Decompositions of Complete Digraphs into Small Tripartite Digraphs

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Abstract

Necessary and sufficient conditions on n are established for the existence of a (K_n^*, D)-design where D is any digraph obtained by orienting the edges of a triangle with a pendant edge.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, then $[a, b] = \varnothing$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_m$ the group of integers modulo m. For a graph (or digraph) H, let $V(H)$ and $E(H)$ denote the vertex set of H and the edge (or arc) set of H, respectively. The *order* and the *size* of a (di-)graph H are $|V(H)|$ and $|E(H)|$, respectively.

We denote the complete multipartite graph with parts of sizes a_i for $1 \leq i \leq m$ by $K_{a_1, a_2, \ldots, a_m}$. If $a_i = a$ for all $i \in [1, m]$, then we use the notation $K_{m \times a}$. Throughout this paper, let $V(K_{m \times a}) = \mathbb{Z}_{ma}$ with vertex partition $\{V_0, V_1, \ldots, V_{m-1}\}$ where $V_i = \{j \in \mathbb{Z}_{ma} \colon j \equiv i \pmod{m}\}$. Then $E(K_{m \times a})$ consists of all edges $\{i, j\}$ such that $i \not\equiv j \pmod{m}$.

The *complete graph of order n with a hole of size t*, denoted $K_n \setminus K_t$, is the graph with vertex set V and edge set $\{\{a, b\} \colon a \in V, b \in V \setminus U, a \neq b\}$

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where $|V| = n$, $U \subseteq V$, and $|U| = t$. The vertices in U are said to be the *vertices in the hole*.

Let tG denote the graph consisting of t vertex-disjoint copies of G. The *join* of two vertex-disjoint graphs G and H, denoted $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{a, b\}: a \in V(G), b \in V(H)\}$.

Let H be a graph and let $\mathcal{G}$ be a set of subgraphs of H. We will refer to a graph $G \in \mathcal{G}$ as a *G-block*. A *$\mathcal{G}$-decomposition* of H is a set $\Delta = \{G_1, G_2, \ldots, G_r\}$ of pairwise edge-disjoint subgraphs of H such that for every $i \in [1, r]$, $G_i \cong G$ for some $G \in \mathcal{G}$ and such that $E(H) = \bigcup_{i=1}^{r} E(G_i)$. Of particular importance is when $\mathcal{G} = \{G\}$, in which case we write "G-decomposition of H" instead of "$\{G\}$-decomposition of H." A G-decomposition of K_n is also known as a *(K_n, G)-design*. The set of all n for which K_n admits a G-decomposition is called the *spectrum of G*. The spectrum has been determined for many classes of graphs, including for all graphs on at most 4 vertices [4] and all graphs on 5 vertices (see [3] and [9]). We direct the reader to [2] and [5] for recent surveys on graph decompositions.

Similar concepts to the ones defined above for undirected graphs can be defined for digraphs. First, we introduce additional notation. For an undirected graph G, let G^* denote the digraph obtained from G by replacing each edge $\{u, v\} \in E(G)$ with the arcs (u, v) and (v, u). Thus K_n^*, the *complete digraph of order n*, is the digraph on n vertices with the arcs (u, v) and (v, u) between every pair of distinct vertices u and v.

Let H and D be digraphs such that D is a subgraph of H. A *D-decomposition* of H is a set $\Delta = \{D_1, D_2, \ldots, D_r\}$ of pairwise arc-disjoint subgraphs of H each of which is isomorphic to D and such that $E(H) = \bigcup_{i=1}^{r} E(D_i)$. As with the undirected case, a D-decomposition of K_n^* is also known as a *(K_n^*, D)-design*, and the set of all n for which K_n^* admits a D-decomposition is called the *spectrum of D*.

The spectra for several digraphs of small order have been determined. This includes the spectra for all digraphs on at most 3 vertices [10] and all bipartite digraphs on 4 vertices with up to 5 arcs [6].

In this paper, we extend the known results on small digraphs by determining the spectrum for each of the 8 digraphs obtained by orienting the edges of a triangle with a pendant edge. We use the naming convention found in *An Atlas of Graphs* [11] by Read and Wilson. The digraphs under investigation are shown in Figure 1 where a key that denotes a labeled copy for each of the 8 digraphs of interest is given. For example, D57$[w, x, y, z]$ refers to the digraph with vertex set $\{w, x, y, z\}$ and arc set $\big\{(w, x), (x, y), (z, y), (x, z)\big\}$.

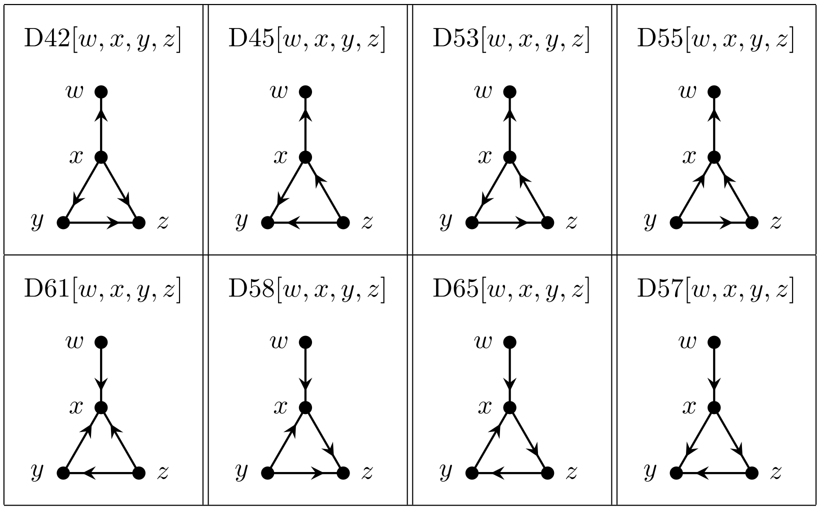

Figure 1: The 8 digraphs obtained by orienting the edges of a triangle with a pendant edge.

