On the $\lambda$-fold Spectra for Bipartite Subgraphs of $^2K_4$


$^1$Armstrong Township High School, Armstrong, IL 61812
$^2$Illinois State University, Normal, IL 61790
$^3$Iowa State University, Ames, IA 50011
$^4$University of Wisconsin-La Crosse, La Crosse, WI 54601
$^5$East Leyden High School, Franklin Park, IL 60131

Abstract

For a graph $H$ and a positive integer $\lambda$, let $^\lambda H$ denote the multigraph obtained by replacing each edge of $H$ with $\lambda$ parallel edges. Let $G$ be a multigraph with edge multiplicity 2 and with $C_4$ as its underlying simple graph. We find necessary and sufficient conditions for the existence of a $G$-decomposition of $^\lambda K_n$ for all positive integers $\lambda$ and $n$.

1 Introduction

If $a$ and $b$ are integers with $a \leq b$, we denote $\{a, a + 1, \ldots, b\}$ by $[a, b]$. Let $\mathbb{Z}_n$ be the group of integers modulo $n$. For a finite set $S$ and a positive integer $\lambda$, we let $^\lambda S$ denote the multiset that contains every element of $S$ exactly $\lambda$ times. For example, $^3[a, b]$ is the multiset $\{a, a, a + 1, a + 1, a + 1, \ldots, b - 1, b - 1, b - 1, b, b, b\}$. Similarly for a graph $H$, we let $^\lambda H$ denote the multigraph obtained by replacing each edge in $H$ with $\lambda$ parallel edges. Thus $^\lambda K_n$ denotes the $\lambda$-fold complete multigraph of order $n$. We note that a multigraph is not required to contain multiple edges. However, our graphs contain no loops. If we wish to emphasize that a given graph does not contain parallel edges, then we refer to it as a simple graph. For positive integers $r$ and $s$, let $K_{r \times s}$ denote the complete multipartite graph.

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with \( r \) parts of cardinality \( s \) each. The order and size of a multigraph \( G \) refer to \( |V(G)| \) and \( |E(G)| \), respectively.

Let \( V(\gamma K_n) = [0,n-1] \). The label of an edge \( \{i,j\} \) in \( \gamma K_n \) is defined to be \( |i-j| \). The length of an edge \( \{i,j\} \) in \( \gamma K_n \) is defined to be \( \min\{|i-j|, n-|i-j|\} \). Thus if the elements of \( V(\gamma K_n) \) are placed in order as vertices of an equisided \( n \)-gon, then the length of edge \( \{i,j\} \) is the shortest distance around the polygon between \( i \) and \( j \). Note that if \( n \) is odd, then \( \gamma K_n \) consists of \( \lambda n \) edges of length \( i \) for \( i \in [1, \frac{n-1}{2}] \). Furthermore if \( n \) is even, then \( \gamma K_n \) consists of \( \lambda n \) edges of length \( i \) for \( i \in [1, \frac{n}{2}] \) and \( \frac{\lambda n}{2} \) edges of length \( \frac{n}{2} \).

Let \( V(\gamma K_n) = \mathbb{Z}_n \) and let \( G \) be a subgraph of \( \gamma K_n \). By rotating \( G \), we mean applying the permutation \( i \mapsto i+1 \) to \( V(G) \). Note that rotating an edge does not change its length.

Alternatively, we may let \( V(\gamma K_n) = \mathbb{Z}_{n-1} \cup \{\infty\} \). As expected, rotating a subgraph \( G \) of \( \gamma K_n \) in this case continues to mean applying the permutation \( i \mapsto i+1 \) to \( V(G) \), with the convention that \( \infty + 1 = \infty \). If neither \( i \) nor \( j \) is the \( \infty \)-vertex, then the label and length of the edge \( e = \{i,j\} \) are defined as if \( e \) is in \( \gamma K_{n-1} \). The label and length of an edge \( \{i,\infty\} \) are both defined to be \( \infty \). Again, rotating an edge does not change its length.

