	[7], [6]
D50	$n \equiv 0$ or $1 \pmod 4$, $n = 1$ or $n \geq 8$	Lemmas 6, 9, and 15
D51	$n \equiv 0$ or $1 \pmod 4$, $n \neq 4$	[6], Lemmas 6 and 16
D52	$n \equiv 0$ or $1 \pmod 4$	Lemmas 10 and 17
D54	$n \equiv 0$ or $1 \pmod 4$	[6], Lemma 18
D56	$n \equiv 0$ or $1 \pmod 4$	[6], Lemma 18
D59	$n \equiv 0$ or $1 \pmod 4$, $n = 1$ or $n \geq 8$	[7]
D60	$n \equiv 0$ or $1 \pmod 4$, $n \neq 5$	[7]
D62	$n \equiv 0$ or $1 \pmod 4$	[6], Lemma 14
D63	$n \equiv 1 \pmod 4$	[6]
D64	$n \equiv 0$ or $1 \pmod 4$	Lemmas 10 and 17
D66	$n \equiv 0$ or $1 \pmod 4$	[6], [1]
D67	$n \equiv 0$ or $1 \pmod 4$, $n \neq 4$	[15]
D70	$n \equiv 0$ or $1 \pmod 5$	[6], Lemma 19
D77	$n \equiv 0$ or $1 \pmod 5$	[6], Lemma 20
D84	$n \equiv 0$ or $1 \pmod 5$	[6], Lemma 21
D86	$n \equiv 0$ or $1 \pmod 5$	[6], Lemma 22
D91	$n \equiv 0$ or $1 \pmod 5$	[6], Lemma 19
D92	$n \equiv 0$ or $1 \pmod 5$	[6], Lemma 20
D99	$n \equiv 0$ or $1 \pmod 5$, $n \neq 6$	Lemmas 6, 11, and 23
D100	$n \equiv 0$ or $1 \pmod 5$	[6], Lemma 24
D104	$n \equiv 0$ or $1 \pmod 5$	[6], Lemma 21

3.1 Decompositions of K^*_{mx+1}

Labeling methods can be used to obtain cyclic (K^*_{mx+1}, D)-designs for 31 of the 42 directed graphs. This is due to the fact that these graphs admit directed ρ^+-labelings as shown in [6].

Theorem 7. *[6] Let D be a bipartite subgraph of K^*_4 of size m, and let x be a positive integer. If D does not belong to the set* {D25*,* D27*,* D32*,* D37*,* D39*,* D50*,* D52*,* D59*,* D60*,* D64*,* D99}*, then there exists a cyclic (K^*_{mx+1}, D)-design.*

Next, we proceed to the digraphs that do not admit a directed ρ^+-labeling. Throughout this section we let the vertex set of K^*_{mx+1} be $[0, mx]$. Furthermore, throughout the entirety of the paper let $K^*_{s,t}$ have vertex bipartition $\{A, B\}$ where $A = [0, s-1]$ and $B = [s, s+t-1]$.

Lemma 8. *For every integer $x \geq 1$ there exists a (K^*_{3x+1}, D)-design for $D \in \{\text{D32}, \text{D39}\}$.*

Proof. Since D39 is the reverse orientation of D32, it suffices to show the existence of a $(K^*_{3x+1}, \text{D32})$-design. For $x = 1$, we have the following $(K^*_4, \text{D32})$-design:

$$\{\text{D32}[0, 3, 2, 1], \text{D32}[1, 2, 3, 0], \text{D32}[3, 2, 1, 0], \text{D32}[2, 3, 0, 1]\}.$$

For $x > 1$, we require the following $(K^*_{3,3}, \text{D32})$-design:

$$\{\text{D32}[0, 3, 4, 2], \text{D32}[1, 4, 5, 0], \text{D32}[2, 5, 3, 1], \\ \text{D32}[4, 1, 0, 5], \text{D32}[5, 2, 1, 3], \text{D32}[3, 0, 2, 4]\}.$$