2 Some Basic Results

The obvious necessary conditions for a digraph D to decompose K_n^* are

(A) $|V(D)| \leq n$,
(B) $|E(D)|$ divides $n(n-1)$, and
(C) both $\gcd\{\text{outdegree}(v)\colon\ v \in V(D)\}$ and $\gcd\{\text{indegree}(v)\colon\ v \in V(D)\}$ divide $n-1$.

Applying these necessary conditions to the 8 digraphs under consideration, we obtain the following necessary conditions. For $D \in \{\text{D42}, \text{D53}, \text{D55}, \text{D57}, \text{D61}, \text{D65}\}$, if a (K_n^*, D)-design exists, then $n \geq 4$ and $n \equiv 0$ or $1 \pmod 4$. For $D \in \{\text{D45}, \text{D58}\}$, if a (K_n^*, D)-design exists, then $n \geq 5$ and $n \equiv 1 \pmod 4$.

Given a digraph D, the *reverse orientation of* D, denoted $\operatorname{Rev} D$, is the digraph with vertex set $V(D)$ and arc set $\{(v,u)\colon (u,v) \in E(D)\}$. We make the following straightforward and useful observation:

Observation 1. Let D and H be digraphs. A D-decomposition of H exists if and only if a $(\operatorname{Rev} D)$-decomposition of $\operatorname{Rev} H$ exists.

The fact that $K_n^* \cong \operatorname{Rev}(K_n^*)$ leads to the following corollary.

Corollary 2. *Let D be a digraph. A (K_n^*, D)-design exists if and only if a $(K_n^*, \operatorname{Rev} D)$-design exists.*

Note that the 8 digraphs of interest in this paper occur in pairs with respect to their reverse orientations (see Figure 1). Namely, D42 $\cong$ Rev(D61), D45 $\cong$ Rev(D58), D53 $\cong$ Rev(D65), and D55 $\cong$ Rev(D57).

The following theorems on decompositions of complete graphs and complete multipartite graphs are fundamental to our main results. Note that these background results concern graphs, as opposed to digraphs. All of these results can be found in the *Handbook of Combinatorial Designs* [7] (see [1] and [8]).

Theorem 3. *A K_3-decomposition of K_n exists if and only if $n \equiv 1$ or $3 \pmod 6$.*

Theorem 4. *A $\{K_3, K_4\}$-decomposition of K_n exists if and only if $n \equiv 0$ or $1 \pmod 3$ and $n \neq 6$.*

Theorem 5. *If $n \equiv 5 \pmod 6$, then a $\{K_3, K_5\}$-decomposition of K_n exists.*

Theorem 6. *The necessary and sufficient conditions for the existence of a K_3-decomposition of $K_{u\times m}$ are (i) $u \geq 3$, (ii) $(u-1)m \equiv 0 \pmod 2$, and (iii) $u(u-1)m^2 \equiv 0 \pmod 6$.*

Theorem 7. *The necessary and sufficient conditions for the existence of a K_4-decomposition of $K_{u\times m}$ are (i) $u \geq 4$, (ii) $(u-1)m \equiv 0 \pmod 3$, and (iii) $u(u-1)m^2 \equiv 0 \pmod{12}$, with the exception of $(u, m) \in \{(4, 2), (4, 6)\}$, in which case no such decomposition exists.*

Let H be a graph and let t be a positive integer. Let $H[t]$ denote the graph obtained from H by replacing each vertex $a \in V(H)$ with $\{a_1, a_2, \ldots, a_t\}$ and replacing each edge $\{a, b\} \in E(H)$ with the complete bipartite graph $K_{t,t}$ with edge set $\{\{a_i, b_j\} \colon 1 \leq i, j \leq t\}$. This is often referred to as the *composition*, or *lexicographic product*, of H and $\overline{K}_t$ (the complement of K_t). We make use of the following straightforward observation.

Observation 8. If there exists a G-decomposition of H, then there exists a $G[t]$-decomposition of $H[t]$ for any positive integer t.

Since $K_r \cong K_{r\times 1}$ and $K_{u\times m}[t] \cong K_{u\times mt}$, we have the following corollary.

Corollary 9. *If there exists a $(K_{u\times m}, K_r)$-design, then there exists a $(K_{u\times mt}, K_{r\times t})$-design for any positive integer t.*

3 Examples of Small Designs

We now turn our attention to the designs of small order which will be used for the general constructions.

Given a digraph represented by the notation $D[a, b, c, d]$ and some $i \in \mathbb{Z}_n$, we define $D[a, b, c, d] + i = D[a + i, b + i, c + i, d + i]$ where all addition is performed in $\mathbb{Z}_n$. By convention, define $\infty + 1 = \infty$.

Example 1. *There exists a* (K_4^*, D)*-design for* $D \in \{\mathrm{D42}, \mathrm{D53}, \mathrm{D61}, \mathrm{D65}\}$.

Let $V(K_4^*) = \mathbb{Z}_3 \cup \{\infty\}$.
A $(K_4^*, \mathrm{D61})$-design is given by $\{\mathrm{D61}[1, 2, \infty, 0] + i \colon i \in \mathbb{Z}_3\}$.
A $(K_4^*, \mathrm{D65})$-design is given by $\{\mathrm{D65}[1, 2, , 0, \infty] + i \colon i \in \mathbb{Z}_3\}$.
Applying Corollary 2, we obtain the remaining designs.

Example 2. *There exists a* (K_5^*, D)*-design for* $D \in \{\mathrm{D42}, \mathrm{D45}, \mathrm{D53}, \mathrm{D55}, \mathrm{D57}, \mathrm{D58}, \mathrm{D61}, \mathrm{D65}\}$.

