On the λ-fold spectra of tripartite multigraphs of order 4 and size 5

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Abstract

Necessary and sufficient conditions on λ and n are established for the existence of a $({}^{\lambda}K_n, G)$-design where G is any of the three multigraphs with 5 edges and edge multiplicity at most 2 whose underlying graph is a triangle with a pendant edge.

1 Introduction

Let $\mathbb{Z}_n$ be the group of integers modulo n. For a finite set S and a positive integer λ, we let ${}^{\lambda}S$ denote the multiset that contains every element of S exactly λ times. For example, ${}^{3}\{a, b\}$ is the multiset $\{a, a, a, b, b, b\}$. Similarly for a graph G, we let ${}^{\lambda}G$ denote the multigraph obtained by replacing each edge in G with λ parallel edges. Thus ${}^{\lambda}K_n$ denotes the λ-fold complete multigraph of order n. We note that a multigraph is not required to contain multiple edges, thus throughout this we may refer to a multigraph as a graph. However, our graphs contain no loops.

If G and K are multigraphs with $V(G) \subseteq V(K)$ and $E(G) \subseteq E(K)$, then we shall refer to G as a *subgraph* of K (in order to avoid having to use terms such as "submultigraph"). For positive integers r and s, let $K_{r\times s}$

*Research supported by National Science Foundation Grant No. A1359300

denote the complete multipartite graph with r parts each of cardinality s. Furthermore, the complete multipartite graph with r parts each of cardinality s and one part of size t will denoted by $K_{r\times s,t}$. The *order* and *size* of a multigraph G refer to $|V(G)|$ and $|E(G)|$, respectively.

For positive integers λ, n, and t with $t \leq n$, the *complete λ-fold multigraph of order n with a hole of order t*, denoted ${}^{\lambda}K_n \setminus {}^{\lambda}K_t$ or ${}^{\lambda}(K_n \setminus K_t)$, is the multigraph with vertex set V and edge multiset ${}^{\lambda}\{\, \{u,v\} : u \in V, v \in V \setminus U \,\}$ where $|V| = n$, $|U| = t$, and $U \subseteq V$. In a $({}^{\lambda}K_n \setminus {}^{\lambda}K_t, G)$-design, we say that the *hole* of the design is U.

Let G and K be graphs with G a subgraph of K. A *G-decomposition* of K is a set (or multiset) $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that each edge of K appears in exactly one G_i. Similarly, if G and H are each subgraphs of K, then a $\{G,H\}$-decomposition of K is defined to be a set $\{H_1, H_2, \ldots, H_t\}$ of subgraphs of K each of which is isomorphic to either G or H and such that each edge of K appears in exactly one H_i. A G-decomposition of K is also known as a *(K,G)-design*. A $({}^{\lambda}K_n, G)$-design is called a *G-design* of *order n* and *index λ*. The study of graph decompositions is generally known as the study of graph designs, or G-designs. For a recent survey on G-designs of index 1 see [1].

Let G be a graph. A classical problem in the study of graph designs is to find necessary and sufficient conditions for the existence of a G-decomposition of ${}^{\lambda}K_n$. This is known as the *spectrum problem* for G. The set of all such n is called the *spectrum for G-designs of index λ*. The spectrum for G-designs of index 1 has been determined for several classes of graphs including cycles, paths, stars, and simple graphs of order at most 5 (see [2]).

In recent years, there have been some investigations of G-designs of index λ where G is a multigraph with edge multiplicity at least 2. For example, in [3] Carter determined the spectrum for G-designs of index λ for all connected cubic multigraphs G of order at most 6. The spectra for G-designs of index λ have been investigated for various bipartite multigraphs G of small order (see for example [5] and [8]). In this paper we consider tripartite multigraphs of small order. In particular, we consider the three multigraphs with 5 edges and edge multiplicity at most 2 whose underlying graph is a triangle with a pendant edge (see Figure 1). In [5] necessary and sufficient conditions for a $({}^{\lambda}K_n, G_2)$-design were given in the case where $n \geq 6$, along with partial results on the existence of $({}^{\lambda}K_n, G_1)$- and $({}^{\lambda}K_n, G_3)$-designs. We settle the spectrum problem for all three graphs. Combining the results of this paper with the results from [5] and [8] settles the spectrum problem for the all connected multigraphs G of order 4 and size 5 with a few exceptions. In particular, there are a few graphs for which it remains to determine whether ${}^{\lambda}K_4$ or ${}^{\lambda}K_5$ admit decompositions for λ being in certain

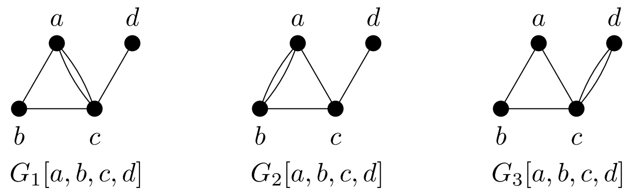

Figure 1: The three multigraphs for which we settle the spectrum problem.

congruence classes modulo 5, and there are two graphs for which more work than mentioned previously is needed.

Let $G \in \{G_1, G_2, G_3\}$. Then $G[a, b, c, d]$ denotes the graph with vertex set $\{a, b, c, d\}$ and edge set as represented in Figure 1. For example, $G_3[2, 4, 6, 8]$ denotes the graph with vertex set $\{2, 4, 6, 8\}$ and edge multiset $\{\{2, 4\}, \{4, 6\}, \{2, 6\}, \{6, 8\}, \{6, 8\}\}$.

Applying the obvious necessary conditions imposed by the degrees and number of edges in the graphs in question, we obtain the following two lemmas.

Lemma 1. *Let $\lambda \geq 2$ and $n \geq 4$ be integers and let $G \in \{G_1, G_2\}$. If there exists a $({}^{\lambda}K_n, G)$-design, then the following necessarily hold:*

1. *if* $\gcd(\lambda, 5) = 1$, *then* $n \equiv 0$ *or* $1 \pmod 5$*;*
2. *if* $\gcd(\lambda, 5) = 5$, *then* $n \geq 4$*.*

Lemma 2. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. If there exists a $({}^{\lambda}K_n, G_3)$-design, then the following necessarily hold:*

1. *if λ is even and* $\gcd(\lambda, 5) = 1$, *then* $n \equiv 0$ *or* $1 \pmod 5$*;*
2. *if λ is even and* $\gcd(\lambda, 5) = 5$, *then* $n \geq 4$*;*
3. *if λ is odd and* $\gcd(\lambda, 5) = 1$, *then* $n \equiv 1$ *or* $5 \pmod{10}$*;*
4. *if λ is odd and* $\gcd(\lambda, 5) = 5$, *then n is odd.*

The focus of this paper is to prove that the necessary conditions in the above two lemmas are sufficient with the exceptions that a $({}^{2}K_5, G_2)$-design does not exist and that a $({}^{10t+5}K_4, G_2)$-design does not exist for any $t \geq 0$. We utilize Wilson's Fundamental Construction (also known as the "blow-up method") for constructing our graph designs.

2 Examples of Small Designs

We first turn our attention to some designs of small order which will be used for the general constructions. Let $G \in \{G_1, G_2, G_3\}$. Recall that we use

$G[a,b,c,d]$ to denote a labeled copy of G, as shown in Figure 1. For example, $G_1[1,2,3,4]$ denotes the graph with vertex set $\{1,2,3,4\}$ with edge multiset $\big\{\{1,2\},\{2,3\},\{1,3\},\{1,3\},\{3,4\}\big\}$. Given a graph $G[a,b,c,d]$ and some $i \in \mathbb{Z}_n$, we define $G[a,b,c,d]+i = G[a+i,b+i,c+i,d+i]$ where all addition is performed in $\mathbb{Z}_n$. We furthermore use the following two conventions: that $\infty + 1 = \infty$ and that any subscript on a vertex is unaffected by adding a quantity to the vertex. For example, $\infty_2 + 1 = \infty_2$ and $0_\ell + 1 = 1_\ell$.

Example 1. There exists a $({}^5K_4, G_1)$-design.

Let $V({}^\lambda K_4) = \mathbb{Z}_3 \cup \{\infty\}$. The following is a $({}^5K_4, G_1)$-design:

$$\textstyle\bigcup_{i\in\mathbb{Z}_3}\{G_1[0,1,2,\infty]+i, G_1[0,1,\infty,2]+i\}.$$

Example 2. A $({}^\lambda K_5, G_i)$-design exists for all $(\lambda, i) \in \{(2,1),(2,3),(3,1),(3,2),(3,3),(4,2),(5,2)\}$.

