Spectrum for multigraph designs on four vertices and six edges

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Abstract

Let G be one of the two multigraphs obtained from $K_4 - e$ by replacing one edge with a double-edge. We find necessary and sufficient conditions on n and λ for the existence of a G-decomposition of ${}^{\lambda}K_n$.

1 Introduction

Throughout this paper, note we may refer to a multigraph as a graph; however, our graphs contain no loops. If we wish to emphasize that a given graph does not contain parallel edges, then we refer to it as a simple graph. For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set (or multiset) of G, respectively. For a simple graph G and a positive integer λ, we use ${}^{\lambda}G$ to denote the graph obtained from G by replacing each edge in $E(G)$ with λ parallel edges. Alternatively, we let λG denote the graph consisting of λ vertex-disjoint copies of G. For edge-disjoint graphs G and H, we use $G \cup H$ to represent the graph with edge set $E(G) \cup E(H)$ and vertex set $V(G) \cup V(H)$. We use $K_{r\times s}$ to denote the complete simple multipartite graph with r parts of size s, and we use $K_{t,\,r\times s}$ to denote the complete simple multipartite graph with one part of size t and r parts of size s. If G is a subgraph of H, we use $H \setminus G$ to denote the graph obtained from H by removing $E(G)$ from $E(H)$.

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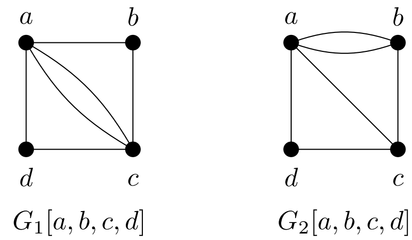

Figure 1: The two multigraphs consisting of $K_4 - e$ with a double edge

Let K and G be graphs with G a subgraph of K. A *G-decomposition* of K is a set (or multiset) $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that each edge of K appears in exactly one G_i. Similarly, if G and H are each subgraphs or K, then a $\{G, H\}$-decomposition of K is defined to be a set $\{H_1, H_2, \ldots, H_t\}$ of subgraphs of K each of which is isomorphic to either G or H and such that each edge of K appears in exactly one H_i. If there exists a G-decomposition of K, then we say G *divides* K and write $G \mid K$. A G-decomposition of K is also known as a *(K, G)-design*.

Let G be a graph. A classical problem in the study of graph designs is to find necessary and sufficient conditions for the existence of a G-decomposition of ${}^{\lambda}K_n$. This is known as the *spectrum problem* for G. The set of all such n is called the *spectrum for G-designs of index λ*. The spectrum for G-designs of index 1 has been determined for several classes of graphs including cycles, paths, stars, and simple graphs of order at most 5 (see [2]).

In recent years, there have been some investigations of G-designs of index λ where G is a multigraph with edge multiplicity at least 2. For example, in [4] Carter determined the spectra for G-designs of index λ for all connected cubic multigraphs G of order at most 6. The spectra for G-designs of index λ have been investigated for various multigraphs G of small order (see for example [8], [3], and [9]). In this paper we consider the two multigraphs with 6 edges obtained by replacing one edge of $K_4 - e$ with a double edge (see Figure 1). We settle the spectrum problem for both graphs.

Let $G \in \{G_1, G_2\}$. Then $G[a, b, c, d]$ denotes the graph with vertex set $\{a, b, c, d\}$ and edge set as represented in Figure 1. For example, $G_2[0, 1, 2, 3]$ denotes the graph with vertex set $\{0, 1, 2, 3\}$ and edge multiset $\{\{0, 1\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{2, 3\}\}$.

The following theorems on decompositions of complete graphs and complete multipartite graphs are used extensively in proving our main results. All of these results can be found in the *Handbook of Combinatorial Designs*

[5] (see [1], [6] and [7]).

Theorem 1.1. *If n is an odd positive integer, then there exists a $\{K_3, K_5\}$-decomposition of K_n.*

Theorem 1.2. *The necessary and sufficient conditions for the existence of a K_3-decomposition of $K_{t\times m}$ are (i) $t \geq 3$, (ii) $(t-1)m \equiv 0 \pmod 2$, and (iii) $t(t-1)m^2 \equiv 0 \pmod 6$.*

Theorem 1.3. *If $t \geq 3$ and $t \equiv 0 \pmod 3$, then there exists a K_3-decomposition of $K_{4,\, t\times 2}$.*

Combining the previous two results, we have the following corollary that that is more directly applicable in our general constructions.

Corollary 1.4. *Let $t \geq 3$. There exists a K_3-decomposition of $K_{t\times 2}$ if $t \equiv 0$ or $1 \pmod 3$ and of $K_{4,\,(t-2)\times 2}$ if $t \equiv 2 \pmod 3$.*

The following is a well-known result that is a special case of Wilson's Fundamental Construction (see [7]).

Theorem 1.5. *Let m, n, r, s, and t be positive integers. If there exists a $(K_{t\times m}, K_n)$-design, then there exists a $(K_{t\times ms}, K_{n\times s})$-design. Similarly, if there exists a $(K_{r,\, t\times m}, K_n)$-design, then there exists a $(K_{rs,\, t\times ms}, K_{n\times s})$-design.*

2 Some Small Examples

In this section we present G_1- and G_2-decompositions of various graphs that are needed for the constructions used in Section 3. A G-decomposition of a graph with vertex set V may be written as a pair (V, B), where B is a collection of copies of G that partition the edge-set of the graph.