We write $K^*_{3x+1} \cong K^*_1 \vee \bigvee_{i=1}^{x} K^*_3$. On each of the x copies of $K^*_1 \vee K^*_3$, we place a $(K^*_4, \text{D32})$-design. The remaining arcs form arc-disjoint copies of $K^*_{3,3}$, on each of which we place a $(K^*_{3,3}, \text{D32})$-design. ■

Lemma 9. *For every integer $x \geq 2$ there exists a $(K^*_{4x+1}, \text{D50})$-design.*

Proof. For $x = 2$, consider the digraph $G = \text{D50}[0, 3, 5, 1] \cup \text{D50}[0, 6, 4, 2]$. It is easy to check that we have a directed ρ-labeling of G, and thus a $(K^*_9, \text{D50})$-design exists. For $x = 3$, consider the directed graph $H = \text{D50}[0, 6, 4, 1] \cup \text{D50}[0, 4, 5, 2] \cup \text{D50}[0, 5, 6, 3]$. It is easy to check that we have a directed ρ-labeling of H, and thus a $(K^*_{13}, \text{D50})$-design exists. For $x > 3$, we require the following $(K^*_{4,4}, \text{D50})$-design:

$$\{\text{D50}[0, 5, 6, 4], \text{D50}[1, 6, 7, 5], \text{D50}[2, 7, 4, 6], \text{D50}[3, 4, 5, 7], \\ \text{D50}[0, 6, 5, 7], \text{D50}[1, 7, 6, 4], \text{D50}[2, 4, 7, 5], \text{D50}[3, 5, 4, 6]\}.$$

Case 1: $x = 2k$ for some integer $k \geq 2$.
We write $K^*_{8k+1} \cong K^*_1 \vee \bigvee_{i=1}^{k} K^*_8$. On each of the k copies of $K^*_1 \vee K^*_8$, we place a $(K^*_9, \mathrm{D50})$-design. The remaining arcs form arc-disjoint copies of $K^*_{8,8}$, each of which can be decomposed into copies of $K^*_{4,4}$. On each of these copies of $K^*_{4,4}$, we place a $(K^*_{4,4}, \mathrm{D50})$-design.
Case 2: $x = 2k+1$ for some integer $k \geq 2$.
We write $K^*_{8k+5} \cong K^*_1 \vee K^*_{12} \vee \bigvee_{i=1}^{k-1} K^*_8$. On the copy of $K^*_1 \vee K^*_{12}$, we place a $(K^*_{13}, \mathrm{D50})$-design. On each of the $k-1$ copies of $K^*_1 \vee K^*_8$, we place a $(K^*_9, \mathrm{D50})$-design. The remaining arcs form arc-disjoint copies of $K^*_{8,12}$ and $K^*_{8,8}$, both of which can be decomposed into copies of $K^*_{4,4}$. On each of these copies of $K^*_{4,4}$, we place a $(K^*_{4,4}, \mathrm{D50})$-design. ■

Lemma 10. *For every integer $x \geq 1$ there exists a (K^*_{4x+1}, D)-design for $D \in \{\mathrm{D52}, \mathrm{D64}\}$.*

Proof. Since D64 is the reverse orientation of D52, it suffices to show the existence of a $(K^*_{4x+1}, \mathrm{D52})$-design. For $x = 1$, we have the following $(K^*_5, \mathrm{D52})$-design:

$$\{\mathrm{D52}[1,0,2,3], \mathrm{D52}[3,4,2,1], \mathrm{D52}[3,1,0,4], \mathrm{D52}[4,1,0,2], \mathrm{D52}[2,4,0,3]\}.$$

For $x > 1$, we require the following $(K^*_{2,2}, \mathrm{D52})$-design:

$$\{\mathrm{D52}[0,3,2,1], \mathrm{D52}[1,3,2,0]\}.$$

We write $K^*_{4x+1} \cong K^*_1 \vee \bigvee_{i=1}^{x} K^*_4$. On each of the x copies of $K^*_1 \vee K^*_4$, we place a $(K^*_5, \mathrm{D52})$-design. The remaining arcs form arc-disjoint copies of $K^*_{4,4}$, each of which can be decomposed into copies of $K^*_{2,2}$. On each of these copies of $K^*_{2,2}$, we place a $(K^*_{2,2}, \mathrm{D52})$-design. ■