Let $V({}^\lambda K_5) = \mathbb{Z}_4 \cup \{\infty\}$. The following is a $({}^2K_5, G_1)$-design:

$$\textstyle\bigcup_{i\in\mathbb{Z}_4}\{G_1[0,\infty,1,3]+i\}.$$

The following is a $({}^2K_5, G_3)$-design:

$$\big\{G_3[1,\infty,0,2], G_3[1,\infty,0,3], G_3[1,2,3,\infty], G_3[1,3,2,\infty]\big\}.$$

The following is a $({}^3K_5, G_1)$-design:

$$\big\{G_1[3,1,2,0], G_1[\infty,3,2,0], G_1[0,3,\infty,2], \\ G_1[1,\infty,0,3], G_1[1,\infty,3,0], G_1[2,0,1,\infty]\big\}.$$

The following is a $({}^3K_5, G_2)$-design:

$$\big\{G_2[2,0,\infty,3], G_2[\infty,0,3,2], G_2[\infty,2,1,3], \\ G_2[\infty,1,3,2], G_2[1,2,0,3], G_2[0,1,3,2]\big\}.$$

The following is a $({}^3K_5, G_3)$-design:

$$\big\{G_3[0,1,\infty,2], G_3[0,\infty,1,3], G_3[1,2,0,3], \\ G_3[3,\infty,0,2], G_3[1,\infty,3,2], G_3[\infty,3,2,1]\big\}.$$

The following is a $({}^4K_5, G_2)$-design:

$$\textstyle\bigcup_{i\in\mathbb{Z}_4}\{G_2[0,\infty,1,3]+i, G_2[0,1,2,\infty]+i\}.$$

Now, let $V({}^\lambda K_5) = \mathbb{Z}_5$. The following is a $({}^5K_5, G_2)$-design:

$$\textstyle\bigcup_{i\in\mathbb{Z}_5}\{G_2[0,2,1,3]+i, G_2[0,1,2,4]+i\}.$$

Example 3. A $({}^3K_6, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^3K_6) = \{1,2,3,4,5,6\}$. The following is a $({}^3K_6, G_1)$-design:

$$\big\{G_1[2,1,4,5], G_1[1,5,3,2], G_1[5,1,3,4], G_1[1,4,2,3], G_1[2,6,5,1],$$
$$G_1[6,3,2,5], G_1[5,4,6,1], G_1[3,6,4,5], G_1[1,4,6,3]\big\}.$$

The following is a $({}^3K_6, G_2)$-design:

$$\big\{G_2[2,3,5,6], G_2[1,6,3,2], G_2[1,2,5,6], G_2[1,3,6,2], G_2[4,2,5,1],$$
$$G_2[4,1,2,6], G_2[5,3,4,1], G_2[6,4,5,1], G_2[4,3,6,2]\big\}.$$

Example 4. A $({}^5K_7, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^5K_7) = \mathbb{Z}_7$. The following is a $({}^5K_7, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_7}\{G_1[1,0,3,5]+i, G_1[2,0,3,1]+i, G_1[1,0,4,3]+i\}.$$

The following is a $({}^5K_7, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_7}\{G_2[0,1,4,3]+i, G_2[0,5,3,2]+i, G_2[0,2,3,6]+i\}.$$

Example 5. A $({}^5K_8, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^5K_8) = \mathbb{Z}_7 \cup \{\infty\}$. The following is a $({}^5K_8, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_7}\{G_1[3,\infty,2,0]+i, G_1[3,\infty,1,0]+i,$$
$$G_1[6,4,3,\infty]+i, G_1[6,4,3,0]+i\}.$$

The following is a $({}^5K_8, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_7}\{G_2[0,3,\infty,1]+i, G_2[0,2,1,\infty]+i,$$
$$G_2[0,2,1,4]+i, G_2[0,3,1,\infty]+i\}.$$

Example 6. A $({}^5K_9, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^5K_9) = \mathbb{Z}_9$. The following is a $({}^5K_9, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_9}\{G_1[0,1,2,3]+i, G_1[0,2,4,6]+i,$$
$$G_1[0,4,3,6]+i, G_1[0,4,3,7]+i\}.$$

The following is a $({}^5K_9, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_9}\{G_2[0,1,2,4]+i, G_2[0,2,4,7]+i,$$
$$G_2[0,3,4,7]+i, G_2[0,4,3,7]+i\}.$$

Example 7. A $({}^3K_{10}, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^3K_{10}) = \mathbb{Z}_9 \cup \{\infty\}$. The following is a $({}^3K_{10}, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_9}\{G_1[\infty,0,1,2]+i, G_1[3,4,0,2]+i, G_1[4,2,0,3]+i\}.$$

The following is a $({}^3K_{10}, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_9}\{G_2[\infty,6,2,0]+i, G_2[0,1,3,7]+i, G_2[0,3,1,5]+i\}.$$

Example 8. A $({}^3K_{11}, G_i)$-design exists for all $i \in \{1,2,3\}$.

Let $V({}^3K_{11}) = \mathbb{Z}_{11}$. The following is a $({}^3K_{11}, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_{11}}\{G_1[2,7,0,1]+i, G_1[7,2,0,1]+i, G_1[3,8,0,1]+i\}.$$

The following is a $({}^3K_{11}, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_{11}}\{G_2[0,5,10,6]+i, G_2[0,10,7,4]+i, G_2[0,2,9,6]+i\}.$$

The following is a $({}^3K_{11}, G_3)$-design:

$$\bigcup_{i\in\mathbb{Z}_{11}}\{G_3[0,2,4,3]+i, G_3[0,2,10,5]+i, G_3[0,3,8,4]+i\}.$$

Example 9. A $({}^5K_{12}, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^5K_{12}) = \mathbb{Z}_{11} \cup \{\infty\}$. The following is a $({}^5K_{12}, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_{11}}\{G_1[\infty,0,1,6]+i, G_1[0,\infty,1,6]+i, G_1[0,1,2,7]+i,$$
$$G_1[0,2,4,8]+i, G_1[0,4,8,5]+i, G_1[0,3,6,4]+i\}.$$

The following is a $({}^5K_{12}, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_{11}}\{G_2[0,5,\infty,1]+i, G_2[0,5,7,\infty]+i, G_2[0,4,7,\infty]+i,$$
$$G_2[0,1,2,4]+i, G_2[0,2,3,6]+i, G_2[0,3,4,9]+i\}.$$

Example 10. A $({}^5K_{13}, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^5K_{13}) = \mathbb{Z}_{13}$. The following is a $({}^5K_{13}, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_{13}}\{G_1[0,6,5,1]+i, G_1[0,5,3,2]+i, G_1[0,6,4,3]+i,$$
$$G_1[0,5,6,3]+i, G_1[0,5,2,1]+i, G_1[0,6,4,1]+i\}.$$

The following is a $({}^5K_{13}, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_{13}}\{G_2[0,5,6,3]+i, G_2[0,3,5,4]+i, G_2[0,4,6,5]+i,$$
$$G_2[0,6,5,2]+i, G_2[0,2,5,4]+i, G_2[0,4,6,2]+i\}.$$

Example 11. A $({}^5K_{14}, G_i)$-design exists for all $i \in \{1, 2\}$.

Let $V({}^5K_{14}) = \mathbb{Z}_{13} \cup \{\infty\}$. The following is a $({}^5K_{14}, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_{13}} \{G_1[\infty, 8, 4, 0] + i, G_1[0, 3, 6, \infty] + i, G_1[0, 6, 2, \infty] + i, G_1[0, 3, 2, 1] + i, G_1[0, 4, 3, 1] + i, G_1[0, 6, 5, 1] + i, G_1[0, 6, 5, 10] + i\}.$$

The following is a $({}^5K_{14}, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_{13}} \{G_2[\infty, 8, 4, 0] + i, G_2[0, 6, 3, \infty] + i, G_2[0, 2, 6, \infty] + i, G_2[0, 2, 3, 1] + i, G_2[0, 3, 4, 5] + i, G_2[0, 5, 6, 2] + i, G_2[0, 5, 6, 1] + i\}.$$

Example 12. A $({}^3K_{20}, G_i)$-design exists for all $i \in \{1, 2\}$.