Given the graphs represented by the notation $G[a, b, c, d]$ and some $i \in \mathbb{Z}_n$, we define $G[a, b, c, d] + i = G[a + i, b + i, c + i, d + i]$ where all addition is performed in $\mathbb{Z}_n$. By convention, define $\infty + 1 = \infty$.

2.1 Small designs of index 2

Example 2.1. Let $V = \mathbb{Z}_4$ and let $B_1 = \{G_1[0, 1, 2, 3], G_1[3, 2, 1, 0]\}$ and $B_2 = \{G_2[0, 1, 2, 3], G_2[3, 1, 2, 0]\}$. Then (V, B_1) and (V, B_2) are respectively G_1- and G_2-decompositions of ${}^2K_4$.

Example 2.2. Let $V = \mathbb{Z}_5 \cup \{\infty\}$ and let $B_1 = \{G_1[0, \infty, 2, 1] + i : i \in \mathbb{Z}_5\}$ and $B_2 = \{G_2[0, 2, 1, \infty] + i : i \in \mathbb{Z}_5\}$. Then (V, B_1) and (V, B_2) are respectively G_1- and G_2-decompositions of ${}^2K_6$.

Example 2.3. Let $V = \mathbb{Z}_7$ and let $B_1 = \{G_1[0,3,1,5] + i : i \in \mathbb{Z}_7\}$ and $B_2 = \{G_2[0,2,3,4] + i : i \in \mathbb{Z}_7\}$. Then (V, B_1) and (V, B_2) are respectively G_1- and G_2-decompositions of ${}^2K_7$.

Example 2.4. Let $V = \mathbb{Z}_3 \times \mathbb{Z}_3$ and let

$$\begin{aligned} B = &\{G_1[(0,i),(0,1+i),(1,i),(2,2+i)] : i \in \mathbb{Z}_3\} \\ &\cup \{G_1[(0,i),(0,2+i),(2,i),(2,1+i)] : i \in \mathbb{Z}_3\} \\ &\cup \{G_1[(1,i),(2,i),(1,1+i),(0,2+i)] : i \in \mathbb{Z}_3\} \\ &\cup \{G_1[(1,i),(2,i),(2,1+i),(0,2+i)] : i \in \mathbb{Z}_3\}. \end{aligned}$$

Then (V, B) is a G_1-decomposition of ${}^2K_9$.

Example 2.5. Let $V = \mathbb{Z}_2 \times \mathbb{Z}_4 \cup \{\infty\}$ and let

$$\begin{aligned} B = &\{G_2[\infty,(0,i),(1,i),(1,2+i)] : i \in \mathbb{Z}_4\} \\ &\cup \{G_2[(0,i),(1,2+i),(1,3+i),(1,i)] : i \in \mathbb{Z}_4\} \\ &\cup \{G_2[(0,i),(1,1+i),(0,2+i),(0,1+i)] : i \in \mathbb{Z}_4\}. \end{aligned}$$

Then (V, B) is a G_2-decomposition of ${}^2K_9$.

Example 2.6. Let $V = \mathbb{Z}_{10}$ and let

$$\begin{aligned} B_1 = \{&G_1[0,1,9,8], G_1[0,2,7,6], G_1[0,3,5,4], G_1[1,0,8,9], G_1[2,0,6,1], \\ &G_1[3,0,4,1], G_1[1,2,7,3], G_1[1,4,5,6], G_1[2,4,8,5], G_1[2,5,3,9], \\ &G_1[4,2,9,7], G_1[3,6,8,7], G_1[6,3,9,5], G_1[4,7,6,8], G_1[5,8,7,9]\}, \\ B_2 = \{&G_2[0,9,6,8], G_2[0,7,4,3], G_2[0,5,2,1], G_2[4,2,3,0], G_2[2,6,1,0], \\ &G_2[8,1,6,0], G_2[1,3,7,5], G_2[1,4,9,5], G_2[7,2,9,1], G_2[9,8,2,5], \\ &G_2[3,5,8,2], G_2[6,3,9,4], G_2[7,8,3,9], G_2[4,8,5,6], G_2[7,6,5,4]\}. \end{aligned}$$

Then (V, B_1) and (V, B_2) are G_1- and G_2-decompositions of ${}^2K_{10}$, respectively.

Example 2.7. Let $V = \mathbb{Z}_{11} \cup \{\infty\}$ and let $B_1 = \{G_1[0,\infty,4,5] + i : i \in \mathbb{Z}_{11}\} \cup \{G_1[0,2,3,5] + i : i \in \mathbb{Z}_{11}\}$ and $B_2 = \{G_2[0,3,4,\infty] + i : i \in \mathbb{Z}_{11}\} \cup \{G_2[0,2,7,1] + i : i \in \mathbb{Z}_{11}\}$. Then (V, B_1) and (V, B_2) are respectively G_1- and G_2-decompositions of ${}^2K_{12}$.