In this case, if \( n \) is odd, then \( \gamma K_n \) consists of \( \lambda (n-1) \) edges of length \( \infty \) along with \( \lambda (n-1) \) edges of length \( i \) for \( i \in [1, \frac{n-3}{2}] \) and \( \frac{\lambda (n-1)}{2} \) edges of length \( \infty \). Furthermore if \( n \) is even, then then \( \gamma K_n \) consists of \( \lambda (n-1) \) edges of length \( \infty \) along with \( \lambda (n-1) \) edges of length \( i \) for \( i \in [1, \frac{n-2}{2}] \).

Let \( K \) and \( G \) be graphs with \( G \) a subgraph of \( K \). A \( G \)-decomposition of \( K \) is a set (or multiset) \( \Delta = \{G_1, G_2, \ldots, G_t\} \) of subgraphs of \( K \) each of which is isomorphic to \( G \) (and is called a \( G \)-block) and such that each edge of \( K \) appears in exactly one \( G \)-block. If there exists a \( G \)-decomposition of \( K \), then we say \( G \) divides \( K \) and write \( G \rvert K \). A \( G \)-decomposition of \( K \) is also known as a \((K,G)\)-design. A \((\gamma K_n,G)\)-design is called a \( \gamma \)-design of order \( n \) and index \( \lambda \). A \((\gamma K_n,G)\)-design \( \Delta \) is said to be cyclic if rotating a \( G \)-block in \( \Delta \) yields another \( G \)-block in \( \Delta \). If \( V(\gamma K_n) = \mathbb{Z}_{n-1} \cup \{\infty\} \), then a cyclic \((\gamma K_n,G)\)-design is also called a 1-rotational \((\gamma K_n,G)\)-design. The study of graph decompositions is generally known as the study of graph designs, or \( G \)-designs. For recent surveys on \( G \)-designs of index 1, see [1] and [3].

Let \( G \) be a graph. A classical problem in the study of graph designs is to find necessary and sufficient conditions for the existence of a \( G \)-decomposition of \( \gamma K_n \). This is known as the spectrum problem for \( G \). The set of all such \( n \) is called the spectrum for \( G \)-designs of index \( \lambda \), or alternatively the index \( \lambda \) spectrum for \( G \). The spectra for \( G \)-designs of index 1 has been determined for several classes of graphs including cycles, paths, stars and all graphs of order at most 5 (see [1]).
In recent years, there have been some investigations of \( G \)-designs of index \( \lambda \) where \( G \) is a multigraph with edge multiplicity at least 2. For example, in [6] Carter determined the spectrum for \( G \)-designs of index \( \lambda \) for all connected cubic multigraphs \( G \) of order at most 6. Sarvate and various co-authors have investigated \( G \)-designs of index \( \lambda \) for various multigraphs \( G \) of small order (see for example [7], [10], [12], and [13]). See also [5] and [8] for the spectrum for \( G \)-designs where \( G \) is a multigraph of small order.

In this article, we investigate \( G \)-decompositions of \( \lambda K_n \), where \( G \) is a multigraph with edge multiplicity 2 and with \( C_4 \) as the simple graph underlying \( G \). Figure 1 shows the five possibilities for such a \( G \). We find necessary and sufficient conditions for the existence of a \( G \)-decomposition of \( \lambda K_n \) for all integers \( \lambda \geq 2 \).

![Figure 1: The five multigraphs with edge multiplicity 2 and \( C_4 \) as the underlying simple graph.](image)

2 Main Results

The index \( \lambda \) spectra for \( G_1 \) and \( G_2 \) are settled in [12] and in [6], respectively. Thus we will focus on the three remaining multigraphs. The case \( \lambda = 2 \) for all bipartite subgraphs of \( ^2K_4 \) is settled in [2].

2.1 \( (\lambda K_n, G_3) \)-designs

We begin with some obvious necessary conditions.

**Lemma 1.** Let \( \lambda \geq 2 \) and \( n \geq 4 \) be integers. If there exists a \( (\lambda K_n, G_3) \)-design, then the following necessarily hold:

1. if \( \gcd(\lambda, 6) = 1 \), then \( n \equiv 0, 1, 4, \) or \( 9 \) (mod 12);
2. if \( \gcd(\lambda, 6) = 2 \), then \( n \equiv 0 \) or \( 1 \) (mod 3);
3. if \( \gcd(\lambda, 6) = 3 \), then \( n \equiv 0 \) or \( 1 \) (mod 4);
4. if \( \gcd(\lambda, 6) = 6 \), then \( n \geq 4 \).