Let $V({}^3K_{20}) = \mathbb{Z}_{19} \cup \{\infty\}$. The following is a $({}^3K_{20}, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_{19}} \{G_1[\infty, 0, 1, 2] + i, G_1[3, 4, 0, 2] + i, G_1[4, 2, 0, 3] + i, G_1[0, 5, 10, 17] + i, G_1[0, 5, 11, 3] + i, G_1[0, 6, 12, 3] + i\}.$$

The following is a $({}^3K_{20}, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_{19}} \{G_2[\infty, 6, 2, 0] + i, G_2[0, 1, 3, 7] + i, G_2[0, 3, 1, 5] + i, G_2[0, 6, 11, 2] + i, G_2[0, 8, 13, 4] + i, G_2[0, 7, 12, 3] + i\}.$$

Example 13. A $({}^3K_{21}, G_i)$-design exists for all $i \in \{1, 2, 3\}$.

Let $V({}^3K_{21}) = \mathbb{Z}_{21}$. The following is a $({}^3K_{21}, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_{21}} \{G_1[0, 1, 2, 8] + i, G_1[0, 2, 3, 9] + i, G_1[0, 3, 7, 13] + i, G_1[0, 4, 8, 16] + i, G_1[0, 5, 10, 19] + i, G_1[0, 7, 12, 1] + i\}.$$

The following is a $({}^3K_{21}, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_{21}} \{G_2[0, 5, 10, 6] + i, G_2[0, 10, 7, 4] + i, G_2[0, 2, 9, 6] + i, G_2[0, 1, 8, 14] + i, G_2[0, 4, 6, 12] + i, G_2[0, 8, 9, 18] + i\}.$$

The following is a $({}^3K_{21}, G_3)$-design:

$$\bigcup_{i\in\mathbb{Z}_{21}} \{G_3[0, 1, 2, 5] + i, G_3[0, 1, 6, 4] + i, G_3[0, 4, 12, 7] + i, G_3[0, 4, 8, 2] + i, G_3[0, 3, 11, 2] + i, G_3[0, 7, 14, 4] + i\}.$$

Example 14. A $({}^5K_{22}, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^5K_{22}) = \mathbb{Z}_{21} \cup \{\infty\}$. The following is a $({}^5K_{22}, G_1)$-design:

$$\begin{aligned}\textstyle\bigcup_{i\in\mathbb{Z}_{21}}\{&G_1[0,\infty,10,1]+i, G_1[0,\infty,9,1]+i, G_1[0,5,8,\infty]+i,\\ &G_1[0,2,5,1]+i, G_1[0,3,7,4]+i, G_1[0,9,10,3]+i,\\ &G_1[0,10,8,6]+i, G_1[0,9,6,5]+i, G_1[0,2,6,5]+i,\\ &G_1[0,6,7,5]+i, G_1[0,4,5,1]+i\}.\end{aligned}$$

The following is a $({}^5K_{22}, G_2)$-design:

$$\begin{aligned}\textstyle\bigcup_{i\in\mathbb{Z}_{21}}\{&G_2[1,2,3,\infty]+i, G_2[0,2,4,\infty]+i, G_2[0,3,6,\infty]+i,\\ &G_2[0,6,9,2]+i, G_2[0,7,10,\infty]+i, G_2[0,9,10,\infty]+i,\\ &G_2[0,5,10,4]+i, G_2[0,8,10,4]+i, G_2[0,4,8,1]+i,\\ &G_2[0,8,7,17]+i, G_2[0,5,9,18]+i\}.\end{aligned}$$

Example 15. A $({}^5K_{23}, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^5K_{23}) = \mathbb{Z}_{23}$. The following is a $({}^5K_{23}, G_1)$-design:

$$\begin{aligned}\textstyle\bigcup_{i\in\mathbb{Z}_{23}}\{&G_1[0,2,1,11]+i, G_1[0,5,2,12]+i, G_1[0,7,3,12]+i,\\ &G_1[0,10,4,13]+i, G_1[0,11,5,14]+i, G_1[0,11,6,14]+i,\\ &G_1[0,11,7,15]+i, G_1[0,11,8,15]+i, G_1[0,11,9,16]+i,\\ &G_1[0,8,10,16]+i, G_1[0,4,1,6]+i\}.\end{aligned}$$

The following is a $({}^5K_{23}, G_2)$-design:

$$\begin{aligned}\textstyle\bigcup_{i\in\mathbb{Z}_{23}}\{&G_2[0,1,2,13]+i, G_2[0,2,3,14]+i, G_2[0,3,4,15]+i,\\ &G_2[0,4,6,17]+i, G_2[0,3,5,15]+i, G_2[0,5,10,20]+i,\\ &G_2[0,6,10,19]+i, G_2[0,6,11,20]+i, G_2[0,9,19,11]+i,\\ &G_2[0,8,15,7]+i, G_2[0,7,16,9]+i\}.\end{aligned}$$

Example 16. A $({}^5K_{24}, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^5K_{24}) = \mathbb{Z}_{23} \cup \{\infty\}$. The following is a $({}^5K_{24}, G_1)$-design:

$$\begin{aligned}\textstyle\bigcup_{i\in\mathbb{Z}_{23}}\{&G_1[0,\infty,9,18]+i, G_1[\infty,0,6,12]+i, G_1[0,11,1,3]+i,\\ &G_1[0,11,2,5]+i, G_1[0,11,3,7]+i, G_1[0,11,4,9]+i,\\ &G_1[0,11,5,12]+i, G_1[0,10,1,8]+i, G_1[0,10,2,9]+i,\\ &G_1[0,10,3,11]+i, G_1[0,10,4,12]+i, G_1[0,6,5,13]+i\}.\end{aligned}$$

The following is a $({}^5K_{24}, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_{23}}\{G_2[1,2,3,\infty]+i, G_2[0,2,11,\infty]+i, G_2[0,3,6,\infty]+i,$$
$$G_2[0,6,9,2]+i, G_2[0,7,10,\infty]+i, G_2[0,9,11,\infty]+i,$$
$$G_2[0,5,10,4]+i, G_2[0,8,10,4]+i, G_2[0,4,8,1]+i,$$
$$G_2[0,8,7,11]+i, G_2[0,5,9,18]+i, G_2[0,11,22,12]\}.$$

Example 17. A $({}^\lambda K_{3\times 5}, G_i)$-design exists for all $(\lambda, i) \in \{(3,1),(3,2),(3,3),(5,1),(5,2)\}$.

Let $V({}^\lambda K_{3\times 5}) = \{0,3,6,9,12\} \cup \{1,4,7,10,13\} \cup \{2,5,8,11,14\}$. Note that any addition done on these vertices is done modulo 15. The following is a $({}^3K_{3\times 5}, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_{15}}\{G_1[0,2,1,3]+i, G_1[0,8,4,9]+i, G_1[0,7,5,12]+i\}.$$

The following is a $({}^3K_{3\times 5}, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_{15}}\{G_2[0,2,7,6]+i, G_2[0,4,5,1]+i, G_2[0,7,2,1]+i\}.$$

The following is a $({}^3K_{3\times 5}, G_3)$-design:

$$\bigcup_{i\in\mathbb{Z}_{15}}\{G_3[0,8,4,9]+i, G_3[0,2,1,3]+i, G_3[0,5,1,8]+i\}.$$

The following is a $({}^5K_{3\times 5}, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_{15}}\{G_1[0,1,2,6]+i, G_1[0,4,8,12]+i, G_1[0,5,10,9]+i,$$
$$G_1[0,14,7,2]+i, G_1[0,4,2,1]+i\}.$$

The following is a $({}^5K_{3\times 5}, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_{15}}\{G_2[0,1,5,4]+i, G_2[0,2,7,3]+i, G_2[0,4,2,9]+i,$$
$$G_2[0,5,4,11]+i, G_2[0,7,5,4]+i\}.$$

Example 18. A $({}^\lambda K_{5\times 5}, G_i)$-design exists for all $(\lambda, i) \in \{(3,1),(3,2),(3,3),(5,1),(5,2)\}$

The existence of a $(K_{5\times 5}, K_5)$-design easily follows from the existence of an affine plane of order 5. Furthermore, Example 2 shows that a $({}^\lambda K_5, G_i)$-design exists for all pairs $(\lambda, i) \in \{(3,1),(3,2),(3,3),(5,1),(5,2)\}$. Thus, for all such pairs (λ, i) a $({}^\lambda K_{5\times 5}, G_i)$-design exists.

Example 19. A $({}^2K_4 \setminus {}^2K_2, G_i)$-design exists for all $i \in \{1,2,3\}$.