Let $V(K_5^*) = \mathbb{Z}_5$.
A $(K_5^*, \mathrm{D57})$-design is given by $\{\mathrm{D57}[2, 3, 0, 1] + i \colon i \in \mathbb{Z}_5\}$.
A $(K_5^*, \mathrm{D58})$-design is given by $\{\mathrm{D58}[1, 3, 0, 4] + i \colon i \in \mathbb{Z}_5\}$.
A $(K_5^*, \mathrm{D61})$-design is given by $\{\mathrm{D61}[3, 1, 2, 0] + i \colon i \in \mathbb{Z}_5\}$.
A $(K_5^*, \mathrm{D65})$-design is given by

$$\{\mathrm{D65}[1, 4, 3, 2], \mathrm{D65}[4, 1, 3, 0], \mathrm{D65}[0, 2, 3, 1], \mathrm{D65}[4, 0, 2, 1], \mathrm{D65}[2, 4, 0, 3]\}.$$

Applying Corollary 2, we obtain the remaining designs.

Example 3. *There exists a* (K_8^*, D)*-design for* $D \in \{\mathrm{D42}, \mathrm{D53}, \mathrm{D55}, \mathrm{D57}, \mathrm{D61}, \mathrm{D65}\}$.

Let $V(K_8^*) = \mathbb{Z}_7 \cup \{\infty\}$.
A $(K_8^*, \mathrm{D57})$-design is given by

$$\{\mathrm{D57}[3, 0, 6, \infty] + i \colon i \in \mathbb{Z}_7\} \cup \{\mathrm{D57}[2, 0, 3, 1] + i \colon i \in \mathbb{Z}_7\}.$$

A $(K_8^*, \mathrm{D61})$-design is given by

$$\{\mathrm{D61}[0, 2, \infty, 6] + i \colon i \in \mathbb{Z}_7\} \cup \{\mathrm{D61}[0, 4, 3, 5] + i \colon i \in \mathbb{Z}_7\}.$$

A $(K_8^*, \mathrm{D65})$-design is given by

$$\{\mathrm{D65}[1, 6, 0, \infty] + i \colon i \in \mathbb{Z}_7\} \cup \{\mathrm{D65}[1, 4, 0, 6] + i \colon i \in \mathbb{Z}_7\}.$$

Applying Corollary 2, we obtain the remaining designs.

Example 4. *There exists a* (K_9^*, D)*-design for* $D \in \{\mathrm{D42}, \mathrm{D45}, \mathrm{D53}, \mathrm{D55}, \mathrm{D57}, \mathrm{D58}, \mathrm{D61}, \mathrm{D65}\}$.

Let $V(K_9^*) = \mathbb{Z}_9$.
A $(K_9^*, \mathrm{D57})$-design is given by

$$\{\mathrm{D57}[2,3,0,5] + i : i \in \mathbb{Z}_9\} \cup \{\mathrm{D57}[7,6,0,4] + i : i \in \mathbb{Z}_9\}.$$

A $(K_9^*, \mathrm{D58})$-design is given by

$$\{\mathrm{D58}[2,3,4,0] + i : i \in \mathbb{Z}_9\} \cup \{\mathrm{D58}[8,2,7,0] + i : i \in \mathbb{Z}_9\}.$$

A $(K_9^*, \mathrm{D61})$-design is given by

$$\{\mathrm{D61}[3,6,2,0] + i : i \in \mathbb{Z}_9\} \cup \{\mathrm{D61}[3,8,1,0] + i : i \in \mathbb{Z}_9\}.$$

A $(K_9^*, \mathrm{D65})$-design is given by

$$\{\mathrm{D65}[1,3,0,8] + i : i \in \mathbb{Z}_9\} \cup \{\mathrm{D65}[1,8,0,3] + i : i \in \mathbb{Z}_9\}.$$

Applying Corollary 2, we obtain the remaining designs.

Example 5. *There exists a* (K_{12}^*, D)*-design for* $D \in \{\mathrm{D55}, \mathrm{D57}\}$.

Let $V(K_{12}^*) = \mathbb{Z}_{11} \cup \{\infty\}$.
A $(K_{12}^*, \mathrm{D57})$-design is given by

$$\{\mathrm{D57}[2,0,4,\infty] + i : i \in \mathbb{Z}_{11}\} \cup \{\mathrm{D57}[6,0,7,1] + i : i \in \mathbb{Z}_{11}\} \cup \{\mathrm{D57}[8,0,10,2] + i : i \in \mathbb{Z}_{11}\}.$$

Applying Corollary 2, we obtain the remaining designs.

Example 6. *There exists a* (K_{24}^*, D)*-design for* $D \in \{\mathrm{D42}, \mathrm{D53}, \mathrm{D55}, \mathrm{D57}, \mathrm{D61}, \mathrm{D65}\}$.

Let $V(K_{24}^*) = \mathbb{Z}_{23} \cup \{\infty\}$.
A $(K_{24}^*, \mathrm{D57})$-design is given by

$$\{\mathrm{D57}[7,0,17,\infty] + i : i \in \mathbb{Z}_{23}\} \cup \{\mathrm{D57}[8,0,1,2] + i : i \in \mathbb{Z}_{23}\} \cup \{\mathrm{D57}[9,0,3,5] + i : i \in \mathbb{Z}_{23}\} \cup \{\mathrm{D57}[11,0,4,7] + i : i \in \mathbb{Z}_{23}\} \cup \{\mathrm{D57}[12,0,6,10] + i : i \in \mathbb{Z}_{23}\} \cup \{\mathrm{D57}[14,0,8,13] + i : i \in \mathbb{Z}_{23}\}.$$