**Proof.** Let \( \lambda \) and \( n \) be as stated and suppose there exists a \((\lambda K_n, G_3)\)-design.

Since the number of edges in \( G_3 \) is 6, we must have that \( 6|\lambda n(n-1)/2 \), and thus \( 12|\lambda n(n-1) \). If \( \gcd(\lambda, 6) = 1 \), then \( 12|n(n-1) \), and thus \( n \equiv 0, 1, 4, \) or \( 9 \) (mod 12). If \( \gcd(\lambda, 6) = 2 \), then \( 6|n(n-1) \), and thus \( n \equiv 0 \) or \( 1 \) (mod 3). Similarly, if \( \gcd(\lambda, 6) = 3 \), then \( 4|n(n-1) \), and thus \( n \equiv 0 \) or \( 1 \) (mod 4). Finally, if \( \gcd(\lambda, 6) = 6 \), then \( 2|n(n-1) \), which is always true. ■

From Allen et al. [2], we have the following for index 2.

**Lemma 2.** There exists a \((^2K_n, G_3)\)-design for all \( n \equiv 0 \) or \( 1 \) (mod 3) where \( n \neq 3 \).

Next, we settle both the index 3 and index 6 spectra for \( G_3 \).

**Lemma 3.** There exists a \((^3K_n, G_3)\)-design for all \( n \equiv 0 \) or \( 1 \) (mod 4).

**Proof.** We consider two cases.

**Case 1:** \( n \equiv 0 \) (mod 4).
Let \( n = 4x \) and let \( V(^3K_{4x}) = \mathbb{Z}_{4x-1} \cup \{\infty\} \). Let

\[
\Delta = \{G_3[\infty, j, 2+j, 1+j]: 0 \leq j \leq 4x-2\} \\
\cup \{G_3[4i+j, j, 4i+2+j, 1+j]: 1 \leq i \leq x-1, 0 \leq j \leq 4x-2\}.
\]

It is easily checked that \( \Delta \) is a 1-rotational \((^3K_{4x}, G_3)\)-design.

**Case 2:** \( n \equiv 1 \) (mod 4).
Let \( n = 4x+1 \) and let \( V(^3K_{4x+1}) = \mathbb{Z}_{4x+1} \). Let

\[
\Delta = \{G_3[4i+j, j, 4i-2+j, 1+j]: 1 \leq i \leq x, 0 \leq j \leq 4x\}.
\]

It is easily checked that \( \Delta \) is a cyclic \((^3K_{4x}, G_3)\)-design. ■

**Lemma 4.** There exists a \((^6K_n, G_3)\)-design for all \( n \geq 4 \).

**Proof.** We consider four cases.

**Case 1:** \( n \equiv 0 \) or \( 1 \) (mod 3).
By Lemma 2 there exists a \((^2K_n, G_3)\)-design. Hence, we can obtain a \((^6K_n, G_3)\)-design from three copies of a \((^2K_n, G_3)\)-design.

**Case 2:** \( n \equiv 5 \) or \( 8 \) (mod 12).
By Lemma 3 there exists a \((^3K_n, G_3)\)-design. Hence, we can obtain a \((^6K_n, G_3)\)-design from two copies of a \((^3K_n, G_3)\)-design.

**Case 3:** \( n \equiv 2 \) (mod 12).
Let \( n = 12x + 14 \). Then we are looking to show that \( G_3 \) divides \(^6K_{12x+14} \).
We view our $6K_{12x+14}$ as $6K_6 \cup 6K_{12x+8} \cup 6K_{6,12x+8}$. It is proved in the above cases that $G_3|6K_6$ and $G_3|6K_{12x+8}$. We now must show that $G_3|6K_{6,12x+8}$. Clearly $2K_{3,2}$ divides $6K_{6,12x+8}$, so all that remains is to show that $G_3|2K_{3,2}$. Let $2K_{3,2}$ have vertex partition $\{\{u_1, u_2, u_3\}, \{v_1, v_2\}\}$. Then $\{G_3[u_1, u_1, v_2, v_3], G_3[v_1, u_2, v_2, u_3]\}$ is a $(2K_{3,2}, G_3)$-design.

