On the Index 2 Spectra of Bipartite Subgraphs of $^2K_4$

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Abstract

Let ^2K_n denote the complete 2-fold multigraph of order n and let G be a bipartite subgraph of $^2K_4$. We find necessary and sufficient conditions for the existence of a G-decomposition of ^2K_n.

1 Introduction

If a and b are integers with $a \le b$, we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$. Let $\mathbb{Z}_n$ be the group of integers modulo n. For a finite set S and a positive integer λ, we let $^\lambda S$ denote the multiset that contains every element of S exactly λ times. For example $^2[a, b]$ is the multiset $\{a, a, a+1, a+1, \ldots, b, b\}$. Similarly for a graph G, we let $^\lambda G$ denote the multigraph obtained by replacing each edge in G with λ parallel edges. Thus $^\lambda K_n$ denotes the λ-fold complete multigraph of order n. We note that a multigraph is not required to contain multiple edges. Thus a graph is a multigraph. If G and K are multigraphs with $V(G) \subseteq V(K)$ and $E(G) \subseteq E(K)$, then we shall refer to G as a *subgraph* of K (in order to avoid having to use terms such as "submultigraph"). For a multigraph G and a positive integer r, we let rG denote the vertex-disjoint union of r copies of G. For positive integers r and s, let $K_{r\times s}$ denote the complete multipartite graph with r parts of cardinality s each. The *order* and *size* of a multigraph G refer to $|V(G)|$ and $|E(G)|$, respectively.

Let $V(^\lambda K_n) = [0, n-1]$. The *label* of an edge $\{i, j\}$ in $^\lambda K_n$ is defined to be $|i-j|$. The *length* of an edge $\{i, j\}$ in $^\lambda K_n$ is defined to be $\min\{|i-$

*Research supported by National Science Foundation Grant No. A1063038

$j|, n - |i - j|\}$. Thus if the elements of $V({}^{\lambda}K_n)$ are placed in order as vertices of an equisided n-gon, then the length of edge $\{i, j\}$ is the shortest distance around the polygon between i and j. Note that if n is odd, then ${}^{\lambda}K_n$ consists of λn edges of length i for $i \in [1, \frac{n-1}{2}]$, and if n is even, then ${}^{\lambda}K_n$ consists of λn edges of length i for $i \in [1, \frac{n}{2} - 1]$, and $\lambda n/2$ edges of length $n/2$.

Let $V({}^{\lambda}K_n) = \mathbb{Z}_n$ and let G be a subgraph of ${}^{\lambda}K_n$. By *clicking* G, we mean applying the permutation $i \mapsto i + 1$ to $V(G)$. Note that clicking an edge does not change its length.

Alternatively, we may let $V({}^{\lambda}K_n) = \mathbb{Z}_{n-1} \cup \{\infty\}$. As expected, clicking a subgraph G of ${}^{\lambda}K_n$ in this case continues to mean applying the permutation $i \mapsto i + 1$ to $V(G)$, with the convention that $\infty + 1 = \infty$. If $i, j \in \mathbb{Z}_{n-1}$, then the label and length of the edge $\{i, j\}$ are defined as if $\{i, j\}$ were an edge in ${}^{\lambda}K_{n-1}$. The label and length of an edge $\{i, \infty\}$ are both defined to be ∞. Again, clicking an edge does not change its length.

Let K and G be multigraphs with G a subgraph of K. A *G-decomposition* of K is a collection $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that each edge of K appears in exactly one G_i. The elements of Δ are called *G-blocks*. A G-decomposition of K is also known as a *(K, G)-design*. If there exists a (K, G)-design, we often say G *divides* K, or simply write $G \mid K$. Conversely, we may write $G \nmid K$ if G does not divide K. A $({}^{\lambda}K_n, G)$-design is called a *G-design* of *order* n and *index* λ. A $({}^{\lambda}K_n, G)$-design Δ is said to be *cyclic* if clicking is an automorphism of Δ. If $V({}^{\lambda}K_n) = \mathbb{Z}_{n-1} \cup \{\infty\}$, then a cyclic $({}^{\lambda}K_n, G)$-design is also called a *1-rotational $({}^{\lambda}K_n, G)$-design*. The study of graph decompositions is generally known as the study of graph designs, or G-designs. For recent surveys on G-designs of index 1, see [1] and [2].