Example 2.8. Let $V = \mathbb{Z}_{13}$ and let $B_1 = \{G_1[0,4,3,5] + i : i \in \mathbb{Z}_{13}\} \cup \{G_1[0,2,6,5] + i : i \in \mathbb{Z}_{13}\}$ and $B_2 = \{G_2[0,6,5,3] + i : i \in \mathbb{Z}_{13}\} \cup \{G_2[0,4,3,5] + i : i \in \mathbb{Z}_{13}\}$. Then (V, B_1) and (V, B_2) are respectively G_1- and G_2-decompositions of ${}^2K_{13}$.

Example 2.9. Let $V = \mathbb{Z}_2 \times \mathbb{Z}_7 \cup \{\infty\}$ and let

$$\begin{aligned} B = &\{G_1[(0,i), \infty, (1, 6+i), (1, 1+i)] : i \in \mathbb{Z}_7\} \\ &\cup \{G_1[(0,i), \infty, (1, 5+i), (0, 5+i)] : i \in \mathbb{Z}_7\} \\ &\cup \{G_1[(0,i), (0, 1+i), (0, 3+i), (1, i)] : i \in \mathbb{Z}_7\} \\ &\cup \{G_1[(1,i), (0, 6+i), (1, 3+i), (1, 1+i)] : i \in \mathbb{Z}_7\} \\ &\cup \{G_1[(0,i), (0, 1+i), (1, 3+i), (1, 2+i)] : i \in \mathbb{Z}_7\}. \end{aligned}$$

Then (V, B) is a G_1-decomposition of ${}^2K_{15}$.

Example 2.10. Let $V = \mathbb{Z}_3 \times \mathbb{Z}_5$ and let

$$\begin{aligned} B = &\{G_2[(0,i), (1, 4+i), (2, i), (1, 1+i)] : i \in \mathbb{Z}_5\} \\ &\cup \{G_2[(0,i), (0, 2+i), (2, 2+i), (1, i)] : i \in \mathbb{Z}_5\} \\ &\cup \{G_2[(1,i), (1, 2+i), (0, 4+i), (0, 3+i)] : i \in \mathbb{Z}_5\} \\ &\cup \{G_2[(2,i), (2, 4+i), (1, 3+i), (0, 1+i)] : i \in \mathbb{Z}_5\} \\ &\cup \{G_2[(1,i), (1, 1+i), (2, 4+i), (0, i)] : i \in \mathbb{Z}_5\} \\ &\cup \{G_2[(0,i), (2, 3+i), (2, 1+i), (0, 4+i)] : i \in \mathbb{Z}_5\} \\ &\cup \{G_2[(1,i), (2, i), (2, 3+i), (0, 2+i)] : i \in \mathbb{Z}_5\}. \end{aligned}$$

Then (V, B) is a G_2-decomposition of ${}^2K_{15}$.

Example 2.11. Consider that an affine plane of order 4 yields a K_4-decomposition of K_{16}, and thus ${}^2K_4 \mid {}^2K_{16}$. By Example 2.1, we have that both G_1 and G_2 divide ${}^2K_4$; hence, there exist both G_1- and G_2-decompositions of ${}^2K_{16}$.

Example 2.12. Let $V = \mathbb{Z}_6 \cup \{\infty_1, \infty_2, \infty_3\}$ and let

$$\begin{aligned} B_1 = \{&G_1[\infty_1, 0, 5, 2], G_1[\infty_1, 0, 3, 4], G_1[\infty_1, 2, 1, 4], \\ &G_1[\infty_2, 0, 2, 3], G_1[\infty_2, 0, 4, 1], G_1[1, \infty_2, 5, 0], G_1[3, \infty_2, 5, 4], \\ &G_1[\infty_3, 1, 0, 3], G_1[\infty_3, 2, 5, 4], G_1[1, \infty_3, 3, 2], G_1[2, \infty_3, 4, 0]\}, \\ B_2 = \{&G_2[\infty_1, 0, 2, 1], G_2[\infty_1, 5, 1, 3], G_2[\infty_1, 4, 2, 3], \\ &G_2[\infty_2, 1, 0, 2], G_2[\infty_2, 5, 4, 0], G_2[\infty_2, 3, 4, 2], \\ &G_2[\infty_3, 2, 1, 0], G_2[\infty_3, 4, 5, 0], G_2[\infty_3, 3, 5, 1], \\ &G_2[5, 2, 3, 0], G_2[4, 1, 3, 0]\}. \end{aligned}$$

Then (V, B_1) and (V, B_2) are respectively G_1- and G_2-decompositions of ${}^2K_9 \setminus {}^2K_3$ where the removed ${}^2K_3$ has vertex set $\{\infty_1, \infty_2, \infty_3\}$.