Case 4: $n \equiv 11$ (mod 12).

Let $n = 12x + 11$. Then we are looking to show that $G_3$ divides $6K_{12x+11}$. We view our $6K_{12x+11}$ as $6K_5 \cup 6K_{12x+6} \cup 6K_{5,12x+6}$. It is proved in the above cases that $G_3|6K_5$ and $G_3|6K_{12x+6}$. We now must show that $G_3|6K_{5,12x+6}$. Clearly $3K_{5,2}$ divides $6K_{5,12x+6}$, so all that remains is to show that $G_3|3K_{5,2}$. Let $3K_{5,2}$ have vertex partition $\{\{u_1, u_2, u_3, u_4, u_5\}, \{v_1, v_2\}\}$. Then $\{G_3[v_1, u_1, v_2, u_5], G_3[v_1, u_2, v_2, u_1], G_3[v_1, u_3, v_2, u_2], G_3[v_1, u_4, v_2, u_3], G_3[v_1, v_2, u_3, u_4]\}$ is a $(3K_{5,2}, G_3)$-design.

Finally, we have all the necessary building blocks to settle the index $\lambda$ spectrum for $G_3$.

**Theorem 5.** For any positive integers $\lambda \geq 2$ and $n \geq 4$, there exists a $(\lambda K_n, G_3)$-design if and only if $12|\lambda(n - 1)$.

**Proof.** The necessary conditions are established by the fact that the number of edges in $G_3$ must divide the number of edges in $\lambda K_n$. To show sufficiency, we use the following 4-case breakdown prescribed by Lemma 1.

**Case 1:** $\lambda \equiv 0$ (mod 6).

Let $\lambda = 6t$. By Lemma 1, we need to show that $G_3$ divides $6tK_n$ for $n \geq 4$. By Lemma 4 there exists a $(6tK_n, G_3)$-design. Hence, we can obtain a $(6tK_n, G_3)$-design from $t$ copies of a $(6K_n, G_3)$-design.

**Case 2:** $\lambda \equiv 1$ or 5 (mod 6).

We note that $\lambda = 5$ is the least possible edge multiplicity that meets the criterion for this case of the proof. Thus $\lambda = 2t + 3$ for some integer $t \geq 1$. By Lemma 1, we need to show that $G_3$ divides $2t+3K_n$ for $n \equiv 0, 1, 4, 9$ (mod 12). By Lemmas 2 and 3 there exist both a $(2K_n, G_3)$-design and a $(3K_n, G_3)$-design. Hence, we can obtain a $(2t+3K_n, G_3)$-design from $t$ copies of a $(2K_n, G_3)$-design and a single $(3K_n, G_3)$-design.

**Case 3:** $\lambda \equiv 2$ or 4 (mod 6).

Let $\lambda = 2t$ such that $t \equiv 0$ (mod 3). By Lemma 1, we need to show that $G_3$ divides $2tK_n$ for $n \equiv 0$ or 1 (mod 3). By Lemma 2 there exists a $(2K_n, G_3)$-design. Hence, we can obtain a $(2tK_n, G_3)$-design from $t$ copies of a $(2K_n, G_3)$-design.

**Case 4:** $\lambda \equiv 3$ (mod 6).

Let $\lambda = 6t + 3$. By Lemma 1, we need to show that $G_3$ divides $6t+3K_n$ for $n \equiv 0$ or 1 (mod 4). By Lemma 3 there exists a $(3K_n, G_3)$-design. Hence,
we can obtain a \((6t+3K_n, G_3)\)-design from \(2t + 1\) copies of a \((3K_n, G_3)\)-design.

2.2 \((\lambda K_n, G_4)\)-designs

Again, we begin with some necessary conditions.

Lemma 6. Let \(\lambda \geq 2\) and \(n \geq 4\) be integers. If there exists a \((\lambda K_n, G_4)\)-design, then the following necessarily hold:

1. if \(\text{gcd}(\lambda, 7) = 1\), then \(n \equiv 0\) or \(1 \mod 7\);
2. if \(\text{gcd}(\lambda, 7) = 7\), then \(n \geq 4\).