Let G be a graph. A primary question in the study of graph designs is, *"For what values of n does there exist a $({}^{\lambda}K_n, G)$-design?"* The set of all such n is called the *spectrum for G-designs of index λ*. The spectrum for G-designs of index 1 has been determined for several classes of graphs including cycles, paths, stars, and complete graphs of order at most 5. If G is a graph of order at most 5, the spectrum for G-designs of index 1 has been determined for all but 11 values of n (see [1]).

In recent years, there have been some investigations of G-designs of index λ where G is a multigraph with edge multiplicity at least 2. For example, in [5] Carter determined the spectrum for G-designs of index λ for all connected cubic multigraphs G of order at most 6. Sarvate and various co-authors have investigated G-designs of index λ for various multigraphs G of small order (see for example [6], [11], [12], and [14]). See also [4] and [7] for the spectrum for G-designs where G is a multigraph of small order.

In this article, we focus on G-designs of index 2, where G is a bipartite subgraph of ${}^{2}K_4$ (see Table 1). We determine the spectrum for G-designs of index 2 for each of the 24 such subgraphs. We note that not all of the results in this paper are new. For example, the spectrum for G8 is settled in [11] and the spectra for G15, G16, G17, and G18 are settled in [12]. However, we include these graphs in our results for the sake of completeness.

2 Necessary Conditions and Graph Labelings

Let G of size m be a subgraph of ${}^2K_4$. There are 3 necessary conditions for a G-design of order n and index 2 to exist. First is the *size condition*: the number of edges in ${}^2K_n$ must be divisible by the number of edges in G. In other words m must divide $n(n-1)$. Second is the *degree condition*: the degree of each vertex of ${}^2K_n$ must be divisible by the greatest common divisor (gcd) of the degrees of the vertices of G. Therefore, $\gcd(\{\deg(v) : v \in V(G)\})$ must divide $2(n-1)$, where $\deg(v)$ indicates the degree of the vertex v. Third is the *order condition*: if there exists a G-design of order $n > 1$, then we must have $n \geq |V(G)|$.

It follows from the first condition above that for each subgraph we must consider the cases $n \equiv 0$ or $1 \pmod{m}$, unless the second or third condition is violated. If m is a power of a prime, then $n \equiv 0$ or $1 \pmod{m}$ are the only two possibilities. Since a bipartite subgraph of ${}^2K_4$ has at most 8 edges, we additionally consider the cases $n \equiv 3$ or $4 \pmod{6}$ for the four bipartite subgraphs of size 6.

For the most part, the cases $n \equiv 0$ or $1 \pmod{m}$ can be settled via two types of multigraph labelings which we define next.

Let G be a subgraph of ${}^2K_{m+1}$ such that $|E(G)| = m$. A *2-fold ρ-labeling* of G is a one-to-one function $f\colon V(G) \to [0, m]$ such that the multiset

$$\Big\{\min\big\{|f(u) - f(v)|, m + 1 - |f(u) - f(v)|\big\}\colon \{u, v\} \in E(G)\Big\} = \begin{cases} {}^2[1, \frac{m}{2}] & \text{if } m \text{ is even,} \\ {}^2[1, \frac{m-1}{2}] \cup \{\frac{m+1}{2}\} & \text{if } m \text{ is odd.} \end{cases}$$

Thus a 2-fold ρ-labeling of such a G induces an embedding of G in ${}^2K_{m+1}$ so that either (i) there are two edges of G of length i for each $i \in [1, \frac{m}{2}]$ when m is even or (ii) there are two edges of G of length i for each $i \in [1, \frac{m-1}{2}]$ and one edge of length $\frac{m+1}{2}$ when m is odd.

If f is a 2-fold ρ-labeling of a bipartite multigraph G with vertex bipartition $\{A, B\}$ and if for each edge $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$ we have $f(a) < f(b)$, then f is called an *ordered 2-fold ρ-labeling* and is denoted by ρ^+.