Lemma 11. *For every integer $x \geq 2$ there exists a $(K^*_{5x+1}, \mathrm{D99})$-design.*

Proof. For $x = 2$, we have the following $(K^*_{11}, \mathrm{D99})$-design:

$$\begin{aligned}\{&\mathrm{D99}[0,1,4,7], \mathrm{D99}[7,3,2,6], \mathrm{D99}[3,0,8,5], \mathrm{D99}[9,5,2,4], \mathrm{D99}[9,8,3,6],\\ &\mathrm{D99}[5,10,8,3], \mathrm{D99}[7,8,4,1], \mathrm{D99}[0,2,5,1], \mathrm{D99}[0,6,7,5], \mathrm{D99}[1,3,5,2],\\ &\mathrm{D99}[2,10,1,6], \mathrm{D99}[3,4,2,5], \mathrm{D99}[4,6,2,8], \mathrm{D99}[4,9,8,1], \mathrm{D99}[6,7,2,9],\\ &\mathrm{D99}[6,9,5,3], \mathrm{D99}[8,0,2,7], \mathrm{D99}[8,10,4,0], \mathrm{D99}[9,0,1,10],\\ &\mathrm{D99}[10,4,6,1], \mathrm{D99}[10,7,3,5], \mathrm{D99}[10,9,1,7]\}.\end{aligned}$$

For $x = 3$, consider the digraph $G = \mathrm{D99}[0,7,2,1] \cup \mathrm{D99}[0,13,8,3] \cup \mathrm{D99}[0,4,5,6]$. It is easy to check that we have a directed ρ-labeling of

G, and thus a $(K^*_{16}, \mathrm{D99})$-design exists. For $x > 3$, we require the following $(K^*_{5,10}, \mathrm{D99})$-design:

$$\{\mathrm{D99}[0,5,6,2], \mathrm{D99}[1,6,7,3], \mathrm{D99}[2,7,8,4], \mathrm{D99}[3,8,9,0], \mathrm{D99}[4,9,5,1], \mathrm{D99}[5,3,4,14], \mathrm{D99}[6,4,0,10], \mathrm{D99}[7,0,1,11], \mathrm{D99}[8,1,2,12], \mathrm{D99}[9,2,3,13], \mathrm{D99}[12,0,4,8], \mathrm{D99}[13,1,0,9], \mathrm{D99}[14,2,1,5], \mathrm{D99}[10,3,2,6], \mathrm{D99}[11,4,3,7], \mathrm{D99}[0,14,11,1], \mathrm{D99}[1,10,12,2], \mathrm{D99}[2,11,13,3], \mathrm{D99}[3,12,14,4], \mathrm{D99}[4,13,10,0]\};$$

and the argument proceeds similarly as in the proof for Lemma 9. ■

3.2 Decompositions of K^*_{mx}

Necessary condition (A) implies that there is no (K^*_3, D)-design for any digraph $D \in \{\mathrm{D28}, \mathrm{D31}, \mathrm{D32}, \mathrm{D33}, \mathrm{D36}, \mathrm{D37}, \mathrm{D38}, \mathrm{D39}, \mathrm{D40}\}$. Furthermore, there is no $(K^*_2, \mathrm{D27})$-design.

Now, let $D \in \{\mathrm{D7}, \mathrm{D10}, \mathrm{D28}, \mathrm{D40}, \mathrm{D44}, \mathrm{D63}\}$ and suppose D has m arcs. Then necessary condition (C) implies that there is no (K^*_{mx}, D)-design for any positive integer x.

Throughout this section we let $V(K^*_m) = [0, m-2] \cup \{\infty\}$.