A $(K_{24}^*, \mathrm{D61})$-design is given by

$$\{\mathrm{D61}[0,6,\infty,7] + i : i \in \mathbb{Z}_{23}\} \cup \{\mathrm{D61}[11,9,8,0] + i : i \in \mathbb{Z}_{23}\} \cup \{\mathrm{D61}[16,12,10,0] + i : i \in \mathbb{Z}_{23}\} \cup \{\mathrm{D61}[19,14,11,0] + i : i \in \mathbb{Z}_{23}\} \cup \{\mathrm{D61}[1,17,13,0] + i : i \in \mathbb{Z}_{23}\} \cup \{\mathrm{D61}[13,20,15,0] + i : i \in \mathbb{Z}_{23}\}.$$

A $(K^*_{24}, \mathrm{D65})$-design is given by

$$\begin{aligned}&\{\mathrm{D65}[22,1,0,\infty]+i\colon i\in\mathbb{Z}_{23}\}\cup\{\mathrm{D65}[14,22,0,20]+i\colon i\in\mathbb{Z}_{23}\}\\ \cup\,&\{\mathrm{D65}[7,20,0,16]+i\colon i\in\mathbb{Z}_{23}\}\cup\{\mathrm{D65}[4,18,0,12]+i\colon i\in\mathbb{Z}_{23}\}\\ \cup\,&\{\mathrm{D65}[17,9,0,19]+i\colon i\in\mathbb{Z}_{23}\}\cup\{\mathrm{D65}[13,6,0,18]+i\colon i\in\mathbb{Z}_{23}\}.\end{aligned}$$

Applying Corollary 2, we obtain the remaining designs.

Example 7. *There exists a* (K^*_{25}, D)*-design for* $D \in \{\mathrm{D42}, \mathrm{D45}, \mathrm{D53}, \mathrm{D55}, \mathrm{D57}, \mathrm{D58}, \mathrm{D61}, \mathrm{D65}\}$.

Let $D \in \{\mathrm{D42}, \mathrm{D45}, \mathrm{D53}, \mathrm{D55}, \mathrm{D57}, \mathrm{D58}, \mathrm{D61}, \mathrm{D65}\}$. A (K_{25}, K_5)-design can be obtained from an affine plane of order 5. Thus, there exists a (K^*_{25}, K^*_5)-design. Since a (K^*_5, D)-design exists by Example 2, the desired (K^*_{25}, D)-design exists.

Example 8. *There exist* $(K^*_{28}, \mathrm{D55})$*- and* $(K^*_{28}, \mathrm{D57})$*-designs.*

Let $V(K^*_{28}) = \mathbb{Z}_{27} \cup \{\infty\}$.
A $(K^*_{28}, \mathrm{D57})$-design is given by

$$\begin{aligned}&\{\mathrm{D57}[19,0,7,\infty]+i\colon i\in\mathbb{Z}_{27}\}\cup\{\mathrm{D57}[14,0,21,17]+i\colon i\in\mathbb{Z}_{27}\}\\ \cup\,&\{\mathrm{D57}[18,0,26,25]+i\colon i\in\mathbb{Z}_{27}\}\cup\{\mathrm{D57}[17,0,24,22]+i\colon i\in\mathbb{Z}_{27}\}\\ \cup\,&\{\mathrm{D57}[12,0,19,14]+i\colon i\in\mathbb{Z}_{27}\}\cup\{\mathrm{D57}[16,0,23,20]+i\colon i\in\mathbb{Z}_{27}\}\\ &\cup\{\mathrm{D57}[11,0,18,12]+i\colon i\in\mathbb{Z}_{27}\}.\end{aligned}$$

Applying Corollary 2, we obtain a $(K^*_{28}, \mathrm{D55})$-design.

Example 9. *There exists a* $(K^*_{3\times 2}, D)$*-design for* $D \in \{\mathrm{D55}, \mathrm{D57}\}$.

Let $V(K^*_{3\times 2}) = \mathbb{Z}_6$ with vertex partition $\{\{0,3\},\{1,4\},\{2,5\}\}$.
A $(K^*_{3\times 2}, \mathrm{D57})$-design is given by

$$\{\mathrm{D57}[4,0,5,1]+i\colon i\in\mathbb{Z}_6\}.$$

Applying Corollary 2, we obtain a $(K^*_{3\times 2}, \mathrm{D55})$-design.

Example 10. *There exists a* $(K^*_{3\times 4}, D)$*-design for* $D \in \{\mathrm{D42}, \mathrm{D45}, \mathrm{D53}, \mathrm{D55}, \mathrm{D57}, \mathrm{D58}, \mathrm{D61}, \mathrm{D65}\}$.

Let $V(K^*_{3\times 4}) = \mathbb{Z}_{12}$ with vertex partition $\{V_i\colon i \in \mathbb{Z}_3\}$, where $V_i = \{j \in \mathbb{Z}_{12}\colon j \equiv i \pmod 3\}$.
A $(K^*_{3\times 4}, \mathrm{D57})$-design is given by

$$\{\mathrm{D57}[2,0,8,1]+i\colon i\in\mathbb{Z}_{12}\}\cup\{\mathrm{D57}[10,0,4,5]+i\colon i\in\mathbb{Z}_{12}\}.$$

A $(K^*_{3\times 4}, \mathrm{D58})$-design is given by

$$\{\mathrm{D58}[9,2,0,10]+i\colon i\in\mathbb{Z}_{12}\}\cup\{\mathrm{D58}[3,4,0,11]+i\colon i\in\mathbb{Z}_{12}\}.$$

A $(K^*_{3\times 4}, \mathrm{D61})$-design is given by

$$\{\mathrm{D61}[9,10,2,0]+i\colon i\in\mathbb{Z}_{12}\}\cup\{\mathrm{D61}[6,11,4,0]+i\colon i\in\mathbb{Z}_{12}\}.$$

A $(K^*_{3\times 4}, \mathrm{D65})$-design is given by

$$\{\mathrm{D65}[10,0,5,1]+i\colon i\in\mathbb{Z}_{12}\}\cup\{\mathrm{D65}[2,0,1,5]+i\colon i\in\mathbb{Z}_{12}\}.$$

Applying Corollary 2, we obtain the remaining designs.