The following results are proved in [3].

Theorem 1. *Let G of size m be a subgraph of ${}^2K_{m+1}$. There exists a cyclic $({}^2K_{m+1}, G)$-design if and only if G admits a 2-fold ρ-labeling.*

Theorem 2. *Let G of size m be a bipartite subgraph of ${}^2K_{m+1}$. If G admits a 2-fold ρ^+-labeling, then there exists a cyclic $({}^2K_{mx+1}, G)$-design for each positive integer x.*

We illustrate how Theorem 2 works. Let $\{A, B\}$ be a bipartition of $V(G)$ and let f be a 2-fold ρ^+-labeling of G such that $f(a) < f(b)$ for every edge $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$. Let $A = \{u_1, u_2, \ldots, u_r\}$ and $B = \{v_1, v_2, \ldots, v_s\}$ and let x be a positive integer. For $1 \le i \le x$, let G_i be a copy of G with vertex bipartition $\{A, B_i\}$ where $B_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,s}\}$ and $v_{i,j}$ corresponds to v_j in B. Let $G(x) = G_1 \cup G_2 \cup \cdots \cup G_x$. Thus $G(x)$ is bipartite with vertex bipartition $\{A, B_1 \cup B_2 \cup \cdots \cup B_x\}$. Define a labeling f' of $G(x)$ as follows: $f'(a) = f(a)$ for each $a \in A$ and $f'(v_{i,j}) = f(v_j) + (i-1)m$ for $1 \le i \le x$ and $1 \le j \le s$. It is easy to see that f' is a 2-fold ρ^+-labeling of $G(x)$, and thus Theorem 1 applies. Figure 1 demonstrates how Theorem 2 works with a particular multigraph of size 5.

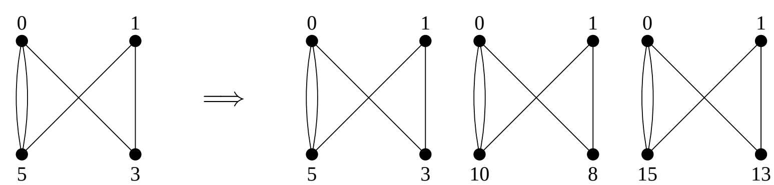


Figure 1: A 2-fold ρ^+-labeling of a multigraph G of size 5 and three starters for a cyclic G-decomposition of ${}^2K_{16}$.

Next, let G of size m be a subgraph of ${}^2K_m$. Let w be a vertex in G of degree 2 and let u and v be the neighbors of w (u and v need not be distinct). A *1-rotational 2-fold ρ-labeling* of G is a one-to-one function $f\colon V(G) \to \mathbb{Z}_{m-1} \cup \{\infty\}$ such that f restricted to $G - w$ is a 2-fold ρ-labeling of $G - w$, $f(w) = \infty$, and $\{f(u), f(v)\} \subseteq \{0, 1\}$. If in addition G is bipartite and f restricted to $G - w$ is a 2-fold ρ^+-labeling of $G - w$, then we call f *ordered*.

The following two theorems are also from [3].

Theorem 3. *Let G of size m be a subgraph of ${}^2K_m$. There exists a 1-rotational G-decomposition of ${}^2K_m$ if and only if G admits a 1-rotational 2-fold ρ-labeling.*

Theorem 4. *Let G of size m be a bipartite subgraph of ${}^2K_m$. If G admits an ordered 1-rotational 2-fold ρ-labeling, then there exists a 1-rotational G-decomposition of ${}^2K_{mx}$ for every positive integer x.*