Example 2.13. Let $V = \mathbb{Z}_6 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ and let

$$\begin{aligned} B_1 = \{&G_1[\infty_1, 1, 0, 3], G_1[\infty_1, 1, 2, 5], G_1[\infty_1, 3, 4, 5], \\ &G_1[\infty_2, 0, 3, 2], G_1[\infty_2, 0, 5, 4], G_1[\infty_2, 2, 1, 4], \\ &G_1[\infty_3, 0, 1, 4], G_1[\infty_3, 0, 5, 2], G_1[\infty_3, 2, 3, 4] \\ &G_1[\infty_4, 2, 0, 4], G_1[\infty_4, 3, 1, 5], G_1[2, \infty_4, 4, 0], G_1[3, \infty_4, 5, 1]\}, \end{aligned}$$

$$\begin{aligned} B_2 = \{&G_2[\infty_1, 0, 1, 4], G_2[\infty_1, 2, 1, 3], G_2[\infty_1, 5, 3, 4], \\ &G_2[\infty_2, 4, 2, 0], G_2[\infty_2, 5, 0, 3], G_2[\infty_2, 1, 2, 3], \\ &G_2[\infty_3, 0, 2, 4], G_2[\infty_3, 1, 5, 4], G_2[\infty_3, 3, 2, 5], \\ &G_2[\infty_4, 1, 3, 0], G_2[\infty_4, 2, 5, 0], G_2[\infty_4, 4, 3, 5], G_2[4, 0, 1, 5]\}. \end{aligned}$$

Then (V, B_1) and (V, B_2) are respectively G_1- and G_2-decompositions of ${}^2K_{10} \setminus {}^2K_4$ where the removed ${}^2K_4$ has vertex set $\{\infty_1, \infty_2, \infty_3, \infty_4\}$.

Example 2.14. Let $V = \mathbb{Z}_6$ with partition $\big\{\{0, 3\}, \{1, 4\}, \{2, 5\}\big\}$ and let $B = \{G_1[0, 2, 1, 5], G_1[0, 2, 4, 5], G_1[3, 5, 4, 2], G_1[3, 5, 1, 2]\}$. Then (V, B) is a G_1-decomposition of ${}^2K_{3\times 2}$.

Example 2.15. Let $V = \mathbb{Z}_3 \times \mathbb{Z}_3$ and let

$$\begin{aligned} B = &\{G_1[(0, i), (2, i), (1, i), (2, 2 + i)] : i \in \mathbb{Z}_3\} \\ &\cup \{G_1[(1, i), (0, 1 + i), (2, 1 + i), (0, 2 + i)] : i \in \mathbb{Z}_3\} \\ &\cup \{G_1[(2, i), (1, i), (0, 2 + i), (1, 1 + i)] : i \in \mathbb{Z}_3\}. \end{aligned}$$

Then (V, B) is a G_1-decomposition of ${}^2K_{3\times 3}$.

Example 2.16. Let $V = \mathbb{Z}_9$ with partition $\big\{\{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 6\}\big\}$ and let $B = \{G_2[0, 4, 2, 1] + i : i \in \mathbb{Z}_9\}$. Then (V, B) is a G_2-decomposition of ${}^2K_{3\times 3}$.

Example 2.17. Let $V = \mathbb{Z}_{15}$ with the partition $\big\{\{0, 5, 10\}, \{1, 6, 11\}, \{2, 7, 12\}, \{3, 8, 13\}, \{4, 9, 14\}\big\}$ and let $B_1 = \{G_1[0, 6, 4, 7] + i : i \in \mathbb{Z}_{15}\} \cup \{G_1[0, 3, 1, 7] + i : i \in \mathbb{Z}_{15}\}$ and $B_2 = \{G_2[0, 7, 6, 4] + i : i \in \mathbb{Z}_{15}\} \cup \{G_2[0, 3, 2, 6] + i : i \in \mathbb{Z}_{15}\}$ Then (V, B_1) and (V, B_2) are respectively G_1- and G_2-decompositions of ${}^2K_{5\times 3}$.

2.2 Small designs of index 3

Example 2.18. Let $V = \mathbb{Z}_3 \cup \{\infty\}$ and let $B = \{G_2[0, 1, \infty, 2] + i : i \in \mathbb{Z}_3\}$. Then (V, B) is a G_2-decomposition of ${}^3K_4$.

Example 2.19. Let $V = \mathbb{Z}_5$ and let $B_1 = \{G_1[0, 2, 4, 3] + i : i \in \mathbb{Z}_5\}$ and $B_2 = \{G_2[0, 1, 3, 2] + i : i \in \mathbb{Z}_5\}$. Then (V, B_1) and (V, B_2) are respectively G_1- and G_2-decompositions of ${}^3K_5$.

Example 2.20. Let $V = \mathbb{Z}_7 \cup \{\infty\}$ and let $B = \{G_2[0, 1, \infty, 2] + i : i \in \mathbb{Z}_7\} \cup \{G_2[0, 3, 5, 1] + i : i \in \mathbb{Z}_7\}$. Then (V, B) is a G_2-decomposition of ${}^3K_8$.

Example 2.21. Let $V = \mathbb{Z}_9$ and let $B_1 = \{G_1[0, 1, 2, 5] + i : i \in \mathbb{Z}_9\} \cup \{G_1[0, 7, 3, 8] + i : i \in \mathbb{Z}_9\}$ and $B_2 = \{G_2[0, 1, 7, 4] + i : i \in \mathbb{Z}_9\} \cup \{G_2[0, 4, 2, 3] + i : i \in \mathbb{Z}_9\}$. Then (V, B_1) and (V, B_2) are respectively G_1- and G_2-decompositions of ${}^3K_9$.