Example 11. *There exists a* $(K^*_{4\times 4}, D)$*-design for* $D\in\{\mathrm{D42},\mathrm{D45},\mathrm{D53},\mathrm{D55},\mathrm{D57},\mathrm{D58},\mathrm{D61},\mathrm{D65}\}$.

First, let $D\in\{\mathrm{D42},\mathrm{D53},\mathrm{D61},\mathrm{D65}\}$. A $(K_{4\times 4}, K_4)$-design can be obtained by removing one parallel class from an affine plane of order 4. Thus, there exists a $(K^*_{4\times 4}, K^*_4)$-design. Since a (K^*_4, D)-design exists by Example 1, the desired $(K^*_{4\times 4}, D)$-design exists.

Now, let $D\in\{\mathrm{D45},\mathrm{D55},\mathrm{D57},\mathrm{D58}\}$. Let $V(K^*_{4\times 4})=\mathbb{Z}_{16}$ with vertex partition $\{V_i\colon i\in\mathbb{Z}_4\}$, where $V_i=\{j\in\mathbb{Z}_{16}\colon j\equiv i \pmod 4\}$.
A $(K^*_{4\times 4}, \mathrm{D57})$-design is given by

$$\{\mathrm{D57}[5,0,6,1]+i\colon i\in\mathbb{Z}_{16}\}\cup\{\mathrm{D57}[1,0,9,2]+i\colon i\in\mathbb{Z}_{16}\}\\ \cup\{\mathrm{D57}[2,0,13,3]+i\colon i\in\mathbb{Z}_{16}\}.$$

A $(K^*_{4\times 4}, \mathrm{D58})$-design is given by

$$\{\mathrm{D58}[1,6,0,7]+i\colon i\in\mathbb{Z}_{16}\}\cup\{\mathrm{D58}[10,9,0,11]+i\colon i\in\mathbb{Z}_{16}\}\\ \cup\{\mathrm{D58}[12,10,0,13]+i\colon i\in\mathbb{Z}_{16}\}.$$

Applying Corollary 2, we obtain the remaining designs.

Example 12. *There exists a* $(K^*_{5\times 4}, D)$*-design for* $D\in\{\mathrm{D42},\mathrm{D45},\mathrm{D53},\mathrm{D55},\mathrm{D57},\mathrm{D58},\mathrm{D61},\mathrm{D65}\}$.

Let $V(K^*_{5\times 4})=\mathbb{Z}_{20}$ with vertex partition $\{V_i\colon i\in\mathbb{Z}_5\}$, where $V_i=\{j\in\mathbb{Z}_{20}\colon j\equiv i \pmod 5\}$.
A $(K^*_{5\times 4}, \mathrm{D57})$-design is given by

$$\{\mathrm{D57}[3,0,7,6]+i\colon i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D57}[6,0,11,9]+i\colon i\in\mathbb{Z}_{20}\}\\ \cup\{\mathrm{D57}[2,0,16,13]+i\colon i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D57}[1,0,12,8]+i\colon i\in\mathbb{Z}_{20}\}.$$

A $(K^*_{5\times 4}, \mathrm{D58})$-design is given by

$$\{\mathrm{D58}[17,6,0,7]+i\colon i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D58}[14,13,0,17]+i\colon i\in\mathbb{Z}_{20}\}$$
$$\cup\{\mathrm{D58}[16,12,0,14]+i\colon i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D58}[10,8,0,11]+i\colon i\in\mathbb{Z}_{20}\}.$$

A $(K^*_{5\times 4}, \mathrm{D61})$-design is given by

$$\{\mathrm{D61}[19,7,6,0]+i\colon i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D61}[19,11,9,0]+i\colon i\in\mathbb{Z}_{20}\}$$
$$\cup\{\mathrm{D61}[19,16,13,0]+i\colon i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D61}[19,18,14,0]+i\colon i\in\mathbb{Z}_{20}\}.$$

A $(K^*_{5\times 4}, \mathrm{D65})$-design is given by

$$\{\mathrm{D65}[12,0,17,19]+i\colon i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D65}[11,0,13,17]+i\colon i\in\mathbb{Z}_{20}\}$$
$$\cup\{\mathrm{D65}[9,0,19,13]+i\colon i\in\mathbb{Z}_{20}\}\cup\{\mathrm{D65}[8,0,18,14]+i\colon i\in\mathbb{Z}_{20}\}.$$

Applying Corollary 2, we obtain the remaining designs.

Example 13. *There exists a $(K^*_{3\times 8}, D)$-design for $D\in\{\mathrm{D42},\mathrm{D45},\mathrm{D53},\mathrm{D55},\mathrm{D57},\mathrm{D58},\mathrm{D61},\mathrm{D65}\}$.*

Let $D\in\{\mathrm{D42},\mathrm{D45},\mathrm{D53},\mathrm{D55},\mathrm{D57},\mathrm{D58},\mathrm{D61},\mathrm{D65}\}$. By Theorem 6, there exists a $(K_{3\times 2}, K_3)$-design. Furthermore, by Corollary 9, there exists a $(K_{3\times 8}, K_{3\times 4})$-design. Thus, a $(K^*_{3\times 8}, K^*_{3\times 4})$-design exists. Since a $(K^*_{3\times 4}, D)$-design exists by Example 10, the desired $(K^*_{3\times 8}, D)$-design exists.

Example 14. *There exists a $(K^*_{4\times 8}, D)$-design for $D\in\{\mathrm{D42},\mathrm{D45},\mathrm{D53},\mathrm{D55},\mathrm{D57},\mathrm{D58},\mathrm{D61},\mathrm{D65}\}$.*

First, let $D\in\{\mathrm{D42},\mathrm{D53},\mathrm{D61},\mathrm{D65}\}$. By Theorem 7, there exists a $(K_{4\times 8}, K_4)$-design. Thus, there exists a $(K^*_{4\times 8}, K^*_4)$-design. Furthermore, since a (K^*_4, D)-design exists by Example 1, the desired $(K^*_{4\times 8}, D)$-design exists.