We illustrate how Theorem 4 works. Let $\{A, B\}$ be a bipartition of $V(G)$ and let $w \in B$ with neighbors $u, v \in A$ be as in the definition of an ordered 1-rotational 2-fold ρ-labeling. Let f be such a labeling of G. Let $B = \{w, v_1, v_2, \ldots, v_s\}$. Let x be a positive integer. For $1 \le i \le x$, let G_i be a copy of G with bipartition $\{A, B_i\}$ where $B_i = \{w_i, v_{i,1}, v_{i,2}, \ldots, v_{i,s}\}$ and w_i corresponds to w and $v_{i,j}$ corresponds to v_j in B. Let $G(x) = G_1 \cup G_2 \cup \cdots \cup G_x$. Thus $G(x)$ is bipartite with bipartition $\{A, B_1 \cup B_2 \cup \cdots \cup B_x\}$. Define a labeling f' of $G(x)$ as follows: $f'(a) = f(a)$ for each $a \in A$, $f'(b) = f(b)$ for each $b \in B_1$, and for $2 \le i \le x$, let $f'(w_i) = (i-1)m$ and $f'(v_{i,j}) = f(v_j) + (i-1)m$. Then f' is a 1-rotational 2-fold ρ-labeling of $G(x)$, and thus Theorem 3 applies. Figure 2 demonstrates how Theorem 4 works with a particular multigraph of size 5.

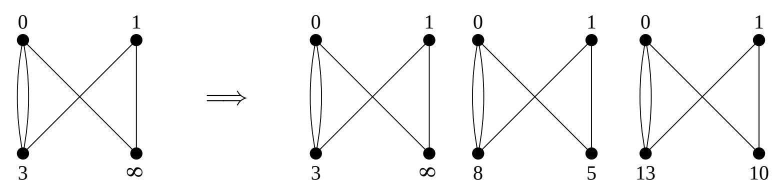


Figure 2: An ordered 1-rotational 2-fold ρ-labeling of a multigraph G of size 5 and three starters for a 1-rotational G-decomposition of ${}^2K_{15}$.

3 Main Results

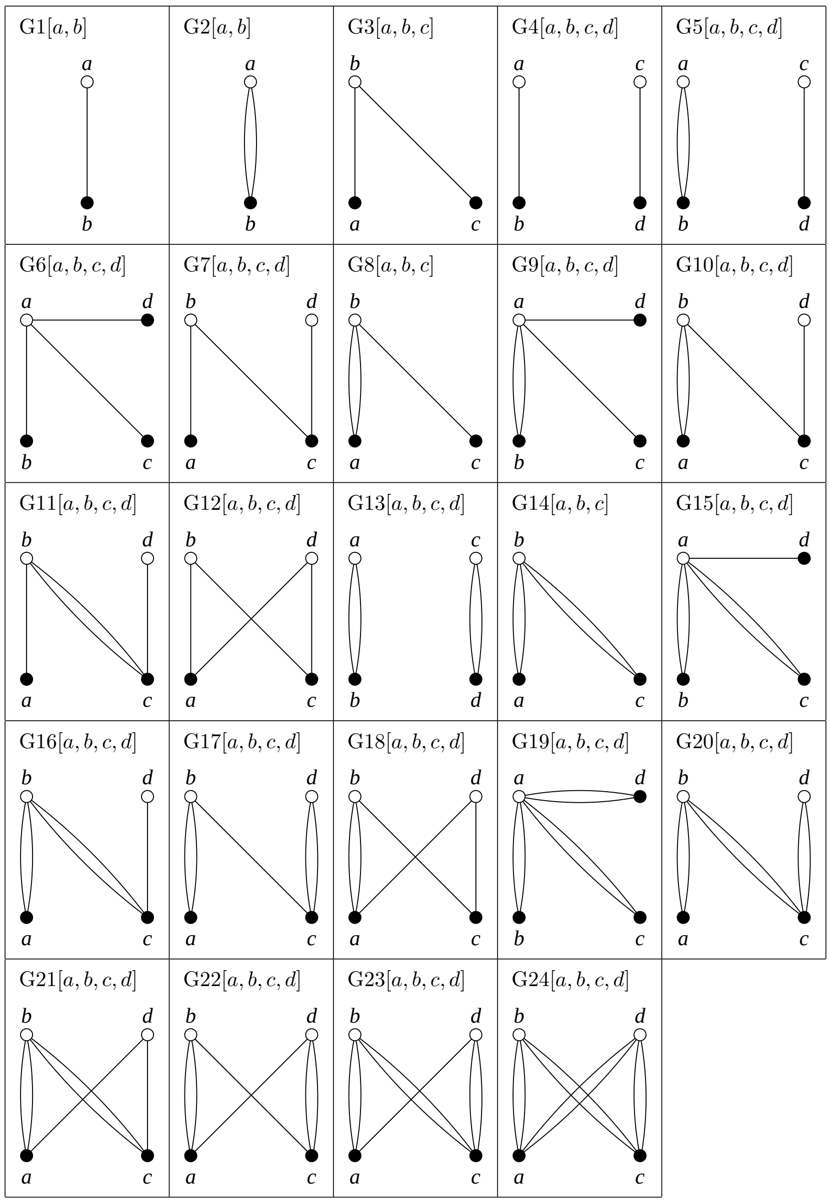
The 24 non-isomorphic bipartite subgraphs of ${}^2K_4$ are shown in Table 1 and are denoted by G1, G2, ..., G24. In Table 1 we also give a way to denote a labeled copy for each of these multigraphs. For example, G8$[a, b, c]$ refers to the multigraph with three vertices labeled a, b, and c with two edges between a and b and a single edge between b and c.