Example 2.22. Let $V = \mathbb{Z}_{17}$ and let $B = \{G_1[0, 1, 2, 9] + i : i \in \mathbb{Z}_{17}\} \cup \{G_1[0, 5, 3, 9] + i : i \in \mathbb{Z}_{17}\} \cup \{G_1[0, 6, 5, 9] + i : i \in \mathbb{Z}_{17}\} \cup \{G_1[0, 7, 4, 11] + i : i \in \mathbb{Z}_{17}\}$. Then (V, B) is a G_1-decomposition of ${}^3K_{17}$.

Example 2.23. Let $V = \mathbb{Z}_6$ with partition $\big\{\{0, 3\}, \{1, 4\}, \{2, 5\}\big\}$ and let $B = \{G_2[0, 1, 2, 4] + i : i \in \mathbb{Z}_6\}$. Then (V, B) is a G_2-decomposition of ${}^3K_{3\times 2}$.

Example 2.24. Let $V = \mathbb{Z}_{12}$ with the partition $\big\{\{0, 3, 6, 9\}, \{1, 4, 7, 10\}, \{2, 5, 8, 11\}\big\}$ and let

$$\begin{aligned} B = \{&G_1[1, 0, 2, 3], G_1[0, 4, 2, 7], G_1[5, 0, 10, 6], G_1[0, 1, 5, 10], G_1[0, 4, 11, 10],\\ &G_1[1, 0, 8, 9], G_1[4, 0, 8, 9], G_1[7, 0, 11, 3], G_1[2, 1, 3, 10], G_1[3, 8, 10, 11],\\ &G_1[7, 0, 8, 3], G_1[3, 1, 5, 7], G_1[6, 1, 11, 4], G_1[1, 8, 6, 11], G_1[9, 7, 11, 10],\\ &G_1[6, 2, 7, 5], G_1[7, 2, 9, 5], G_1[1, 5, 9, 11], G_1[2, 4, 6, 10], G_1[9, 2, 10, 8]\\ &G_1[4, 2, 9, 5], G_1[4, 3, 5, 6], G_1[3, 8, 4, 11], G_1[6, 7, 8, 10]\}. \end{aligned}$$

Then (V, B) is a G_1-decomposition of ${}^3K_{3\times 4}$.

Example 2.25. Let $V = \mathbb{Z}_{20}$ with partition $\big\{\{0, 5, 10, 15\}, \{1, 6, 11, 16\}, \{2, 7, 12, 17\}\big\}$ and let $B = \{G_1[0, 1, 8, 2] + i : i \in \mathbb{Z}_{20}\} \cup \{G_1[0, 4, 1, 12] + i : i \in \mathbb{Z}_{20}\} \cup \{G_1[0, 4, 6, 9] + i : i \in \mathbb{Z}_{20}\} \cup \{G_1[2, 0, 9, 6] + i : i \in \mathbb{Z}_{20}\}$. Then (V, B) is a G_1-decomposition of ${}^3K_{5\times 4}$.

2.3 Small designs of index 6

Example 2.26. Let $V = \mathbb{Z}_7 \cup \{\infty\}$ and let $B = \{G_1[\infty, 0, 3, 1] + i : i \in \mathbb{Z}_7\} \cup \{G_1[0, \infty, 1, 3] + i : i \in \mathbb{Z}_7\} \cup \{G_1[0, 1, 2, 4] + i : i \in \mathbb{Z}_7\} \cup \{G_1[0, 3, 4, 5] + i : i \in \mathbb{Z}_7\}$. Then (V, B) is a G_1-decomposition of ${}^6K_8$.

Example 2.27. Let $V = \mathbb{Z}_3 \cup \{\infty_1, \infty_2\}$ and let

$$\begin{aligned} B_1 = \{&G_1[\infty_1, 0, 1, 2], G_1[\infty_1, 0, 2, 1], G_1[\infty_1, 1, 0, 2], G_1[\infty_1, 1, 0, 2],\\ &G_1[\infty_2, 0, 1, 2], G_1[\infty_2, 0, 2, 1], G_1[\infty_2, 1, 0, 2], G_1[\infty_2, 1, 0, 2],\\ &G_1[1, \infty_1, 2, \infty_2], \} \end{aligned}$$

$$\begin{aligned} B_2 = \{&G_2[\infty_1, 0, 1, 2], G_2[\infty_1, 0, 1, 2], G_2[\infty_1, 1, 2, 0], G_2[\infty_1, 2, 0, 1],\\ &G_2[\infty_2, 0, 1, 2], G_2[\infty_2, 0, 2, 1], G_2[\infty_2, 2, 0, 1], G_2[0, 2, \infty_2, 1],\\ &G_2[1, \infty_2, 2, \infty_1]\}. \end{aligned}$$

Then (V, B_1) and (V, B_2) are respectively G_1- and G_2-decompositions of ${}^6K_5 \setminus {}^2K_2$ where the removed ${}^2K_2$ has vertex set $\{\infty_1, \infty_2\}$.