Now, let $D\in\{\mathrm{D45},\mathrm{D55},\mathrm{D57},\mathrm{D58}\}$. Let $V(K^*_{4\times 8})=\mathbb{Z}_{32}$ with vertex partition $\{V_i\colon i\in\mathbb{Z}_4\}$, where $V_i=\{j\in\mathbb{Z}_{32}\colon j\equiv i \pmod 4\}$.
A $(K^*_{4\times 8}, \mathrm{D57})$-design is given by

$$\{\mathrm{D57}[23,0,31,30]+i\colon i\in\mathbb{Z}_{32}\}\cup\{\mathrm{D57}[22,0,29,27]+i\colon i\in\mathbb{Z}_{32}\}$$
$$\cup\{\mathrm{D57}[21,0,26,23]+i\colon i\in\mathbb{Z}_{32}\}\cup\{\mathrm{D57}[14,0,25,19]+i\colon i\in\mathbb{Z}_{32}\}$$
$$\cup\{\mathrm{D57}[19,0,22,17]+i\colon i\in\mathbb{Z}_{32}\}\cup\{\mathrm{D57}[17,0,21,14]+i\colon i\in\mathbb{Z}_{32}\}.$$

A $(K^*_{4\times 8}, \mathrm{D58})$-design is given by

$$\{\mathrm{D58}[12,30,0,31]+i\colon i\in\mathbb{Z}_{32}\}\cup\{\mathrm{D58}[12,27,0,29]+i\colon i\in\mathbb{Z}_{32}\}$$
$$\cup\{\mathrm{D58}[10,23,0,26]+i\colon i\in\mathbb{Z}_{32}\}\cup\{\mathrm{D58}[8,19,0,25]+i\colon i\in\mathbb{Z}_{32}\}$$
$$\cup\{\mathrm{D58}[7,17,0,22]+i\colon i\in\mathbb{Z}_{32}\}\cup\{\mathrm{D58}[5,14,0,21]+i\colon i\in\mathbb{Z}_{32}\}.$$

Applying Corollary 2, we obtain the remaining designs.

Example 15. *There exists a* $((K_8 \setminus K_4)^*, D)$*-design for* $D \in \{\mathrm{D55}, \mathrm{D57}\}$.

Let $V((K_8 \setminus K_4)^*) = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup \mathbb{Z}_4$, where the vertices in the hole are ∞_1, ∞_2, ∞_3, and ∞_4.
A $(K_8 \setminus K_4)^*, \mathrm{D57})$-design is given by

$$\{\mathrm{D57}[\infty_1, 0, 1, \infty_2], \mathrm{D57}[2, 3, \infty_3, 0], \mathrm{D57}[\infty_1, 1, \infty_4, 0], \mathrm{D57}[\infty_2, 3, \infty_1, 1],$$
$$\mathrm{D57}[\infty_2, 0, \infty_1, 2], \mathrm{D57}[1, \infty_3, 0, 2], \mathrm{D57}[\infty_1, 3, 2, \infty_2], \mathrm{D57}[\infty_1, 2, \infty_2, 1],$$
$$\mathrm{D57}[2, \infty_3, 3, 1], \mathrm{D57}[2, \infty_4, 3, 0], \mathrm{D57}[3, \infty_4, 2, 1]\}.$$

Applying Corollary 2, we obtain a $((K_8 \setminus K_4)^*, \mathrm{D55})$-design.

4 Main Results

We finally address the general constructions needed to piece together the small designs mentioned above to show (near) sufficiency of the necessary conditions.

Theorem 10. *If* $n \equiv 1 \pmod 4$ *and* $n \geq 5$*, then a* (K_n^*, D)*-design exists for* $D \in \{\mathrm{D42}, \mathrm{D45}, \mathrm{D53}, \mathrm{D55}, \mathrm{D57}, \mathrm{D58}, \mathrm{D61}, \mathrm{D65}\}$.

Proof. Let $D \in \{\mathrm{D42}, \mathrm{D45}, \mathrm{D53}, \mathrm{D55}, \mathrm{D57}, \mathrm{D58}, \mathrm{D61}, \mathrm{D65}\}$ and let $n = 4x+1$ for some positive integer x. When x is 1, 2, or 6, the result follows from Examples 2, 4, and 7, respectively. The remainder of the proof breaks into three cases.

First, suppose that $x \equiv 0$ or $1 \pmod 3$ with $x \geq 3$ and $x \neq 6$. By Theorem 4 there exists a $\{K_3, K_4\}$-decomposition of K_x. Thus, by Corollary 9, there exists a $\{K_{3\times 4}, K_{4\times 4}\}$-decomposition of $K_{x\times 4}$. Note that $K_{4x+1} = K_{x\times 4} \cup (xK_4 \vee K_1) = K_{x\times 4} \cup \bigcup_{i=1}^{x} K_5$. Thus, $K_n^* = K_{x\times 4}^* \cup \bigcup_{i=1}^{x} K_5^*$. Since there exists a $(K_{3\times 4}^*, D)$-design (by Example 10) and there exists a $(K_{4\times 4}^*, D)$-design (by Example 11), there exists a $(K_{x\times 4}^*, D)$-design. Since there also exists a (K_5^*, D)-design (by Example 2), there exists a (K_n^*, D)-design.