3.1 Decompositions of ${}^2K_{mx+1}$

If a multigraph G of size m is one of our subgraphs of interest, then the necessary conditions for a G-decomposition of ${}^2K_n$ allow for $n \equiv 1 \pmod{m}$. All but two of our 24 multigraphs admit ρ^+-labelings and thus cyclically decompose ${}^2K_n$ for $n \equiv 1 \pmod{m}$.

Theorem 5. *Let G of size m be a bipartite subgraph of ${}^2K_4$ and let x be a positive integer. There exists a cyclic G-decomposition of ${}^2K_{mx+1}$ unless $x = 1$ and G is either* G4 *or* G5.

Table 1: Bipartite Subgraphs of ${}^2K_4$.



Proof. Since $|V(\mathrm{G4})| > 3$, there cannot exist a G4-decomposition of ${}^2K_3$. Let $x \geq 2$ and let $V({}^2K_{2x+1}) = \mathbb{Z}_{2x+1}$. Consider the following multigraph:

$$\mathrm{G4}^* = \mathrm{G4}[0,1,2,3] \cup \bigcup_{i=2}^{x} \mathrm{G4}[0,2i-2,1,2i-1].$$

It is easy to check that we have a 2-fold ρ-labeling of $\mathrm{G4}^*$. Thus $\mathrm{G4}^*$ divides ${}^2K_{2x+1}$, and since G4 clearly divides $\mathrm{G4}^*$, we have a cyclic G4-decomposition of ${}^2K_{2x+1}$.

As far as G5 is concerned, one can quickly verify that G5 does not decompose ${}^2K_4$. Let $x \geq 2$ and let $V({}^2K_{3x+1}) = \mathbb{Z}_{3x+1}$. Consider the following multigraph:

$$\mathrm{G5}^* = \mathrm{G5}[0,1,2,4] \cup \bigcup_{i=2}^{x} \mathrm{G5}[0,3i-3,2,3i-2].$$

It is easy to check that we have a 2-fold ρ-labeling of $\mathrm{G5}^*$. Thus $\mathrm{G5}^*$ divides ${}^2K_{3x+1}$, and since G5 clearly divides $\mathrm{G5}^*$, we have a cyclic G5-decomposition of ${}^2K_{3x+1}$.

In Table 2, we give a 2-fold ρ^+-labeling for each of the remaining 22 bipartite subgraphs of ${}^2K_4$. By Theorem 2, the result follows. □

Table 2: 2-fold ρ^+-labelings of all but two of the bipartite subgraphs of ${}^2K_4$.