Example 2.28. Let $V = \mathbb{Z}_4 \cup \{\infty_1, \infty_2\}$ and let

$$\begin{aligned} B_1 = \{&G_1[\infty_1, 0, 2, 1], G_1[\infty_1, 0, 3, 1], G_1[\infty_1, 1, 0, 2], G_1[\infty_1, 2, 0, 3], \\ &G_1[\infty_1, 1, 3, 2], G_1[\infty_1, 2, 1, 3], G_1[\infty_2, 0, 1, 3], G_1[\infty_2, 1, 0, 2], \\ &G_1[\infty_2, 1, 0, 3], G_1[\infty_2, 1, 2, 3], G_1[0, \infty_2, 3, 1], G_1[1, \infty_2, 2, 0], \\ &G_1[2, \infty_2, 3, 0], G_1[2, \infty_2, 3, 1]\}, \\ B_2 = \{&G_2[\infty_1, 0, 2, 1], G_2[\infty_1, 0, 2, 3], G_2[\infty_1, 1, 3, 0], G_2[\infty_1, 2, 0, 3], \\ &G_2[\infty_1, 1, 3, 2], G_2[\infty_1, 3, 1, 2], G_2[\infty_2, 0, 1, 2], G_2[\infty_2, 1, 0, 2], \\ &G_2[\infty_2, 2, 0, 3], G_2[\infty_2, 3, 0, 1], G_2[\infty_2, 3, 1, 0], G_2[\infty_2, 2, 3, 1], \\ &G_2[1, 2, 3, 0], G_2[2, 3, 0, 1]\}. \end{aligned}$$

Then (V, B_1) and (V, B_2) are respectively G_1- and G_2-decompositions of ${}^6K_6 \setminus {}^2K_2$ where the removed ${}^2K_2$ has vertex set $\{\infty_1, \infty_2\}$.

3 Main Results

We now establish some necessary conditions for the existence of a G_1- or G_2-decomposition of ${}^\lambda K_n$.

Lemma 3.1. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. If there exists a G_1-decomposition of ${}^\lambda K_n$, then the following hold:*

1. *if* $\gcd(\lambda, 6) = 1$, *then* $n \equiv 1$ *or* $9 \pmod{12}$;
2. *if* $\gcd(\lambda, 6) = 2$, *then* $n \equiv 0$ *or* $1 \pmod 3$;
3. *if* $\gcd(\lambda, 6) = 3$, *then* $n \equiv 1 \pmod 4$;
4. *if* $\gcd(\lambda, 6) = 6$, *then* $n \geq 4$.

Proof. Let λ and n be as stated and suppose there exists an G_1-decomposition of ${}^\lambda K_n$. Since the number of edges in G_1 is 6, we must have that $6 \mid \lambda n(n-1)/2$, and thus $12 \mid \lambda n(n-1)$. Further, since the degree of each vertex in G_1 is even, we must have that $2 \mid \lambda(n-1)$. First, if $\gcd(\lambda, 6) = 1$, then $12 \mid n(n-1)$ and $2 \mid (n-1)$, and thus $n \equiv 1$ or $9 \pmod{12}$. Second, if $\gcd(\lambda, 6) = 2$, then $6 \mid n(n-1)$, and thus $n \equiv 0$ or $1 \pmod 3$. Third, if $\gcd(\lambda, 6) = 3$, then $4 \mid n(n-1)$ and $2 \mid 3(n-1)$, and thus $n \equiv 1 \pmod 4$. Finally, if $\gcd(\lambda, 6) = 6$, then $2 \mid n(n-1)$, which is true for any $n \geq 4$. ■

Lemma 3.2. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. If there exists a G_2-decomposition of ${}^\lambda K_n$, then the following hold:*

1. *if* $\gcd(\lambda, 6) = 1$, *then* $n \equiv 0$, 1, 4, *or* 9 $\pmod{12}$*;*
2. *if* $\gcd(\lambda, 6) = 2$, *then* $n \equiv 0$ *or* 1 $\pmod 3$*;*
3. *if* $\gcd(\lambda, 6) = 3$, *then* $n \equiv 0$ *or* 1 $\pmod 4$*;*
4. *if* $\gcd(\lambda, 6) = 6$, *then* $n \geq 4$.

Proof. Let λ and n be as stated and suppose there exists an G_2-decomposition of ${}^\lambda K_n$. Since the number of edges in G_2 is 6, we must have that $6 \mid \lambda n(n-1)/2$, and thus $12 \mid \lambda n(n-1)$. First, if $\gcd(\lambda, 6) = 1$, then $12 \mid n(n-1)$, and thus $n \equiv 0$, 1, 4, or 9 $\pmod{12}$. Second, if $\gcd(\lambda, 6) = 2$, then $6 \mid n(n-1)$, and thus $n \equiv 0$ or 1 $\pmod 3$. Third, if $\gcd(\lambda, 6) = 3$, then $4 \mid n(n-1)$, and thus $n \equiv 0$ or 1 $\pmod 4$. Finally, if $\gcd(\lambda, 6) = 6$, then $2 \mid n(n-1)$ which is always true. ■

Next, we establish sufficiency for the necessary conditions for small values of λ.