Second, suppose that $x \equiv 2 \pmod 6$ and let $x = 6z+2$ for some integer $z \geq 1$. Hence, $n = 8(3z+1)+1$. By Theorem 4 there exists a $\{K_3, K_4\}$-decomposition of K_{3z+1}. Thus, by Corollary 9, there exists a $\{K_{3\times 8}, K_{4\times 8}\}$-decomposition of $K_{(3z+1)\times 8}$. Note that $K_{24z+9} = K_{(3z+1)\times 8} \cup ((3z+1)K_8 \vee K_1) = K_{(3z+1)\times 8} \cup \bigcup_{i=1}^{3z+1} K_9$. Thus, $K_n^* = K_{(3z+1)\times 8}^* \cup \bigcup_{i=1}^{3z+1} K_9^*$. Since there exists a $(K_{3\times 8}^*, D)$-design (by Example 13) and there exists a $(K_{4\times 8}^*, D)$-design (by Example 14), there exists a $(K_{(3z+1)\times 8}^*, D)$-design. Since there also exists a (K_9^*, D)-design (by Example 4), there exists a (K_n^*, D)-design.

Finally, suppose $x \equiv 5 \pmod 6$ and let $x = 6z + 5$ for some integer $z \geq 0$. Hence, $n = 4(6z + 5) + 1$. By Theorem 5 there exists a $\{K_3, K_5\}$-decomposition of K_{6z+5}. Thus, by Corollary 9, there exists a $\{K_{3\times 4}, K_{5\times 4}\}$-decomposition of $K_{(6z+5)\times 4}$. Note that $K_{24z+21} = K_{(6z+5)\times 4} \cup ((6z + 5)K_4 \vee K_1) = K_{(6z+5)\times 4} \cup \bigcup_{i=1}^{6z+5} K_5$. Thus, $K_n^* = K^*_{(6z+5)\times 4} \cup \bigcup_{i=1}^{6z+5} K_5^*$. Since there exists a $(K^*_{3\times 4}, D)$-design (by Example 10) and there exists a $(K^*_{5\times 4}, D)$-design (by Example 12), there exists a $(K^*_{(6z+5)\times 4}, D)$-design. Since there also exists a (K_5^*, D)-design (by Example 2), there exists a (K_n^*, D)-design. ■

Theorem 11. *If* $n \equiv 0 \pmod 4$ *with* $n \geq 4$*, then a* (K_n^*, D)*-design exists for* $D \in \{\mathrm{D42}, \mathrm{D53}, \mathrm{D61}, \mathrm{D65}\}$.

Proof. Let $D \in \{\mathrm{D42}, \mathrm{D53}, \mathrm{D61}, \mathrm{D65}\}$ and let $n = 4x$ for some positive integer x. When x is 1, 2, or 6, the result follows from Examples 1, 3, and 6, respectively. The remainder of the proof breaks into three cases.

First, suppose that $x \equiv 0$ or $1 \pmod 3$ with $x \geq 3$ and $x \neq 6$. By Theorem 4 there exists a $\{K_3, K_4\}$-decomposition of K_x. Thus, by Corollary 9, there exists a $\{K_{3\times 4}, K_{4\times 4}\}$-decomposition of $K_{x\times 4}$. Note that $K_{4x} = K_{x\times 4} \cup xK_4$. Thus, $K_n^* = K^*_{x\times 4} \cup \bigcup_{i=1}^{x} K_4^*$. Since there exists a $(K^*_{3\times 4}, D)$-design (by Example 10) and there exists a $(K^*_{4\times 4}, D)$-design (by Example 11), there exists a $(K^*_{x\times 4}, D)$-design. Since there also exists a (K_4^*, D)-design (by Example 1), there exists a (K_n^*, D)-design.

Second, suppose that $x \equiv 2 \pmod 6$ and let $x = 6z+2$ for some integer $z \geq 1$. Hence, $n = 8(3z + 1)$. By Theorem 4 there exists a $\{K_3, K_4\}$-decomposition of K_{3z+1}. Thus, by Corollary 9, there exists a $\{K_{3\times 8}, K_{4\times 8}\}$-decomposition of $K_{(3z+1)\times 8}$. Note that $K_{24z+8} = K_{(3z+1)\times 8} \cup (3z + 1)K_8$. Thus, $K_n^* = K^*_{(3z+1)\times 8} \cup \bigcup_{i=1}^{3z+1} K_8^*$. Since there exists a $(K^*_{3\times 8}, D)$-design (by Example 13) and there exists a $(K^*_{4\times 8}, D)$-design (by Example 14), there exists a $(K^*_{(3z+1)\times 8}, D)$-design. Since there also exists a (K_8^*, D)-design (by Example 3), there exists a (K_n^*, D)-design.

Finally, suppose $x \equiv 5 \pmod 6$ and let $x = 6z + 5$ for some integer $z \geq 0$. Hence, $n = 4(6z + 5)$. By Theorem 5 there exists a $\{K_3, K_5\}$-decomposition of K_{6z+5}. Thus, by Corollary 9, there exists a $\{K_{3\times 4}, K_{5\times 4}\}$-decomposition of $K_{(6z+5)\times 4}$. Note that $K_{24z+20} = K_{(6z+5)\times 4} \cup (6z + 5)K_4$. Thus, $K_n^* = K^*_{(6z+5)\times 4} \cup \bigcup_{i=1}^{6z+5} K_4^*$. Since there exists a $(K^*_{3\times 4}, D)$-design (by Example 10) and there exists a $(K^*_{5\times 4}, D)$-design (by Example 12), there exists a $(K^*_{(6z+5)\times 4}, D)$-design. Since there also exists a (K_4^*, D)-design (by Example 1), there exists a (K_n^*, D)-design. ■

Theorem 12. *If* $n \equiv 0 \pmod 4$ *with* $n \geq 8$*, then a* (K_n^*, D)*-design exists for* $D \in \{\mathrm{D55}, \mathrm{D57}\}$.

Proof. It is easily verified that no $(K_4^*, \mathrm{D55})$- or $(K_4^*, \mathrm{D57})$-design exists. Let $n = 4x$ for some integer $x \geq 2$. When x is 2, 3, 6, or 7, the result follows from Examples 3, 5, 6, and 8, respectively. The remainder of the proof breaks into two cases.