G1[0, 1]	G2[0, 1]	G3[2, 0, 1]	$\mathrm{G4} \nmid {}^2K_3$
$\mathrm{G5} \nmid {}^2K_4$	G6[0, 3, 2, 1]	G7[3, 0, 2, 1]	G8[1, 0, 2]
G9[0, 2, 4, 1]	G10[4, 0, 3, 1]	G11[3, 1, 2, 0]	G12[4, 0, 2, 1]
G13[0, 3, 1, 2]	G14[2, 0, 1]	G15[0, 1, 2, 3]	G16[2, 1, 3, 0]
G17[4, 0, 3, 2]	G18[5, 0, 3, 1]	G19[0, 3, 2, 1]	G20[3, 0, 2, 1]
G21[4, 2, 3, 0]	G22[3, 0, 2, 1]	G23[4, 1, 2, 0]	G24[4, 0, 2, 1]

3.2 Decompositions of ${}^2K_{mx}$

If a multigraph G of size m is one of our subgraphs of interest, then the size condition for a G-decomposition of ${}^2K_n$ allows for $n \equiv 0 \pmod{m}$. However, the degree condition rules out the existence of such G-decomposition if G is isomorphic to either G22 or G24. Moreover, the order condition rules out the existence of a G-decomposition of ${}^2K_m$ if G is isomorphic to any of the subgraphs in $\{\text{G1}, \text{G3}, \text{G4}, \text{G5}, \text{G6}, \text{G7}\}$. Of the remaining multigraphs, only G11 fails to decompose ${}^2K_m$.

Lemma 6. *Let G of size m be a bipartite subgraph of ${}^2K_4$. The necessary conditions for a G-decomposition of ${}^2K_m$ are sufficient if and only if G is not isomorphic to* G11*.*

Proof. One can quickly verify that $\text{G11} \nmid {}^2K_4$. If G is isomorphic to G23, then we let $V({}^2K_m) = \mathbb{Z}_7$ and use the following G-blocks for a G-decomposition of ${}^2K_m$: $\text{G23}[0,3,4,1]$, $\text{G23}[0,6,3,1]$, $\text{G23}[0,4,5,2]$, $\text{G23}[0,5,3,2]$, $\text{G23}[1,6,4,2]$, and $\text{G23}[1,5,6,2]$. In Table 3, we give an ordered 1-rotational 2-fold ρ-labeling for the remaining bipartite subgraphs of ${}^2K_4$ where the necessary conditions for a G-decomposition of ${}^2K_m$ are satisfied. By Theorem 3, the result follows. □

Table 3: Ordered 1-rotational 2-fold ρ-labelings of bipartite subgraphs of ${}^2K_4$.

$\text{G1} \nmid {}^2K_1$	$\text{G2}[0,\infty]$	$\text{G3} \nmid {}^2K_2$	$\text{G4} \nmid {}^2K_2$
$\text{G5} \nmid {}^2K_3$	$\text{G6} \nmid {}^2K_3$	$\text{G7} \nmid {}^2K_3$	$\text{G8}[\infty,0,1]$
$\text{G9}[0,\infty,2,1]$	$\text{G10}[2,0,\infty,1]$	$\text{G11} \nmid {}^2K_4$	$\text{G12}[\infty,0,2,1]$
$\text{G13}[0,\infty,1,2]$	$\text{G14}[\infty,0,1]$	$\text{G15}[0,\infty,1,2]$	$\text{G16}[\infty,0,3,1]$
$\text{G17}[\infty,0,2,1]$	$\text{G18}[3,0,\infty,1]$	$\text{G19}[0,\infty,2,1]$	$\text{G20}[\infty,0,2,1]$
$\text{G21}[0,2,1,\infty]$			

As noted in Table 3, not all bipartite subgraphs of ${}^2K_4$ with size m decompose ${}^2K_m$. However, the necessary conditions for such a decomposition of ${}^2K_{mx}$, where $x \geq 2$, are sufficient for all of the bipartite subgraphs in question (still excluding G22 and G24).

Theorem 7. *Let G of size m be a bipartite subgraph of ${}^2K_4$. If $G \notin \{\text{G22}, \text{G24}\}$, then there exists a G-decomposition of ${}^2K_{mx}$ for every integer $x \geq 2$.*

Proof. Let $x \geq 2$ be an integer. We consider a G1-decomposition of ${}^2K_x$ to be a trivial result. In Table 3, we give an ordered 1-rotational 2-fold ρ-labeling for all $G \notin \{\text{G1}, \text{G3}, \text{G4}, \text{G5}, \text{G6}, \text{G7}, \text{G11}, \text{G23}\}$. By Theorem 4, the result follows for these multigraphs.