Proof. Let \(\lambda\) and \(n\) be as stated and suppose there exists a \((\lambda K_n, G_4)\)-design. Since the number of edges in \(G_4\) is 7, we must have that \(7|\lambda n(n-1)/2\), and thus \(14|\lambda n(n-1)\). If \(\text{gcd}(\lambda, 7) = 1\), then \(14|n(n-1)\), and thus \(n \equiv 0, 1, 7, \) or \(8 \mod 14\). If \(\text{gcd}(\lambda, 7) = 7\), then \(2|n(n-1)\), which is always true. ■

From Allen et al. [2], we have the following for index 2.

Lemma 7. There exists a \((2K_n, G_4)\)-design for all \(n \equiv 0\) or \(1 \mod 7\).

Next, we show the only insufficiencies of the necessary conditions in Lemma 6 (i.e., when \(\lambda\) is 3 or 5) before settling the index 7 spectrum for \(G_4\).

Lemma 8. There does not exist a \((3K_n, G_4)\)-design for any \(n\).

Proof. Suppose \(\Delta\) is a \((3K_n, G_4)\)-design. We note that each \(G_4\)-block in \(\Delta\) contains exactly one edge of multiplicity 1 and three edges with multiplicity 2. Since each edge in \(3K_n\) has edge multiplicity 3, each pair of vertices must be incident with at least one edge of multiplicity 1 within a \(G_4\)-block of \(\Delta\). This leads to a contradiction, as the number of vertex pairings in \(3K_n\) (i.e., the size of \(K_n\)) exceeds the number of \(G_4\)-blocks in \(\Delta\). ■

Lemma 9. There does not exist a \((5K_n, G_4)\)-design for any \(n\).

Proof. Suppose \(\Delta\) is a \((5K_n, G_4)\)-design. Then the proof proceeds similarly to that of Lemma 8. ■

Lemma 10. There exists a \((7K_n, G_4)\)-design for all \(n \geq 4\).

Proof. We consider four cases.
Case 1: \( n \equiv 0 \pmod{4} \).
Let \( n = 4x \) and let \( V(7K_{4x}) = \mathbb{Z}_{4x-1} \cup \{\infty\} \). Let
\[
\Delta = \left\{ G_{4x}[\infty, j, 1 + j, 2 + j], G_{4x}[1 + j, j, \infty, 2 + j] : 0 \leq j \leq 4x - 2 \right\}
\]
\[
\cup \left\{ G_{4x}[4i - 1 + j, j, 4i + 1 + j, 1 + j], \right.
\]
\[
G_{4x}[4i + 1 + j, j, 4i - 1 + j, 1 + j] : 1 \leq i \leq x - 1, \ 0 \leq j \leq 4x - 2 \right\}.
\]

It is easily checked that \( \Delta \) is a 1-rotational \((7K_{4x}, G_{4})\)-design.

Case 2: \( n \equiv 1 \pmod{4} \).
Let \( n = 4x + 1 \) and let \( V(7K_{4x+1}) = \mathbb{Z}_{4x+1} \). Let
\[
\Delta = \left\{ G_{4x}[4i - 2 + j, j, 4i + j, 1 + j], G_{4x}[4i + j, j, 4i - 2 + j, 1 + j] : 1 \leq i \leq x, \ 0 \leq j \leq 4x \right\}.
\]

It is easily checked that \( \Delta \) is a cyclic \((7K_{4x+1}, G_{4})\)-design.

Case 3: \( n \equiv 2 \pmod{4} \).
Let \( n = 4x + 2 \) and let \( V(7K_{4x+2}) = \mathbb{Z}_{4x+1} \cup \{\infty\} \). Let
\[
\Delta = \left\{ G_{4x}[\infty, j, 2 + j, 1 + j], G_{4x}[j, 1 + j, \infty, 2 + j], \right.
\]
\[
G_{4x}[2 + j, j, 1 + j, 3 + j] : 0 \leq j \leq 4x \right\}
\]
\[
\cup \left\{ G_{4x}[4i + j, j, 4i + 2 + j, 1 + j], \right.
\]
\[
G_{4x}[4i + 2 + j, j, 4i + j, 1 + j] : 1 \leq i \leq x - 1, \ 0 \leq j \leq 4x \right\}.
\]

It is easily checked that \( \Delta \) is a 1-rotational \((7K_{4x+2}, G_{4})\)-design.