Let $V({}^2K_4 \setminus {}^2K_2) = \{1,2\} \cup \{\infty_1, \infty_2\}$. The hole of the design will be $\{\infty_1, \infty_2\}$. The following is a $({}^2K_4 \setminus {}^2K_2, G_1)$-design:

$$\big\{G_1[\infty_1, 1, 2, \infty_2], G_1[\infty_2, 2, 1, \infty_1]\big\}.$$

The following is a $({}^2K_4 \setminus {}^2K_2, G_2)$-design:

$$\big\{G_2[\infty_1, 1, 2, \infty_2], G_2[\infty_2, 1, 2, \infty_1]\big\}.$$

The following is a $({}^2K_4 \setminus {}^2K_2, G_3)$-design:

$$\big\{G_3[\infty_1, 1, 2, \infty_2], G_3[\infty_1, 2, 1, \infty_2]\big\}.$$

Example 20. A $({}^5K_7 \setminus {}^5K_2, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^5K_7 \setminus {}^5K_2) = \mathbb{Z}_5 \cup \{\infty_1, \infty_2\}$. The hole of the design will be $\{\infty_1, \infty_2\}$. The following is a $({}^5K_7 \setminus {}^5K_2, G_1)$-design:

$$\bigcup_{i \in \mathbb{Z}_5} \{G_1[0, \infty_1, 2, 1] + i, G_1[\infty_1, 1, 0, 2] + i, G_1[1, \infty_2, 2, 0] + i, G_1[\infty_2, 1, 0, 2] + i\}.$$

The following is a $({}^5K_7 \setminus {}^5K_2, G_2)$-design:

$$\bigcup_{i \in \mathbb{Z}_5} \{G_2[\infty_1, 1, 0, \infty_2] + i, G_2[\infty_2, 2, 0, \infty_1] + i, G_2[0, 2, 1, \infty_1] + i, G_2[0, 2, 1, \infty_2] + i\}.$$

Example 21. A $({}^5K_8 \setminus {}^5K_3, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^5K_8 \setminus {}^5K_3) = \mathbb{Z}_5 \cup \{\infty_1, \infty_2, \infty_3\}$. The hole of the design will be $\{\infty_1, \infty_2, \infty_3\}$. The following is a $({}^5K_8 \setminus {}^5K_3, G_1)$-design:

$$\bigcup_{i \in \mathbb{Z}_5} \{G_1[2, 0, \infty_1, 1] + i, G_1[0, \infty_3, 2, \infty_1] + i, G_1[2, 0, \infty_2, 1] + i, G_1[0, \infty_3, 1, \infty_2] + i, G_1[1, 0, 2, \infty_3] + i\}.$$

The following is a $({}^5K_8 \setminus {}^5K_3, G_2)$-design:

$$\big\{G_2[\infty_1, 0, 2, 4], G_2[\infty_2, 0, 3, 4], G_2[4, \infty_2, 1, 0], G_2[0, 4, \infty_1, 1], G_2[\infty_1, 1, 3, 2], G_2[\infty_2, 3, 4, 2], G_2[\infty_1, 1, 2, 3], G_2[\infty_2, 1, 4, 3], G_2[2, \infty_1, 0, 1], G_2[\infty_2, 1, 2, 0], G_2[\infty_3, 4, 1, 0], G_2[\infty_3, 3, 1, 0], G_2[1, 4, 2, 0], G_2[\infty_2, 2, 0, \infty_1], G_2[\infty_3, 0, 3, \infty_1], G_2[\infty_3, 2, 4, \infty_1], G_2[4, 0, 3, \infty_1], G_2[\infty_1, 3, 4, \infty_2], G_2[\infty_3, 3, 0, \infty_2], G_2[\infty_3, 4, 0, \infty_2], G_2[\infty_3, 1, 2, \infty_2], G_2[\infty_1, 4, 2, \infty_3], G_2[\infty_2, 3, 2, \infty_3], G_2[1, 3, 0, \infty_3], G_2[2, 3, 1, \infty_3]\big\}.$$

Example 22. A $({}^5K_9 \setminus {}^5K_4, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^5K_9 \setminus {}^5K_4) = \mathbb{Z}_5 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. The hole of the design will be $\{\infty_1, \infty_2, \infty_3, \infty_4\}$. The following is a $({}^5K_9 \setminus {}^5K_4, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_5}\{G_1[0,\infty_1,2,\infty_3]+i, G_1[0,\infty_1,2,\infty_2]+i, G_1[0,2,\infty_2,1]+i,\\ G_1[0,\infty_3,1,\infty_1]+i, G_1[0,\infty_3,1,\infty_4]+i, G_1[0,1,\infty_4,2]+i\}.$$

The following is a $({}^5K_9 \setminus {}^5K_4, G_2)$-design:

$$\big\{G_2[\infty_1,3,1,0], G_2[\infty_1,2,4,\infty_2], G_2[\infty_1,0,3,\infty_2], G_2[\infty_4,1,0,\infty_2],\\ G_2[\infty_1,4,3,1], G_2[\infty_1,4,0,1], G_2[3,2,\infty_4,1], G_2[\infty_2,2,4,1],\\ G_2[\infty_3,0,4,\infty_4], G_2[\infty_4,1,3,2], G_2[\infty_2,1,0,2], G_2[1,4,\infty_2,2],\\ G_2[\infty_4,2,0,3], G_2[2,1,0,3], G_2[\infty_3,4,1,3], G_2[\infty_4,0,4,3],\\ G_2[\infty_1,1,2,4], G_2[\infty_3,2,0,4], G_2[\infty_1,1,3,4], G_2[\infty_3,1,2,4],\\ G_2[\infty_2,3,0,\infty_1], G_2[\infty_2,2,3,\infty_3], G_2[\infty_2,0,3,\infty_3], G_2[\infty_3,2,1,\infty_2],\\ G_2[\infty_2,4,1,\infty_3], G_2[\infty_3,0,4,\infty_4], G_2[\infty_4,2,3,\infty_3], G_2[\infty_1,2,0,\infty_4],\\ G_2[\infty_4,3,4,2], G_2[\infty_3,3,4,\infty_4]\big\}.$$

Example 23. A $({}^5K_{12} \setminus {}^5K_2, G_i)$-design exists for all $i \in \{1,2\}$.

Let $V({}^5K_{12} \setminus {}^5K_2) = \{0_\ell, 1_\ell, 2_\ell, 3_\ell, 4_\ell\} \cup \{0_r, 1_r, 2_r, 3_r, 4_r\} \cup \{\infty_1, \infty_2\}$. The hole of the design will be $\{\infty_1, \infty_2\}$. The following is a $({}^5K_{12} \setminus {}^5K_2, G_1)$-design:

$$\bigcup_{i\in\mathbb{Z}_5}\{G_1[0_\ell,0_r,\infty_1,1_r]+i, G_1[4_r,0_r,0_\ell,\infty_1]+i,\\ G_1[0_\ell,1_r,\infty_1,2_r]+i, G_1[0_\ell,2_r,\infty_2,0_r]+i, G_1[0_\ell,3_r,2_\ell,1_\ell]+i,\\ G_1[1_\ell,2_\ell,4_r,\infty_1]+i, G_1[1_r,1_\ell,0_\ell,\infty_2]+i, G_1[0_\ell,4_\ell,2_\ell,1_\ell]+i,\\ G_1[0_\ell,3_r,\infty_2,0_r]+i, G_1[0_r,4_r,1_r,3_r]+i, G_1[0_\ell,1_r,0_r,2_\ell]+i,\\ G_1[0_\ell,2_r,4_r,\infty_2]+i, G_1[0_r,3_\ell,2_r,0_\ell]+i\}.$$

The following is a $({}^5K_{12} \setminus {}^5K_2, G_2)$-design:

$$\bigcup_{i\in\mathbb{Z}_5}\{G_2[\infty_1,0_\ell,0_r,2_r]+i, G_2[\infty_1,4_\ell,0_r,0_\ell]+i,\\ G_2[\infty_1,2_r,0_\ell,1_\ell]+i, G_2[1_\ell,2_\ell,0_r,\infty_1]+i, G_2[0_r,1_r,2_r,0_\ell]+i,\\ G_2[\infty_2,4_\ell,0_r,1_\ell]+i, G_2[\infty_2,2_r,0_\ell,2_\ell]+i, G_2[0_\ell,2_\ell,1_r,3_\ell]+i,\\ G_2[\infty_2,0_\ell,0_r,4_\ell]+i, G_2[0_r,2_r,1_r,3_\ell]+i, G_2[0_\ell,2_\ell,0_r,4_\ell]+i,\\ G_2[1_\ell,2_\ell,0_r,\infty_2]+i, G_2[0_\ell,2_r,0_r,1_\ell]+i\}.$$

Example 24. A $({}^5K_{13} \setminus {}^5K_3, G_i)$-design exists for all $i \in \{1, 2\}$.