Lemma 12. *For every integer $x \geq 2$ there exists a (K^*_{3x}, D)-design for $D \in \{\mathrm{D31}, \mathrm{D36}\}$.*

Proof. Since D36 is the reverse orientation of D31, it suffices to show the existence of a $(K^*_{3x}, \mathrm{D31})$-design. Let $x \geq 2$ be an integer. The following is a $(K^*_{3x}, \mathrm{D31})$-design:

$$\{\mathrm{D31}[0,1,\infty,2], \mathrm{D31}[0,\infty,1,3]\} \cup \{\mathrm{D31}[0, 3+3(i-3), 4+3(i-3), 6+3(i-3)] \colon 3 \leq i \leq x\}.$$ ■

Lemma 13. *For every integer $x \geq 2$ there exists a (K^*_{3x}, D)-design for $D \in \{\mathrm{D32}, \mathrm{D39}\}$.*

Proof. Since D39 is the reverse orientation of D32, it suffices to show the existence of a $(K^*_{3x}, \mathrm{D32})$-design. Let $x \geq 2$ be an integer. The following is a $(K^*_{3x}, \mathrm{D32})$-design:

$$\{\mathrm{D32}[0,1,\infty,2]\} \cup \{\mathrm{D32}[0, 3+3(i-2), 4+3(i-2), 1] \colon 2 \leq i \leq x\}.$$ ■

Lemma 14. *For every integer $x \geq 1$ there exists a (K^*_{4x}, D)-design for $D \in \{\mathrm{D41}, \mathrm{D62}\}$.*

Proof. Since D62 is the reverse orientation of D41, it suffices to show the existence of a $(K_{4x}^*, \mathrm{D41})$-design. Let $x \geq 1$ be an integer. The following is a $(K_{4x}^*, \mathrm{D41})$-design:

$$\{\mathrm{D41}[0,1,2,\infty]\} \cup \{\mathrm{D41}[0,1+4(i-1),2+4(i-1),4(i-1)]\colon\ 2 \leq i \leq x\}. \ \blacksquare$$

Lemma 15. *For every integer $x \geq 2$ there exists a $(K_{4x}^*, \mathrm{D50})$-design.*

Proof. Let $x \geq 2$ be an integer.
Case 1: $x = 2k$ for some integer $k \geq 1$. The following is a $(K_{8k}^*, \mathrm{D50})$-design:

$$\begin{aligned} &\{\mathrm{D50}[0,1,2,\infty], \mathrm{D50}[0,6,5,4]\} \\ &\cup \{\mathrm{D50}[0,1+8(i-1),2+8(i-1),8(i-1)]\colon\ 2 \leq i \leq x\} \\ &\cup \{\mathrm{D50}[0,6+8(i-1),5+8(i-1),4+8(i-1)]\colon\ 2 \leq i \leq x\}. \end{aligned}$$

Case 2: $x = 2k+1$ for some positive integer $k \geq 1$. For $k = 1$, consider the digraph $G = \mathrm{D50}[0,7,3,\infty] \cup \mathrm{D50}[0,3,5,1] \cup \mathrm{D50}[0,5,7,2]$. It is easy to check that we have a 1-rotational directed ρ-labeling of G, and thus a $(K_{12}^*, \mathrm{D50})$-design exists. For $k > 1$, we write $K_{8k+4}^* \cong K_{12}^* \vee \bigvee_{i=1}^{k-1} K_8^*$. On the copy of K_{12}^*, we place a $(K_{12}^*, \mathrm{D50})$-design. On each of the $k-1$ copies of K_8^*, we place a $(K_8^*, \mathrm{D50})$-design, which is shown to exist in the proof of Case 1. The remaining arcs form arc-disjoint copies of $K_{8,12}^*$ and $K_{8,8}^*$, both of which can be decomposed into copies of $K_{4,4}^*$. On each of these copies of $K_{4,4}^*$, we place a $(K_{4,4}^*, \mathrm{D50})$-design, which is shown to exist in the proof for Lemma 9. ■

Lemma 16. *For every integer $x \geq 2$ there exists a $(K_{4x}^*, \mathrm{D51})$-design.*