Lemma 3.3. *Let $G \in \{G_1, G_2\}$ and let $n \geq 4$ be an integer. There exists a G-decomposition of ${}^2K_n$ if and only if $n \equiv 0$ or 1 $\pmod 3$.*

Proof. The necessary conditions are established in Lemmas 3.1 and 3.2. For sufficiency, we consider the following cases:

Case 1: $n \equiv 0 \pmod 6$.
Examples 2.2 and 2.7 demonstrate the existence of G-decompositions of both ${}^2K_6$ and ${}^2K_{12}$. Now, let $n = 6t$ for some integer $t \geq 3$. By Corollary 1.4 there exists of K_3-decomposition of either $K_{t\times 2}$ or $K_{4,(t-2)\times 2}$. Thus by Theorem 1.5, either ${}^2K_{3\times 3} \mid {}^2K_{t\times 6}$ or ${}^2K_{3\times 3} \mid {}^2K_{12,(t-2)\times 6}$. By Examples 2.2, 2.7, 2.15, and 2.16, G divides ${}^2K_6$, ${}^2K_{12}$, and ${}^2K_{3\times 3}$, and the result follows.

The remainder of the proof continues similarly to the prior case, incorporating additional vertices as needed. (For brevity of proof, we note that ${}^2K_7 = {}^2K_{6+1} \setminus {}^2K_1$.)

Case 2: $n \equiv 1$, 3, or 4 $\pmod 6$.
Examples 2.1, 2.3, 2.4, 2.5, 2.6, 2.8, 2.9, 2.10, and 2.11 demonstrate the existence of G-decompositions of ${}^2K_4$, ${}^2K_7$, ${}^2K_9$, ${}^2K_{10}$, ${}^2K_{13}$, ${}^2K_{15}$, and ${}^2K_{16}$. Now, let $n = 6t+i$ for some integers $t \geq 3$ and $i \in \{1, 3, 4\}$. By Corollary 1.4 there exists of K_3-decomposition of either $K_{t\times 2}$ or $K_{4,(t-2)\times 2}$. Thus by Theorem 1.5, either ${}^2K_{3\times 3} \mid {}^2K_{t\times 6}$ or ${}^2K_{3\times 3} \mid {}^2K_{12,(t-2)\times 6}$. By Examples 2.3, 2.4, 2.5, 2.6, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.15, and 2.16, G divides ${}^2K_{6+i}$, ${}^2K_{12+i}$, ${}^2K_{6+i} \setminus {}^2K_i$, and ${}^2K_{3\times 3}$, and the result follows. ■

Lemma 3.4. *Let $n \geq 5$ be an integer. There exists an G_1-decomposition of ${}^3K_n$ if and only if $n \equiv 1 \pmod 4$.*

Proof. The necessary conditions are established in Lemma 3.1. For sufficiency, we consider the following cases:

Case 1: $n \equiv 5 \pmod 8$.
Let $n = 8t + 5$ for some integer $t \geq 0$. By Theorem 1.1 a $\{K_3, K_5\}$-decomposition of K_{2t+1} exists, and thus by Theorem 1.5, there exists a $\{K_{3\times4}, K_{5\times4}\}$-decomposition of $K_{(2t+1)\times4}$. We have by Examples 2.19, 2.24, and 2.25 that G_1 divides ${}^3K_5$, ${}^3K_{3\times4}$, and ${}^3K_{5\times4}$, and the result follows.

Case 2: $n \equiv 1 \pmod 8$.
Examples 2.21 and 2.22 demonstrate the existence of G_1-decompositions of both ${}^3K_9$ and ${}^3K_{17}$. Now, let $n = 8t + 1$ for some integer $t \geq 3$. By Corollary 1.4 there exists of K_3-decomposition of either $K_{t\times2}$ or $K_{4,(t-2)\times2}$. Thus by Theorem 1.5, either ${}^3K_{3\times4} \mid {}^3K_{t\times8}$ or ${}^3K_{3\times4} \mid {}^3K_{16,(t-2)\times8}$. By Examples 2.21, 2.22, and 2.24, G_1 divides ${}^3K_9$, ${}^3K_{17}$, and ${}^3K_{3\times4}$, and the result follows. ■

Lemma 3.5. *Let $n \geq 4$ be an integer. There exists an G_2-decomposition of ${}^3K_n$ if and only if $n \equiv 0$ or $1 \pmod 4$.*

Proof. The necessary conditions are established in Lemma 3.2. Examples 2.18, 2.19, 2.20, and 2.21 demonstrate the existence of G_2-decompositions of ${}^3K_4$, ${}^3K_5$, ${}^3K_8$, and ${}^3K_9$. Now, let $n = 4t + i$ for some integers $t \geq 3$ and $i \in \{0, 1\}$. By Corollary 1.4 there exists of K_3-decomposition of either $K_{t\times2}$ or $K_{4,(t-2)\times2}$. Thus by Theorem 1.5, either ${}^3K_{3\times2} \mid {}^3K_{t\times4}$ or ${}^3K_{3\times2} \mid {}^3K_{8,(t-2)\times4}$. By Examples 2.18, 2.19, 2.20, 2.21, and 2.23, G_2 divides ${}^3K_{4+i}$, ${}^3K_{8+i}$, and ${}^3K_{3\times2}$, and the result follows. ■

Lemma 3.6. *Let $G \in \{G_1, G_2\}$ and let n be a positive integer. There exists a G-decomposition of ${}^6K_n$ if and only if $n \geq 4$.*

Proof. The necessary conditions are established in Lemmas 3.1 and 3.2. For sufficiency, we consider the following cases:

Case 1: $n \equiv 0$, 1, 3, 4, 6, 7, 9, or $10 \pmod{12}$.
By Lemma 3.3, G divides ${}^2K_n$, and the result follows because ${}^2K_n \mid {}^6K_n$.