In the case where G is isomorphic to G23, let ${}^2K_{mx} = x({}^2K_7) \cup {}^2K_{x\times 7}$. Since $\text{G23} \mid {}^2K_7$ and ${}^2K_{7,7} \mid {}^2K_{x\times 7}$, it suffices to show that $\text{G23} \mid {}^2K_{7,7}$.

Let $V(^2K_{7,7}) = \mathbb{Z}_7 \times \mathbb{Z}_2$ with the obvious bipartition, then $\big\{\mathrm{G23}[(i,0),(i+3,1),(i+1,0),(i,1)] : i \in \mathbb{Z}_7\big\} \cup \big\{\mathrm{G23}[(i,0),(i+4,1),(i+6,0),(i,1)] : i \in \mathbb{Z}_7\big\}$ is a G23-decomposition of $^2K_{7,7}$.

In all other cases, it suffices to show that there exists a multigraph G^* of size mx such that $G \mid G^*$ and such that G^* admits a 1-rotational 2-fold ρ-labeling. In Table 4, we give such multigraphs with the desired labelings. □

Table 4: 1-rotational 2-fold ρ-labelings of certain subgraphs of $^2K_{mx}$ where $x \geq 2$.

$\mathrm{G3}^* = \mathrm{G3}[0,\infty,1] \cup \bigcup_{i=2}^{x} \mathrm{G3}[0,2i-2,1]$
$\mathrm{G4}^* = \mathrm{G4}[0,\infty,1,2] \cup \mathrm{G4}[0,\infty,1,2] \cup \bigcup_{i=3}^{x} \mathrm{G4}[0,2i-4,1,2i-2]$
$\mathrm{G5}^* = \mathrm{G5}[0,\infty,1,2] \cup \bigcup_{i=2}^{x} \mathrm{G5}[0,3i-3,1,3i-4]$
$\mathrm{G6}^* = \mathrm{G6}[0,\infty,2,1] \cup \mathrm{G6}[0,\infty,2,1] \cup \bigcup_{i=3}^{x} \mathrm{G6}[0,3i-4,3i-5,3i-6]$
$\mathrm{G7}^* = \mathrm{G7}[1,0,\infty,2] \cup \bigcup_{i=2}^{x} \mathrm{G7}[3i-2,0,3i-3,1]$
$\mathrm{G11}^* = \mathrm{G11}[\infty,0,1,3] \cup \mathrm{G11}[\infty,0,3,1] \cup \bigcup_{i=3}^{x} \mathrm{G11}[4i-7,0,4i-5,1]$

3.3 Other Decompositions

As stated in Section 2, for subgraphs G with 6 edges, $n \equiv 3$ or $4 \pmod 6$ also satisfies the size condition for G-decompositions of ^2K_n. For G22, the degree condition rules out the case $n \equiv 3 \pmod 4$. In [5], Carter shows that there exists a G22-decomposition of ^2K_n for all $n \equiv 4 \pmod 6$.

We note that G19 and G20 are the multigraphs $^2K_{1,3}$ and $^2P_4$, respectively. It is well known (see [1]) that if G is either $K_{1,3}$ or P_4, then exists a G-decomposition of K_n if and only if $n \equiv 0,\ 1,\ 3,$ or $4 \pmod 6$. Thus if G is either G19 or G20, then exists a G-decomposition of ^2K_n for all $n \equiv 3$ or $4 \pmod 6$.

Finally we turn our attention to G21 and show that that there exists a G21-decomposition of ${}^2K_n$ for $n \equiv 3$ or $4 \pmod 6$, $n > 4$.

Lemma 8. *There exists a* G21*-decomposition of* ${}^2K_n$ *for* $n \equiv 3$ *or* $4 \pmod 6$*,* $n > 4$*.*

Proof. First, consider $n \equiv 3 \pmod 6$. Because of the order condition, it is necessary to have $n > 3$. Let $n = 6x + 3$ where x is a positive integer. If $x = 1$, then we let $V({}^2K_9) = \mathbb{Z}_9$, and let

$$\begin{aligned}\Delta = \big\{&\mathrm{G21}[0,1,2,3], \mathrm{G21}[0,2,4,3], \mathrm{G21}[0,5,1,6], \mathrm{G21}[0,7,5,8],\\ &\mathrm{G21}[1,4,5,3], \mathrm{G21}[1,7,2,6], \mathrm{G21}[1,8,4,3], \mathrm{G21}[2,5,6,3],\\ &\mathrm{G21}[2,8,3,6], \mathrm{G21}[3,7,8,5], \mathrm{G21}[4,6,8,0], \mathrm{G21}[4,7,6,0]\big\}.\end{aligned}$$

Then Δ is a G21-decomposition of ${}^2K_9$.