First, suppose that $x \equiv 1$ or $2 \pmod 3$ with $x \geq 4$ and $x \neq 7$. By Theorem 4, there exists a $\{K_3, K_4\}$-decomposition of K_{x-1}. Thus, by Corollary 9, there exists a $\{K_{3\times 4}, K_{4\times 4}\}$-decomposition of $K_{(x-1)\times 4}$. Let M be the complete multipartite graph with parts of size 4 and vertex partition $\{V_i\colon i \in [1, x-1]\}$. Let G be the complete graph of order 8 with vertex set $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup V_1$. For $i \in [2, x-1]$, let H_i be the complete graph of order 8 with a hole of size 4 and with vertex set $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup V_i$, where ∞_1, ∞_2, ∞_3, and ∞_4 are the vertices in the hole. Observe that $K_n^* \cong M \cup G \cup H_2 \cup H_3 \cup \cdots \cup H_{x-1}$. Since there exist a $\{K_{3\times 4}, K_{4\times 4}\}$-decomposition of M and a D-decomposition of both $K_{3\times 4}^*$ and of $K_{4\times 4}^*$ (by Examples 10 and 11), there exists a D-decomposition of M^*. Since there also exist D-decompositions of G^* (by Example 3) and of H_i^* for $i \in [2, x-1]$ (by Example 15), there exists a D-decomposition of K_n^*.

Second, suppose that $x \equiv 0 \pmod 3$ and let $x = 3z$ for some integer $z \geq 3$. Hence, $n = 2(6z)$. By Theorem 6, there exists a K_3-decomposition of $K_{z\times 6}$. Thus, by Corollary 9, there exists a $K_{3\times 2}$-decomposition of $K_{z\times 12}$. Let M be the complete multipartite graph with parts of size 12 and vertex partition $\{V_i\colon i \in [1, z]\}$. For $i \in [1, z]$, let G_i denote the complete graph of order 12 with vertex set V_i. Observe that $K_n^* \cong M \cup G_1 \cup G_2 \cup \cdots \cup G_z$. Since there exist a $K_{3\times 2}$-decomposition of M and a D-decomposition of $K_{3\times 2}^*$ (by Example 9), there exists a D-decomposition of M^*. Since there also exists a D-decomposition of G_i^* for $i \in [1, z]$ (by Example 5), there exists a D-decomposition of K_n^*. ■

We combine the results form the previous 3 theorems to give necessary and sufficient conditions for the existence of a D-decomposition of K_n^* for $D \in \{\mathrm{D42}, \mathrm{D45}, \mathrm{D53}, \mathrm{D55}, \mathrm{D57}, \mathrm{D58}, \mathrm{D61}, \mathrm{D65}\}$.

Theorem 13. *For $D \in \{\mathrm{D42}, \mathrm{D53}, \mathrm{D55}, \mathrm{D57}, \mathrm{D61}, \mathrm{D65}\}$, there exists a (K_n^*, D)-design if and only if $n \equiv 0$ or $1 \pmod 4$ and $n \geq 4$ with the exception that there does not exist a $(K_4^*, \mathrm{D55})$- or $(K_4^*, \mathrm{D57})$-design. For $D \in \{\mathrm{D45}, \mathrm{D58}\}$, there exists a (K_n^*, D)-design if and only if $n \equiv 1 \pmod 4$ and $n \geq 5$.*

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References

[1] R. J. R. Abel, F. E. Bennett, and M. Greig, "PBD-Closure," in *Handbook of Combinatorial Designs*, C. J. Colbourn and J. H. Dinitz (Editors), 2nd ed., Chapman & Hall/CRC Press, Boca Raton, FL, 2007, pp. 246–254.

[2] P. Adams, D. Bryant, and M. Buchanan, A survey on the existence of G-designs, *J. Combin. Des.* **16** (2008), 373–410.

[3] J.-C. Bermond, C. Huang, A. Rosa, and D. Sotteau, Decomposition of complete graphs into isomorphic subgraphs with five vertices, *Ars Combin.*, **10** (1980), 211–254.

[4] J.-C. Bermond and J. Schönheim, G-decomposition of K_n, where G has four vertices or less, *Discrete Math.* **19** (1977), 113–120.

[5] D. Bryant and S. El-Zanati, "Graph Decompositions," in *Handbook of Combinatorial Designs*, C. J. Colbourn and J. H. Dinitz (Editors), 2nd ed., Chapman & Hall/CRC Press, Boca Raton, FL, 2007, pp. 477–486.

[6] R. C. Bunge, S. I. El-Zanati, H. J. Fry, K. S. Krauss, D. P. Roberts, C. A. Sullivan, A. A. Unsicker, and N. E. Witt, On the Spectra of Bipartite Directed Subgraphs of K_4^*, *J. Combin. Math. Combin. Comput.*, to appear.

[7] C. J. Colbourn and J. H. Dinitz (Editors), *Handbook of Combinatorial Designs*, 2nd ed., Chapman & Hall/CRC Press, Boca Raton, FL, 2007.

[8] G. Ge, "Group divisible designs," in *Handbook of Combinatorial Designs*, C. J. Colbourn and J. H. Dinitz (Editors), 2nd ed., Chapman & Hall/CRC Press, Boca Raton, FL, 2007, pp. 255–260.

[9] G. Ge, S. Hu, E. Kolotoğlu, and H. Wei, A complete solution to spectrum problem for five-vertex graphs with application to traffic grooming in optical networks, *J. Combin. Des.* **23** (2015), 233–273.

[10] A. Hartman and E. Mendelsohn, The Last of the Triple Systems, *Ars Combin.* **22** (1986), 25–41.

[11] R. C. Read and R. J. Wilson, *An Atlas of Graphs*, Oxford University Press, Oxford, 1998.