Case 2: $n \equiv 5 \pmod{12}$.
By Lemmas 3.4 and 3.5, G divides ${}^3K_n$, and the result follows because ${}^3K_n \mid {}^6K_n$.

Case 3: $n \equiv 8 \pmod{12}$.
Example 2.26 demonstrates the existence of a G_1-decomposition of ${}^6K_8$. By Lemma 3.5, we have that G_2 divides ${}^3K_n$, which divides ${}^6K_n$. Otherwise, let

$n = 12t+8$ for some positive integer t. By Theorem 1.3 a K_3-decomposition of $K_{4,(3t)\times 2}$ exists, and thus by Theorem 1.5, ${}^6K_{3\times 2} \mid {}^6K_{8,(3t)\times 4}$. By Examples 2.1, 2.26, and 2.14, G_1 divides ${}^2K_4$, ${}^6K_8$, and ${}^2K_{3\times 2}$, and the result follows because ${}^2K_4 \mid {}^6K_4$ and ${}^2K_{3\times 2} \mid {}^6K_{3\times 2}$.

Case 4: $n \equiv 2 \pmod{12}$.
Let $n = 12t + 2$ for some positive integer t. By Theorem 1.2 a K_3-decomposition of $K_{(3t)\times 2}$ exists, and thus by Theorem 1.5, ${}^6K_{3\times 2} \mid {}^6K_{(3t)\times 4}$. By Examples 2.2 and 2.28, G divides ${}^2K_6$ and ${}^6K_6 \setminus {}^6K_2$; by Example 2.14, G_1 divides ${}^2K_{3\times 2}$; by Example 2.23, G_2 divides ${}^3K_{3\times 2}$; and the result follows because ${}^2K_6 \mid {}^6K_6$ and both ${}^2K_{3\times 2}$ and ${}^2K_{3\times 2}$ divide ${}^6K_{3\times 2}$. ∎

Finally, we show that the necessary conditions for the existence of G_1- and G_2-decompositions of ${}^\lambda K_n$ are sufficient.

Theorem 3.7. *Let $\lambda \geq 2$ and $n \geq 4$ be integers. There exists a G_1-decomposition of ${}^\lambda K_n$ if and only if $2 \mid \lambda(n-1)$ and $12 \mid \lambda n(n-1)$. Furthermore, there exists a G_2-decomposition of ${}^\lambda K_n$ if and only if $12 \mid \lambda n(n-1)$.*

Proof. The necessary conditions are established in Lemmas 3.1 and 3.2. For sufficiency, let $G \in \{G_1, G_2\}$ and let $12 \mid \lambda n(n-1)$. Moreover, if $G = G_1$, let $2 \mid \lambda(n-1)$. We now consider the following cases:

Case 1: $\lambda \equiv 0 \pmod 6$.
Since $\gcd(\lambda, 6) = 6$, we have that $\lambda \geq 6$ and $n \geq 4$. Let $\lambda = 6t$ for some positive integer t. By Lemma 3.6, G divides ${}^6K_n$, and the result follows because ${}^6K_n \mid {}^\lambda K_n$.

Case 2: $\lambda \equiv 1$ or $5 \pmod 6$.
Since $\gcd(\lambda, 6) = 1$, we have that $\lambda \geq 5$ and either (i) if $G = G_1$, then $n \equiv 1$ or $9 \pmod{12}$ or (ii) if $G = G_2$, then $n \equiv 0$, 1, 4, or $9 \pmod{12}$. Let $\lambda = 6t - 1 + 2i$ for some integers $t \geq 1$ and $i \in \{0, 1\}$. Now, we split ${}^\lambda K_n$ into $(1+i)$ copies of ${}^2K_n$, one copy of ${}^3K_n$, and $(t-1)$ copies of ${}^6K_n$. By Lemmas 3.3, 3.4, 3.5, and 3.6, G divides ${}^2K_n$, ${}^3K_n$, and ${}^6K_n$, and the result follows.

Case 3: $\lambda \equiv 2$ or $4 \pmod 6$.
Since $\gcd(\lambda, 6) = 2$, we have that $\lambda \geq 2$ and $n \equiv 0$ or $1 \pmod 3$. Let $\lambda = 6t + 2 + 2i$ for some integers $t \geq 0$ and $i \in \{0, 1\}$. Now, we split ${}^\lambda K_n$ into $(1+i)$ copies of ${}^2K_n$ and t copies of ${}^6K_n$. By Lemmas 3.3 and 3.6, G divides ${}^2K_n$ and ${}^6K_n$, and the result follows.

Case 4: $\lambda \equiv 3 \pmod 6$.
Since $\gcd(\lambda, 6) = 3$, we have that $\lambda \geq 3$ and either (i) if $G = G_1$, then $n \equiv 1 \pmod 4$ or (ii) if $G = G_2$, then $n \equiv 0$ or $1 \pmod 4$. Let $\lambda = 6t + 3$ for some integer $t \geq 0$. Now, we split ${}^\lambda K_n$ into one copy of ${}^3K_n$ and t copies of ${}^6K_n$. By Lemmas 3.4, 3.5, and 3.6, G divides ${}^3K_n$ and ${}^6K_n$, and the result follows. ∎

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