Let $V({}^5K_{13} \setminus {}^5K_3) = \{0_\ell, 1_\ell, 2_\ell, 3_\ell, 4_\ell\} \cup \{0_r, 1_r, 2_r, 3_r, 4_r\} \cup \{\infty_1, \infty_2, \infty_3\}$. The hole of the design will be $\{\infty_1, \infty_2, \infty_3\}$. The following is a $({}^5K_{13} \setminus {}^5K_3, G_1)$-design:

$$\begin{aligned}\bigcup_{i\in\mathbb{Z}_5}\{&G_1[\infty_1, 0_\ell, 0_r, 1_r] + i, G_1[\infty_2, 0_\ell, 1_r, 2_r] + i, G_1[\infty_3, 0_\ell, 2_r, 3_r] + i,\\ &G_1[\infty_1, 0_r, 0_\ell, 1_\ell] + i, G_1[\infty_2, 1_r, 0_\ell, 2_\ell] + i, G_1[\infty_3, 2_r, 0_\ell, 0_r] + i,\\ &G_1[2_\ell, \infty_1, 0_r, 2_r] + i, G_1[1_\ell, \infty_2, 0_r, 2_r] + i, G_1[3_\ell, \infty_3, 0_r, 2_r] + i,\\ &G_1[3_r, \infty_1, 0_\ell, 0_r] + i, G_1[4_r, \infty_2, 0_\ell, 0_r] + i, G_1[1_r, \infty_3, 0_\ell, 2_r] + i,\\ &G_1[2_\ell, 1_\ell, 0_\ell, 1_r] + i, G_1[2_\ell, 1_\ell, 0_\ell, 3_r] + i, G_1[2_r, 1_r, 0_r, 1_\ell] + i\}.\end{aligned}$$

The following is a $({}^5K_{13} \setminus {}^5K_3, G_2)$-design:

$$\begin{aligned}\bigcup_{i\in\mathbb{Z}_5}\{&G_2[0_\ell, 2_\ell, 1_r, \infty_1] + i, G_2[0_r, 2_r, 1_\ell, \infty_1] + i, G_2[1_\ell, 2_\ell, 3_r, \infty_1] + i,\\ &G_2[4_r, 4_\ell, 2_\ell, \infty_1] + i, G_2[0_\ell, \infty_1, 4_\ell, 3_r] + i, G_2[0_r, \infty_1, 4_r, 1_\ell] + i,\\ &G_2[0_\ell, \infty_2, 0_r, \infty_3] + i, G_2[1_r, 2_r, \infty_2, 3_r] + i, G_2[\infty_3, 3_\ell, 3_r, 0_r] + i,\\ &G_2[1_\ell, \infty_2, 3_r, 0_\ell] + i, G_2[\infty_3, 2_\ell, 3_r, 1_r] + i, G_2[3_r, 0_\ell, 4_\ell, \infty_2] + i,\\ &G_2[\infty_3, 4_r, 2_\ell, 1_\ell] + i, G_2[0_\ell, 2_\ell, 1_r, 3_r] + i, G_2[0_r, 4_r, 2_\ell, 2_r] + i\}.\end{aligned}$$

Example 25. A $({}^5K_{14} \setminus {}^5K_4, G_i)$-design exists for all $i \in \{1, 2\}$.

Let $V({}^5K_{14} \setminus {}^5K_4) = \{0_\ell, 1_\ell, 2_\ell, 3_\ell, 4_\ell\} \cup \{0_r, 1_r, 2_r, 3_r, 4_r\} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. The hole of the design will be $\{\infty_1, \infty_2, \infty_3, \infty_4\}$. The following is a $({}^5K_{14} \setminus {}^5K_4, G_1)$-design:

$$\begin{aligned}\bigcup_{i\in\mathbb{Z}_5}\{&G_1[\infty_1, 0_r, 0_\ell, 1_\ell] + i, G_1[\infty_2, 1_r, 0_\ell, 1_\ell] + i, G_1[\infty_3, 2_r, 0_\ell, 1_\ell] + i,\\ &G_1[\infty_4, 3_r, 0_\ell, 1_\ell] + i, G_1[\infty_1, 4_r, 0_\ell, 1_\ell] + i, G_1[\infty_2, 0_r, 0_\ell, 2_\ell] + i,\\ &G_1[\infty_3, 1_r, 0_\ell, 2_\ell] + i, G_1[\infty_4, 2_r, 0_\ell, 2_\ell] + i, G_1[\infty_1, 2_\ell, 0_r, 1_r] + i,\\ &G_1[\infty_2, 1_\ell, 0_r, 2_r] + i, G_1[\infty_3, 0_\ell, 0_r, 4_\ell] + i, G_1[\infty_4, 4_\ell, 0_r, 0_\ell] + i,\\ &G_1[1_r, 3_\ell, 0_r, \infty_1] + i, G_1[1_r, 2_\ell, 0_r, \infty_2] + i, G_1[2_r, 1_\ell, 0_r, \infty_3] + i,\\ &G_1[2_r, 0_\ell, 0_r, \infty_4] + i, G_1[2_\ell, 4_r, 0_\ell, 3_r] + i\}.\end{aligned}$$

The following is a $({}^5K_{14} \setminus {}^5K_4, G_2)$-design:

$$\begin{aligned}\bigcup_{i\in\mathbb{Z}_5}\{&G_2[\infty_1, 2_\ell, 0_\ell, 3_r] + i, G_2[0_\ell, 0_r, \infty_1, 1_\ell] + i, G_2[\infty_2, 1_\ell, 0_\ell, 0_r] + i,\\ &G_2[0_\ell, 0_r, \infty_2, 2_\ell] + i, G_2[\infty_3, 1_\ell, 0_\ell, 1_r] + i, G_2[0_\ell, 1_r, \infty_3, 3_\ell] + i,\\ &G_2[\infty_4, 1_\ell, 0_\ell, 2_r] + i, G_2[0_\ell, 1_r, \infty_4, 4_\ell] + i, G_2[\infty_1, 1_r, 2_r, 3_r] + i,\\ &G_2[1_\ell, 0_\ell, 4_r, \infty_1] + i, G_2[\infty_2, 1_r, 0_r, 1_\ell] + i, G_2[2_\ell, 0_\ell, 4_r, \infty_2] + i,\\ &G_2[\infty_3, 2_r, 0_r, 2_\ell] + i, G_2[2_r, 0_r, 1_r, \infty_3] + i, G_2[\infty_4, 2_r, 0_r, 2_\ell] + i,\\ &G_2[2_\ell, 0_\ell, 4_r, \infty_4] + i, G_2[3_\ell, 0_r, 2_r, 4_\ell] + i\}.\end{aligned}$$

3 Additional Building Blocks

For many of our general constructions we make use of group divisible designs, or GDDs. Let K be a set of positive integers. A K-GDD, is a triple (V, P, B) where V is a finite set of cardinality v, P is a partition of V whose parts are called *groups*, and B is a family of subsets, called *blocks*, of V such that

(i) if $b \in B$, then $|b| \in K$;

(ii) every pair of distinct points in V occurs in exactly one block or one group, but not both;

(iii) $|P| > 1$.

Furthermore, a K-GDD is said to be of *type* $g^s m^t$ if P consists of exactly s groups of cardinality g and t groups of cardinality m. For brevity, if $K = \{k\}$ then we write k-GDD instead of $\{k\}$-GDD.

It is useful to note the correspondence between GDDs and graph designs. The elements of the finite set on which the GDD is built can be thought of as vertices of a graph. For example, the blocks of a 3-GDD of type 2^x form a $(K_{x\times 2}, K_3)$-design. Similarly, A 3-GDD of type $2^x 4^1$ corresponds to a K_3-decomposition of $K_{x\times 2,4}$. By *blowing up* the vertices of a graph G by some positive integer t, we mean replacing every vertex of G with t independent vertices and replacing every edge in G with a $K_{t,t}$. Revisiting the previous example, after blowing up the vertices of a 3-GDD of type 2^x by 5, our corresponding $(K_{x\times 2}, K_3)$-design becomes a $(K_{x\times 10}, K_{3\times 5})$-design.

Two particularly useful results concerning the existence of GDDs are presented next. Both will play central roles in our general constructions.