Proof. Let $x \geq 2$ be an integer. The following is a $(K_{4x}^*, \mathrm{D51})$-design:

$$\begin{aligned} &\{\mathrm{D51}[0,\infty,1,4], D51[0,3,2,\infty]\} \\ &\cup \{\mathrm{D51}[0,6+4(i-3),7+4(i-3),2]\colon\ 3 \leq i \leq x\}. \end{aligned} \ \blacksquare$$

Lemma 17. *For every integer $x \geq 1$ there exists a (K_{4x}^*, D)-design for $D \in \{\mathrm{D52}, \mathrm{D64}\}$.*

Proof. Since D64 is the reverse orientation of D52, it suffices to show the existence of a $(K_{4x}^*, \mathrm{D52})$-design. For $x = 1$, consider the digraph $\mathrm{D52}[0,1,\infty,2]$. It is easy to check that we have a 1-rotational directed ρ-labeling of D52, and thus a $(K_4^*, \mathrm{D52})$-design exists. For $x > 1$, we write $K_{4x}^* \cong \bigvee_{i=1}^{x} K_4^*$. On each of the x copies of K_4^*, we place a $(K_4^*, \mathrm{D52})$-design. The remaining arcs form arc-disjoint copies of $K_{4,4}^*$, each of which can be decomposed into copies of $K_{2,2}^*$. On each of these copies of $K_{2,2}^*$, we place a $(K_{2,2}^*, \mathrm{D52})$-design, which is shown to exist in the proof for Lemma 10. ■

Lemma 18. *For every integer $x \geq 1$ there exists a (K_{4x}^*, D)-design for $D \in \{\text{D54}, \text{D56}\}$.*

Proof. Since D56 is the reverse orientation of D54, it suffices to show the existence of a $(K_{4x}^*, \text{D54})$-design. Let $x \geq 1$ be an integer. The following is a $(K_{4x}^*, \text{D54})$-design:

$$\{\text{D54}[0, \infty, 2, 1]\} \cup \{\text{D54}[0, 4 + 4(i-2), 6 + 4(i-2), 1]\colon\ 2 \leq i \leq x\}. \quad \blacksquare$$

Lemma 19. *For every integer $x \geq 1$ there exists a (K_{5x}^*, D)-design for $D \in \{\text{D70}, \text{D91}\}$.*

Proof. Since D91 is the reverse orientation of D70, it suffices to show the existence of a $(K_{5x}^*, \text{D70})$-design. Let $x \geq 1$ be an integer. The following is a $(K_{5x}^*, \text{D70})$-design:

$$\{\text{D70}[0, \infty, 2, 1]\} \cup \{\text{D70}[0, 5(i-1), 2+5(i-1), 1+5(i-1)]\colon\ 2 \leq i \leq x\}. \quad \blacksquare$$

Lemma 20. *For every integer $x \geq 1$ there exists a (K_{5x}^*, D)-design for $D \in \{\text{D77}, \text{D92}\}$.*

Proof. Since D92 is the reverse orientation of D77, it suffices to show the existence of a $(K_{5x}^*, \text{D77})$-design. Let $x \geq 1$ be an integer. The following is a $(K_{5x}^*, \text{D77})$-design:

$$\{\text{D77}[0, \infty, 3, 1]\} \cup \{\text{D77}[0, 5(i-1), 3 + 5(i-1), 1]\colon\ 2 \leq i \leq x\}. \quad \blacksquare$$

Lemma 21. *For every integer $x \geq 1$ there exists a (K_{5x}^*, D)-design for $D \in \{\text{D84}, \text{D104}\}$.*

Proof. Since D104 is the reverse orientation of D84, it suffices to show the existence of a $(K_{5x}^*, \text{D84})$-design. Let $x \geq 1$ be an integer. The following is a $(K_{5x}^*, \text{D84})$-design:

$$\{\text{D84}[0, \infty, 2, 1]\} \cup \{\text{D84}[4 + 5(i-2), 0, 2, 5 + 5(i-2)]\colon\ 2 \leq i \leq x\}. \quad \blacksquare$$

Lemma 22. *For every integer $x \geq 1$ there exists a $(K_{5x}^*, \text{D86})$-design.*