Case 4: \( n \equiv 3 \pmod{4} \).
Let $n = 4x + 3$ and let $V(\overline{7}K_{4x+3}) = Z_{4x+3}$. Let

$$
\Delta = \left\{ G_4[3 + j, j, 2 + j, 1 + j], G_4[3 + j, 1 + j, 2 + j, j],
G_4[3 + j, j, 1 + j, 4 + j] : 0 \leq j \leq 4x + 2 \right\}
$$

$$
\cup \left\{ G_4[4i + 1 + j, j, 4i + 3 + j, 1 + j],
G_4[4i + 3 + j, j, 4i + 1 + j, 1 + j] : 1 \leq i \leq x - 1, 0 \leq j \leq 4x + 2 \right\}.
$$

It is easily checked that $\Delta$ is a cyclic $(\overline{7}K_{4x+3}, G_4)$-design.  

Finally, we have all the necessary building blocks to settle the index $\lambda$ spectrum for $G_4$.

**Theorem 11.** For positive integers $\lambda \geq 2$ and $n \geq 4$, there exists a $(\lambda K_n, G_4)$-design if and only if $14 | \lambda n(n - 1)$ and $\lambda \not\in \{3, 5\}$.

**Proof.** The necessary condition that $14 | \lambda n(n - 1)$ is established by the fact that the number of edges in $G_4$ must divide the number of edges in $\lambda K_n$. The latter condition is proved in Lemmas 8 and 9. To show sufficiency, we consider three cases.

**Case 1:** $\lambda \equiv 0 \pmod{7}$.
Let $\lambda = 7t$. By Lemma 6, we need to show that $G_4$ divides $\overline{7}K_n$ for $n \geq 4$. By Lemma 10 there exists a $(\overline{7}K_n, G_4)$-design. Hence, we can obtain a $(\overline{7}K_n, G_4)$-design from $t$ copies of a $(\overline{7}K_n, G_4)$-design.

**Case 2:** $\lambda \not\equiv 0 \pmod{7}$ and $\lambda$ is even.
Let $\lambda = 2t$. By Lemma 6, we need to show that $G_4$ divides $2tK_n$ for $n \equiv 0$ or 1 (mod 7). By Lemma 7 there exists a $(2tK_n, G_4)$-design. Hence, we can obtain a $(2tK_n, G_4)$-design from $t$ copies of a $(2tK_n, G_4)$-design.

**Case 3:** $\lambda \not\equiv 0 \pmod{7}$ and $\lambda$ is odd.
We note that $\lambda = 9$ is the least possible edge multiplicity that meets the criteria for this case of the proof. Thus $\lambda = 2t + 7$ for some integer $t \geq 1$. By Lemma 6, we need to show that $G_4$ divides $2t+7K_n$ for $n \equiv 0$ or 1 (mod 7). By Lemmas 7 and 10 there exist both a $(2tK_n, G_4)$-design and a $(\overline{7}K_n, G_4)$-design. Hence, we can obtain a $(2t+7K_n, G_4)$-design from $t$ copies of a $(2tK_n, G_4)$-design and a single $(\overline{7}K_n, G_4)$-design.

**2.3 $(\lambda K_n, G_5)$-designs**

Since $G_5$ is isomorphic to $\overline{6}C_4$, we first give the index $\lambda$ spectrum for $C_4$ (see [11] and [9]).
Theorem 12. For any positive integers $\lambda$ and $n$, there exists a $(^3K_n, C_4)$-design if and only if (a) $2$ divides $\lambda(n - 1)$, (b) $8$ divides $\lambda n(n - 1)$, and (c) $n \geq 4$.

It is easy to see that for all graphs $G$ and $K$ we have $G|K$ if and only if $2^G|2^K$. Thus, we have the following.

Theorem 13. For any positive integers $\lambda$ and $n$, there exists a $(^3K_n, G_5)$-design if and only if (a) $4$ divides $\lambda(n - 1)$, (b) $16$ divides $\lambda n(n - 1)$, (c) $n \geq 4$, and (d) $\lambda$ is even.

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