For $x \geq 2$, we let ${}^2K_{6x+3} = {}^2K_9 \cup (x-1){}^2K_6 \cup {}^2K_{9,6(x-1)} \cup {}^2K_{(x-1)\times 6}$. Clearly ${}^2K_{3,2}$ divides ${}^2K_{9,6(x-1)}$ and ${}^2K_{(x-1)\times 6}$. Since we already have proved that G21 divides ${}^2K_9$ and ${}^2K_6$, all that remains to be shown is that $\mathrm{G21} \mid {}^2K_{3,2}$. Let $V\big({}^2K_{3,2}\big)$ have bipartition $\big\{\{u_1,u_2,u_3\},\{v_1,v_2\}\big\}$. Then $\big\{\mathrm{G21}[v_1,u_1,v_2,u_2], \mathrm{G21}[v_1,u_3,v_2,u_2]\big\}$ is a G21-decomposition of ${}^2K_{3,2}$.

Finally, consider $n \equiv 4 \pmod 6$. It is easily checked that G21 does not divide ${}^2K_4$, thus let $n = 6x + 4$ where x is a positive integer. If $n = 10$, then let $V({}^2K_n) = \mathbb{Z}_9 \cup \{\infty\}$, and let

$$\begin{aligned}\Delta = &\big\{\mathrm{G21}[i, i+2, i+1, \infty] : i \in \mathbb{Z}_5\big\}\\ &\cup \big\{\mathrm{G21}[i+5, j, i+7, \infty] : i, j \in \mathbb{Z}_2\big\}\\ &\cup \big\{\mathrm{G21}[2,5,7,6], \mathrm{G21}[2,8,6,7], \mathrm{G21}[3,6,5,8],\\ &\quad \mathrm{G21}[3,7,8,5], \mathrm{G21}[5,4,8,3], \mathrm{G21}[6,4,7,2]\big\}.\end{aligned}$$

Then Δ is a G21-decomposition of ${}^2K_{10}$.

If $x > 1$, then we let ${}^2K_{6x+4} = {}^2K_{10} \cup (x-1){}^2K_6 \cup {}^2K_{10,6(x-1)} \cup {}^2K_{(x-1)\times 6}$. Clearly ${}^2K_{2,3}$ divides ${}^2K_{10,6(x-1)}$ and ${}^2K_{(x-1)\times 6}$. Since G21 divides ${}^2K_{10}$, ${}^2K_6$, and ${}^2K_{2,3}$, the result follows. □

3.4 Summary of Results

We summarize our results in a final theorem.

Main Theorem. *Let G be one of the 24 bipartite subgraphs of* ${}^2K_4$ *as listed in Table 1. The obvious necessary conditions for the existence of a G-decomposition of* ${}^2K_n$ *are sufficient with the following four exceptions:* $\mathrm{G5} \nmid {}^2K_4$*,* $\mathrm{G11} \nmid {}^2K_4$*,* $\mathrm{G19} \nmid {}^2K_4$*, and* $\mathrm{G21} \nmid {}^2K_4$*.*

4 Acknowledgement and Final Note

This research is supported by grant number A1063038 from the Division of Mathematical Sciences at the National Science Foundation. This work was done while the first, second, and fourth authors were participants in *REU Site: Mathematics Research Experience for Pre-service and for In-service Teachers* at Illinois State University. The affiliations of these authors at the time were as follows: S. R. Allen: University of Illinois, Champaign-Urbana; J. Bolt: Kankakee High School (Kankakee, IL); S. Burton: Virginia Polytechnic Institute and State University.

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