Proof. For $x = 1$, we have the following $(K_5^*, \text{D86})$-design:

$$\{\text{D86}[0, 1, 2, 3], \text{D86}[0, 3, \infty, 2], \text{D86}[\infty, 1, 0, 2], \text{D86}[\infty, 3, 2, 1]\}.$$

For $x > 1$, we require the following $(K_{5,5}^*, \text{D86})$-design:

$$\{\text{D86}[0, 5, 7, 4], \text{D86}[1, 6, 8, 0], \text{D86}[2, 7, 9, 1], \text{D86}[3, 8, 5, 2], \text{D86}[4, 9, 6, 3], \\ \text{D86}[6, 2, 4, 5], \text{D86}[7, 3, 0, 6], \text{D86}[8, 4, 1, 7], \text{D86}[9, 0, 2, 8], \text{D86}[5, 1, 3, 9]\};$$

and the argument proceeds similarly as in the proof for Lemma 17. $\blacksquare$

Lemma 23. *For every integer $x \geq 1$ there exists a $(K^*_{5x}, \mathrm{D99})$-design.*

Proof. For $x = 1$, consider the digraph D99$[0, 2, \infty, 1]$. It is easy to check that we have the 1-rotational directed ρ-labeling of D99, and thus a $(K^*_5,$ D99)-design exists. For $x = 2$, We have the following $(K^*_{10}, \mathrm{D99})$-design:

$$\{\mathrm{D99}[1,0,4,5], \mathrm{D99}[2,0,5,6], \mathrm{D99}[3,0,6,4], \mathrm{D99}[0,7,4,1], \mathrm{D99}[0,8,5,2], \mathrm{D99}[0,9,6,3], \mathrm{D99}[7,4,1,5], \mathrm{D99}[8,5,2,6], \mathrm{D99}[9,6,3,4], \mathrm{D99}[4,8,2,7], \mathrm{D99}[5,9,3,8], \mathrm{D99}[6,7,1,9], \mathrm{D99}[8,1,7,5], \mathrm{D99}[9,2,8,6], \mathrm{D99}[7,3,9,4], \mathrm{D99}[2,1,4,9], \mathrm{D99}[3,2,5,7], \mathrm{D99}[1,3,6,8]\}.$$

For $x > 2$, we require a $(K^*_{5,10}, \mathrm{D99})$-design, which is shown to exist in the proof for Lemma 11.
Case 1: $x = 2k + 1$ for some integer $k \geq 2$.
We write $K^*_{10k+5} \cong K^*_5 \vee \bigvee_{i=1}^{k} K^*_{10}$. On the copy of K^*_5, we place a $(K^*_5, \mathrm{D9})$-design. On each of the k copies of K^*_{10}, we place a $(K^*_{10}, \mathrm{D99})$-design. The remaining arcs form arc-disjoint copies of $K^*_{5,10}$ and $K^*_{10,10}$, which can be decomposed into copies of $K^*_{5,10}$. On each of these copies of $K^*_{5,10}$, we place a $(K^*_{5,10}, \mathrm{D99})$-design.
Case 2: $x = 2k$ for some integer $k \geq 2$.
We write $K^*_{10k} \cong \bigvee_{i=1}^{k} K^*_{10}$. On each of the k copies of K^*_{10}, we place a $(K^*_{10}, \mathrm{D99})$-design. The remaining arcs form arc-disjoint copies of $K^*_{10,10}$, each of which can be decomposed into copies of $K^*_{5,10}$. On each of these copies of $K^*_{5,10}$, we place a $(K^*_{5,10}, \mathrm{D99})$-design. ■

Lemma 24. *For every integer $x \geq 1$ there exists a $(K^*_{5x}, \mathrm{D100})$-design.*

Proof. Let $x \geq 1$ be an integer. The following is a $(K^*_{5x}, \mathrm{D100})$-design:

$$\{\mathrm{D100}[0, 1, 2, \infty]\} \cup \{\mathrm{D100}[0, 1 + 5(i-1), 2 + 5(i-1), 2] \colon\ 2 \leq i \leq x\}. \quad ■$$

Thus, we have shown that Theorem 1 holds.

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