Theorem 3 ([4])**.** *Let g, u, and m be non-negative integers. There exists a 3-GDD of type $g^u m^1$ if and only if the following conditions are all satisfied:*

1. *if $g > 0$ and $u \in \{1, 2\}$, then $(u, m) \in \{(1, 0), (2, g)\}$;*
2. *$gu = 0$ or $m \leq g(u - 1)$;*
3. *$gu = 0$ or $g(u - 1) + m$ is even;*
4. *gu is even or $m = 0$;*
5. *$\frac{1}{2}g^2u(u - 1) + gum \equiv 0 \pmod 3$.*

Theorem 3 is more general than we need in this setting. The following corollary proclaims the existence of two specific 3-GDDs that are used in our general constructions.

Corollary 4. Let $x \geq 3$ be an integer. If $x \equiv 0$ or $1 \pmod 3$, then there exists a 3-GDD of type 2^x. Otherwise, if $x \equiv 2 \pmod 3$, then there exists a 3-GDD of type $2^{x-2}4^1$.

Theorem 5 ([7])**.** *If n is odd then a $\{3,5\}$-GDD of type 1^n exists.*

The spectrum for $(K_4 - e)$-designs of index λ is known, and will help with our constructions. We note that the multigraph ${}^{\lambda}(K_4-e)$ is isomorphic to ${}^{\lambda}K_4 \setminus {}^{\lambda}K_2$ and will be referred to as such throughout this paper.

Theorem 6 ([6])**.** *A $({}^{\lambda}K_n, K_4-e)$-design exists if and only if $10|\lambda n(n-1)$, $n \geq 4$, and $(\lambda, n) \neq (1,5)$.*

4 Main Results

With all of the the necessary building blocks and conditions established, we now give our main results including our general constructions.

Lemma 7. *There does not exist a $({}^{2}K_5, G_2)$-design nor a $({}^{10t+5}K_4, G_2)$-design for any $t \geq 0$.*

Proof. Note that the degree of each verex in G_2 is either 3 or 1. Since ${}^{2}K_5$ is 8-regular, every vertex must occur twice as a degree-3 vertex and twice as a degree-1 vertex in a $({}^{2}K_5, G_2)$-design. Since G_2 contains only one degree-1 vertex, 10 copies of G_2 are needed. But such a design would contain only 4 copies of G_2.

Now suppose a $({}^{10t+5}K_4, G_2)$-design exists. Let $v \in V({}^{10t+5}K_4)$ and let x denote the number of times v occurs as a degree-3 vertex in the design and let y denote the number of times it occurs as a degree-1 vertex. Since G_2 spans ${}^{10t+5}K_4$, we have that $x + y = 6(2t+1)$, the number of copies of G_2 in the design, and that $3x + y = 3(10t+5)$, the degree of v in ${}^{10t+5}K_4$. Thus $2x = 18t + 9$, which has no integral solution for x. ■

Lemma 8. *Let $G \in \{G_1, G_2, G_3\}$. If $n \equiv 0$ or $1 \pmod 5$, then a $({}^{2}K_n, G)$-design exists with the single exception that a $({}^{2}K_5, G_2)$-design does not exist.*

Proof. Let $G \in \{G_1, G_2, G_3\}$, and let $n \equiv 0$ or $1 \pmod 5$ with $n \geq 5$. By Lemma 7, a $({}^{2}K_5, G_2)$-design does not exist. Otherwise, when $n = 5$ the desired designs were shown to exist in Example 2.

By Theorem 6, a $(K_n, K_4 - e)$-design exists if and only if $n \equiv 0$ or 1 (mod 5) with $n \neq 5$. Doubling each edge of a (K_n, K_4-e)-design, we obtain a $({}^{2}K_n, {}^{2}K_4 \setminus {}^{2}K_2)$-design. By Example 19, a $({}^{2}K_4 \setminus {}^{2}K_2, G)$-design exists. Thus, replacing each block in a $({}^{2}K_n, {}^{2}K_4 \setminus {}^{2}K_2)$-design with a $({}^{2}K_4 \setminus {}^{2}K_2, G)$-design, we obtain the desired result. ■

Lemma 9. *Let $G \in \{G_1, G_2\}$. If $n \equiv 0$ or $1 \pmod 5$ with $n \geq 5$, then a $({}^3K_n, G)$-design exists.*

Proof. Let $G \in \{G_1, G_2\}$ and let $n \equiv 0$, 1, 5, or $6 \pmod{10}$.

Case 1: $n \equiv 0 \pmod{10}$.
Let $n = 10x = 5(2x)$ for some positive integer x. When x is 1 or 2 the result follows from Examples 7 and 12, respectively, so we now consider when $x \geq 3$. Let $H_1, H_2, \ldots, H_x$ be disjoint sets of 2 vertices each.

Subcase 1a: $x \equiv 0$ or $1 \pmod 3$.
Let $\mathcal{B}$ be the set of blocks of a 3-GDD of type 2^x, which exists by Corollary 4, with vertex set $\bigcup_{i=1}^{x} H_i$. For each $i \in \{1, 2, \ldots, x\}$, let H_i' be the set obtained by blowing up each vertex in H_i by 5, and let $V({}^3K_n) = \bigcup_{i=1}^{x} H_i'$. After blowing up the vertices, $\mathcal{B}$ becomes a $K_{3\times5}$-decomposition of $K_{x\times10}$. Let $\mathcal{B}'$ be a $({}^3K_{x\times10}, {}^3K_{3\times5})$-design with vertex set $\bigcup_{i=1}^{x} H_i'$ obtained from $\mathcal{B}$. For each $b \in \mathcal{B}'$, let b' be a $({}^3K_{3\times5}, G)$-design, which exists by Example 17, on the same vertex set as b.

Now, for each $i \in \{1, 2, \ldots, x\}$ let c_i be a $({}^3K_{10}, G)$-design, which exists by Example 7, with vertex set H_i'. Then the set

$$\left(\bigcup_{b\in\mathcal{B}'} b'\right) \cup \bigcup_{i=1}^{x} c_i$$

forms the desired $({}^3K_n, G)$-design.

Subcase 1b: $x \equiv 2 \pmod 3$.
Let $H_0 = H_{x-1} \cup H_x$ and let $\mathcal{B}$ be the blocks of a 3-GDD of type $2^{x-2}4^1$, which exists by Corollary 4, with vertex set $H_0 \cup \bigcup_{i=1}^{x-2} H_i$. For each $i \in \{0, 1, \ldots, x-2\}$, let H_i' be the set obtained by blowing up each vertex in H_i by 5, and let $V({}^3K_n) = \bigcup_{i=0}^{x-2} H_i'$. After blowing up the vertices $\mathcal{B}$ becomes a $K_{3\times5}$-decomposition of the complete multipartite graph with vertex partition $\{H_0, H_1, \ldots, H_{x-2}\}$. Let $\mathcal{B}'$ be a ${}^3K_{3\times5}$-decomposition of the complete 3-fold multipartite graph with vertex set $\bigcup_{i=0}^{x-2} H_i'$ obtained from $\mathcal{B}$. For each $b \in \mathcal{B}'$, let b' be a $({}^3K_{3\times5}, G)$-design, which exists by Example 17, on the same vertex set as b.

Now, for each $i \in \{1, 2, \ldots, x-2\}$ let c_i be a $({}^3K_{10}, G)$-design, which exists by Example 7, with vertex set H_i'. Furthermore, let d be a $({}^3K_{20}, G)$-design, which exists by Example 12, with vertex set H_0'. Then the set

$$\left(\bigcup_{b\in\mathcal{B}'} b'\right) \cup \left(\bigcup_{i=1}^{x-2} c_i\right) \cup d$$

forms the desired $({}^3K_n, G)$-design.

Case 2: $n \equiv 1 \pmod{10}$ where $n = 10x = 5(2x) + 1$ for some positive integer x.
When $x \in \{1, 2\}$ the result follows from Examples 8 and 13, respectively.
Subcase 2a: $x \equiv 0$ or $1 \pmod 3$.
Let $V(^3K_n) = \left(\bigcup_{i=1}^{x} H_i'\right) \cup \{\infty\}$, where each H_i' is defined as in the proof of Subcase 1a. The desired $(^3K_n, G)$-design can be constructed similarly to how one was constructed in the proof of Subcase 1a with the following modification. For each $i \in \{1, 2, \ldots, x\}$, redefine c_i to be a $(^3K_{11}, G)$-design, which exists by Example 8, with vertex set $H_i' \cup \{\infty\}$.
Subcase 2b: $x \equiv 2 \pmod 3$.
Let $V(^3K_n) = \left(\bigcup_{i=1}^{x-1} H_i'\right) \cup \{\infty\}$, where each H_i' is defined as in the proof of Subcase 1b. The desired $(^3K_n, G)$-design can be constructed similarly to how one was constructed in the proof of Subcase 1a with the following two modifications. For each $i \in \{1, 2, \ldots, x-2\}$, redefine c_i to be a $(^3K_{11}, G)$-design, which exists by Example 8, with vertex set $H_i' \cup \{\infty\}$. Furthermore, redefine d to be a $(^3K_{21}, G)$-design, which exists by Example 13, with vertex set $H_{x-1}' \cup \{\infty\}$.
Case 3: $n \equiv 5 \pmod{10}$ where $n = 10x + 5 = 5(2x + 1)$ for some positive integer x.
Let $H_1, H_2, \ldots, H_{2x+1}$ be sets of vertices where $|H_i| = 1$ for every $i \in \{1, 2 \ldots, 2x + 1\}$. Let $\mathcal{B}$ be the blocks of a $\{3, 5\}$-GDD of type 1^{2x+1}, which exists by Theorem 5, with vertex set $\bigcup_{i=1}^{2x+1} H_i$. Moreover, let $\mathcal{B}_3$ and $\mathcal{B}_5$ consist of the blocks in $\mathcal{B}$ of cardinality 3 and 5, respectively. For each $i \in \{1, 2, \ldots, 2x + 1\}$, let H_i' be the set obtained by blowing up each vertex in H_i by 5, and let $V(^3K_n) = \bigcup_{i=1}^{2x+1} H_i'$. After blowing up the vertices $\mathcal{B}$ becomes a $\{K_{3\times5}, K_{5\times5}\}$-decomposition of $K_{(2x+1)\times5}$. Let $\mathcal{B}'$ be a $\{^3K_{3\times5}, {}^3K_{5\times5}\}$-decomposition of $^3K_{(2x+1)\times5}$ with vertex set $\bigcup_{i=1}^{2x+1} H_i'$ obtained from $\mathcal{B}$. Let $\mathcal{B}_3'$ and $\mathcal{B}_5'$ be the sets of $^3K_{3\times5}$- and $^3K_{5\times5}$-blocks in $\mathcal{B}'$, respectively. For each $b \in \mathcal{B}_3'$ define b' to be a $(^3K_{3\times5}, G)$-design, which exists by Example 17, on the same vertex set as b. For each $b \in \mathcal{B}_5'$ define b' to be a $(^3K_{5\times5}, G)$-design, which exists by Example 18, on the same vertex set as b.

Now, for each $i \in \{1, 2, \ldots, 2x + 1\}$ let c_i be a $(^3K_5, G)$-design, which exists by Example 2, with vertex set H_i'. Then the set

$$\left(\bigcup_{b\in\mathcal{B}'} b'\right) \cup \left(\bigcup_{i=1}^{2x+1} c_i\right)$$

forms the desired $(^3K_n, G)$-design.
Case 4: $n \equiv 6 \pmod{10}$ where $n = 10x + 6 = 5(2x + 1) + 1$ for some

positive integer x.
Let $V({}^3K_n) = \left(\bigcup_{i=1}^{2x+1} H_i'\right) \cup \{\infty\}$, where each H_i' is defined as in the proof of Case 3. The desired $({}^3K_n, G)$-design can be constructed similarly to how one was constructed in the proof of Case 3 with the following modification. For each $i \in \{1, 2, \ldots, 2x+1\}$, redefine c_i to be a $({}^3K_6, G)$-design, which exists by Example 3, with vertex set $H_i' \cup \{\infty\}$. ■

Lemma 10. *If* $n \equiv 1$ *or* $5 \pmod{10}$*, then a* $({}^3K_n, G_3)$*-design exists.*

Proof. The proof proceeds similarly to that of Cases 2 and 3 in the proof of Lemma 9. ■

Lemma 11. *Let* $G \in \{G_1, G_2, G_3\}$*. If* $n \equiv 0$ *or* $1 \pmod 5$*, then a* $({}^4K_n, G)$*-design exists.*

Proof. This follows from Example 2 and Lemma 8. ■

Lemma 12. *Let* $G \in \{G_1, G_2\}$*. If* $n \geq 4$*, then a* $({}^5K_n, G)$*-design exists, with the single exception that no* $({}^5K_4, G_2)$*-design exists.*

Proof. Let $G \in \{G_1, G_2\}$. If $n \equiv 0$ or $1 \pmod 5$ then the result follows from Lemmas 8 and 9. We allow the remainder of the proof to break into the cases $n \equiv 2$, 3, or $4 \pmod 5$.

Case 1: $n \equiv 2 \pmod 5$ where $n = 5x + 2$ for some positive integer x.
When $x \in \{1, 2\}$ the result follows from Examples 4 and 9, respectively.

Subcase 1a: $x = 2y + 1$ for some positive integer y.
Let $V({}^5K_n) = \bigcup_{i=1}^{2y+1} H_i' \cup \{\infty_1, \infty_2\}$ where $|H_i'| = 1$. Begin with a $\{3,5\}$-GDD of type 1^{2y+1} and blow up each vertex by 5. The argument proceeds similarly to the proof of Case 3 of Lemma 9 with the following modifications. For each $i \in \{1, 2, \ldots, 2y\}$, redefine c_i to be a $({}^5K_7 \setminus {}^5K_2, G)$-design with vertex set $H_i' \cup \{\infty_1, \infty_2\}$ where the hole of the design is $\{\infty_1, \infty_2\}$. Furthermore, redefine c_{2y+1} to be a $({}^5K_7, G)$-design with vertex set $H_{2y+1}' \cup \{\infty_1, \infty_2\}$.

Subcase 1b: $x = 2y$ where $y \equiv 0$ or $1 \pmod 3$.
Let $V({}^5K_n) = \bigcup_{i=1}^{y} H_i' \cup \{\infty_1, \infty_2\}$ where $|H_i'| = 10$. Begin with a 3-GDD of type 2^y and blow up each vertex by 5. The argument proceeds similarly to the proof of Subcase 1a of Lemma 9 with the following modifications. For each $i \in \{1, 2, \ldots, y-1\}$ redefine c_i to be a $({}^5K_{12} \setminus {}^5K_2, G)$-design, which exists by Example 23, with vertex set $H_i' \cup \{\infty_1, \infty_2\}$ where the hole of the design is $\{\infty_1, \infty_2\}$. Furthermore, redefine c_y to be a $({}^5K_{12}, G)$-design, which exists by Example 9, with vertex set $H_y' \cup \{\infty_1, \infty_2\}$.

Subcase 1c: $x = 2y$ where $y \equiv 2 \pmod 3$.
The argument proceeds as in the previous case by beginning with a 3-GDD

of type $2^{y-2}4^1$ and blowing up each vertex by 5 with the following modifications. For each $i \in \{1, 2, \ldots, y-2\}$ redefine c_i to be a $({}^5K_{12} \setminus {}^5K_2, G)$-design with vertex set $H'_i \cup \{\infty_1, \infty_2\}$ where the hole of the design is $\{\infty_1, \infty_2\}$. Redefine c_{y-1} to be a $({}^5K_{22}, G)$-design, which exists by Example 14, with vertex set $H'_{y-1} \cup \{\infty_1, \infty_2\}$.

CASE 2: $n \equiv 3 \pmod 5$ where $n = 5x + 3$ for some positive integer x.
When $x \in \{1, 2\}$ the result follows from Examples 5 and 10, respectively.

Subcase 2a: $x = 2y + 1$ for some positive integer y.
The argument proceeds similarly to the argument given in the proof of Subcase 1a of the current lemma by using $({}^5K_8, G)$- and $({}^5K_8 \setminus {}^5K_3, G)$-designs, which exist by Examples 5 and 21, in place of $({}^5K_7, G)$- and $({}^5K_7 \setminus {}^5K_2, G)$-designs, respectively.

Subcase 2b: $x = 2y$ where $y \equiv 0$ or $1 \pmod 3$.
The argument proceeds similarly to the argument given in the proof of Subcase 1b of the current lemma by using $({}^5K_{13}, G)$- and $({}^5K_{13} \setminus {}^5K_3, G)$-designs, which exist by Examples 10 and 24, in place of $({}^5K_{12}, G)$- and $({}^5K_{12} \setminus {}^5K_2, G)$-designs, respectively.

Subcase 2c: $x = 2y$ where $y \equiv 2 \pmod 3$.
The argument proceeds similarly to the argument given in the proof of Subcase 1c of the current lemma by using $({}^5K_{23}, G)$- and $({}^5K_{13} \setminus {}^5K_3, G)$-designs, which exist by Examples 15 and 24, in place of $({}^5K_{22}, G)$- and $({}^5K_{12} \setminus {}^5K_2, G)$-designs, respectively.

CASE 3: $n \equiv 4 \pmod 5$ where $n = 5x + 4$ for some non-negative integer x.
No $({}^5K_4, G_2)$-design exists by Lemma 7. A $({}^5K_4, G_1)$-design was shown to exist in Example 1. When $x \in \{1, 2\}$ the results follow from Examples 6 and 11, respectively.

Subcase 3a: $x = 2y + 1$ for some positive integer y.
The argument proceeds similarly to the argument given in the proof of Subcase 1a of the current lemma by using $({}^5K_9, G)$- and $({}^5K_9 \setminus {}^5K_4, G)$-designs, which exist by Examples 6 and 22, in place of $({}^5K_7, G)$- and $({}^5K_7 \setminus {}^5K_2, G)$-designs, respectively.

Subcase 3b: $x = 2y$ where $y \equiv 0$ or $1 \pmod 3$.
The argument proceeds similarly to the argument given in the proof of Subcase 1b of the current lemma by using $({}^5K_{14}, G)$- and $({}^5K_{14} \setminus {}^5K_4, G)$-designs, which exist by Examples 11 and 25, in place of $({}^5K_{12}, G)$- and $({}^5K_{12} \setminus {}^5K_2, G)$-designs, respectively.

Subcase 3c: $x = 2y$ where $y \equiv 2 \pmod 3$.
The argument proceeds similarly to the argument given in the proof of Subcase 1c of the current lemma by using $({}^5K_{24}, G)$- and $({}^5K_{14} \setminus {}^5K_4, G)$-designs, which exist by Examples 16 and 25, in place of $({}^5K_{22}, G)$- and $({}^5K_{12} \setminus {}^5K_2, G)$-designs, respectively. ■

Lemma 13. *If n is odd and $n \geq 5$, then a $({}^5K_n, G_3)$-design exists.*

Proof. Let $n = 2x + 1$ and let $V({}^5K_n) = \mathbb{Z}_n$. Then

$$\bigcup_{i \in \mathbb{Z}_n} \Big(\{G_3[0, x, 2x, 2x-1] + i\} \cup \big\{G_3[0, j, 2j, x+j+1] + i : 1 \leq j \leq x-1\big\} \Big)$$

is a $({}^5K_n, G_3)$-design. ∎

Lemma 14. *If $n \geq 4$, then a $({}^{10}K_n, G_3)$-design exists.*

Proof. By Theorem 6, a $({}^5K_n, K_4 - e)$-design exists for $n \geq 4$. Doubling each edge of such a design, we obtain a $({}^{10}K_n, {}^2K_4 \setminus {}^2K_2)$-design. By Example 19, a $({}^2K_4 \setminus {}^2K_2, G_3)$-design exists. Thus, replacing each block in a $({}^{10}K_n, {}^2K_4 \setminus {}^2K_2)$-design with a $({}^2K_4 \setminus {}^2K_2, G_3)$-design, we obtain the desired result. ∎

We finally prove that the necessary conditions in Lemmas 1 and 2 are sufficient with the aforementioned exceptions.

Theorem 15. *Let λ and n be integers and let $G \in \{G_1, G_2\}$. There exists a $({}^\lambda K_n, G)$-design if and only if $\lambda \geq 2$ and one of the following holds:*

1. *$n = 4$, $G = G_1$, and $\gcd(\lambda, 5) = 5$;*
2. *$n = 4$, $G = G_2$, and $\lambda \equiv 0 \pmod{10}$;*
3. *$n \geq 5$ and $\gcd(\lambda, 5) = 5$; or*
4. *$n \geq 5$, $\gcd(\lambda, 5) = 1$, and $n \equiv 0$ or $1 \pmod 5$*

with the exception that a $({}^2K_5, G_2)$-design does not exist.

Proof. Since G has order 4 and contains an edge of multiplicity 2, we must have $n \geq 4$ and $\lambda \geq 2$. Also, since G has size 5, we must have 5 divides $\lambda n(n-1)/2$, the size of ${}^\lambda K_n$. Furthermore, by Lemma 7, if $n = 4$ and $G = G_2$, then $\lambda \not\equiv 5 \pmod{10}$. Thus the necessary conditions are established. For sufficiency, we consider 3 cases. Note that it was shown in Lemma 7 that a $({}^2K_5, G_2)$-design does not exist.

Case 1: $n = 4$, $\lambda \equiv 0 \pmod{10}$, and $G = G_2$.
The argument proceeds similarly to the argument given in the proof of Lemma 13.

Case 2: $n \geq 4$, $\lambda \equiv 0 \pmod 5$, and $(n, G) \neq (4, G_2)$.
Let $\lambda = 5t$. We have a $({}^5K_n, G)$-design by Lemma 12. Since ${}^5K_n$ necessarily decomposes ${}^{5t}K_n$, the result follows.

Case 3: $n \geq 4$ and $\lambda \not\equiv 0 \pmod 5$.
We necessarily have that $n \equiv 0$ or $1 \pmod 5$ and $\lambda \geq 2$. If $\lambda \in \{2, 3, 4\}$, then the desired designs exist by Lemmas 8, 9, and 11, respectively. If

$\lambda = 6$, we get a $({}^6K_n, G)$-design by combining two copies of a $({}^3K_n, G)$-design. Finally, if $\lambda \geq 7$, then let $\lambda = 5t + r$ for some positive integer t with $r \in \{2, 3, 4, 6\}$. We get a $({}^{5t+r}K_n, G)$-design by combining t copies of a $({}^5K_n, G)$-design with a $({}^rK_n, G)$-design. ■

Theorem 16. *Let λ and n be integers. There exists a $({}^\lambda K_n, G_3)$-design if and only if $\lambda \geq 2$, $n \geq 4$, and one of the following holds:*

1. $\gcd(\lambda, 10) = 1$ *and* $n \equiv 1$ *or* $5 \pmod{10}$*;*
2. $\gcd(\lambda, 10) = 2$ *and* $n \equiv 0$ *or* $1 \pmod 5$*;*
3. $\gcd(\lambda, 10) = 5$ *and n is odd;*
4. $\gcd(\lambda, 10) = 10$.

Proof. As in the previous theorem, we must have $n \geq 4$, $\lambda \geq 2$, and 5 divides $\lambda n(n-1)/2$. Moreover, since every vertex in G_3 has even degree, we must have $\lambda(n-1)$, the degree of every vertex in ${}^\lambda K_n$, is even. Thus the necessary conditions are established. For sufficiency, we consider 4 cases.

Case 1: $\lambda \equiv 0 \pmod{10}$.
Let $\lambda = 10t$. We have a $({}^{10}K_n, G_3)$-design by Lemma 14. Since ${}^{10}K_n$ necessarily decomposes ${}^{10t}K_n$, the result follows.

Case 2: $\lambda \equiv 1$, 3, 7, or $9 \pmod{10}$.
We necessarily have that $n \equiv 1$ or $5 \pmod{10}$ and $\lambda \geq 2$. Let $\lambda = 2t+3$ for some nonnegative t. We get a $({}^{2t+3}K_n, G_3)$-design by combining t copies of a $({}^2K_n, G_3)$-design (using Lemma 8) with a $({}^3K_n, G_3)$-design (using Lemma 10).

Case 3: $\lambda \equiv 2$, 4, 6, or $8 \pmod{10}$.
We necessarily have that $n \equiv 0$ or $1 \pmod 5$ Let $\lambda = 2t$. We have a $({}^2K_n, G_3)$-design by Lemma 8. Since ${}^2K_n$ necessarily decomposes ${}^{2t}K_n$, the result follows.

Case 4: $\lambda \equiv 5 \pmod{10}$.
We necessarily have that n must be odd; hence, $n \geq 5$. Let $\lambda = 10t+5$. We have a $({}^5K_n, G_3)$-design by Lemma 13. Since ${}^5K_n$ necessarily decomposes ${}^{10t+5}K_n$, the result follows. ■

Acknowledgements

This research is supported by grant number A1359300 from the Division of Mathematical Sciences at the National Science Foundation. This work was done while the third, fourth, sixth, and seventh authors were participants in *REU Site: Mathematics Research Experience for Pre-service and for In-service Teachers* at Illinois State University.

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