Homework Problem Set 2 Solutions

1. For each of the following functions of \( x \) and \( y \), determine the partial derivatives

\[
\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \text{ and } \frac{\partial^2 f}{\partial y \partial x}.
\]

a.) \( f(x,y) = 5e^x y + y \)

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 5e^x y \\
\frac{\partial f}{\partial y} &= 5e^x + 1
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= 5e^x y \\
\frac{\partial^2 f}{\partial y \partial x} &= 5e^x
\end{align*}
\]

b.) \( f(x,y) = y \ln(x) + x \ln(x) \)

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{y}{x} + 1 + \ln(x) \\
\frac{\partial f}{\partial y} &= \ln(x)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= -\frac{y}{x^2} + \frac{1}{x} \\
\frac{\partial^2 f}{\partial y \partial x} &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{1}{x} \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{1}{x}
\end{align*}
\]

c.) \( f(x,y) = (xy^3)^{1/2} \) \text{ or } \( x^{1/2}y^{3/2} \)

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{1}{2} x^{-1/2} y^{3/2} \\
\frac{\partial f}{\partial y} &= \frac{3}{2} x^{1/2} y^{1/2}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= -\frac{1}{4} x^{-3/2} y^{3/2} \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{3}{4} x^{1/2} y^{-1/2}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{3}{4} x^{-1/2} y^{1/2} \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{3}{4} x^{1/2} y^{-1/2}
\end{align*}
\]
1.) continued

d.) \( f(x,y) = 3x^2 \cos y + xy^3 \)

\[
\left( \frac{\partial f}{\partial x} \right)_y = 6x \cos y + y^3 \\
\left( \frac{\partial f}{\partial y} \right)_x = -3x^2 \sin y + 3xy^2
\]

\[
\left( \frac{\partial^2 f}{\partial x^2} \right)_y = 6 \cos y \\
\left( \frac{\partial^2 f}{\partial y^2} \right)_x = -3x^2 \cos y + 6xy
\]

\[
\left( \frac{\partial^2 f}{\partial x \partial y} \right) = -6x \sin y + 3y^2 \\
\left( \frac{\partial^2 f}{\partial y \partial x} \right) = -6x \sin y + 3y^2
\]

e.) \( f(x,y) = e^{-3x} \left( 1 - x^2 \right) y^3 \ln(y) \)

\[
\left( \frac{\partial f}{\partial x} \right)_y = \left[ -3e^{-3x} \left( 1 - x^2 \right) - 6xe^{-3x} \right] y^3 \ln y = \left( 3x^2 - 6x - 3 \right) e^{-3x} y^3 \ln y
\]

\[
\left( \frac{\partial f}{\partial y} \right)_x = e^{-3x} \left( 1 - x^2 \right) \left( 3y^2 \ln y + y^2 \right)
\]

\[
\left( \frac{\partial^2 f}{\partial x^2} \right)_y = (6x - 2) e^{-3x} y^3 \ln y - 3 \left( 3x^2 - 2x - 3 \right) e^{-3x} y^3 \ln y = \left( -9x^2 + 12x + 7 \right) e^{-3x} y^3 \ln y
\]

\[
\left( \frac{\partial^2 f}{\partial y^2} \right)_x = e^{-3x} \left( 1 - x^2 \right) \left( 6y \ln y + 5y \right)
\]

\[
\left( \frac{\partial^2 f}{\partial x \partial y} \right) = \left[ -3e^{-3x} \left( 1 - x^2 \right) - 6xe^{-3x} \right] \left( 3y^2 \ln y + y^2 \right) = \left( 3x^2 - 6x - 3 \right) e^{-3x} \left( 3y^2 \ln y + y^2 \right)
\]

\[
\left( \frac{\partial^2 f}{\partial y \partial x} \right) = \left( 3x^2 - 2x - 3 \right) e^{-3x} \left( 3y^2 \ln y + y^2 \right)
\]
2. For each of the following functions of two variables, evaluate the two first partial derivatives. [Where it appears in the expressions below, \( R \) corresponds to the gas constant.]

a.) \( H(T,P) = \frac{3}{2} R \ln T - P \ln P + \frac{3T}{2P} \)

\[ \left( \frac{\partial H}{\partial T} \right)_P = \frac{3R}{2T} + \frac{3}{2P} \]

\[ \left( \frac{\partial H}{\partial P} \right)_T = -\ln P - 1 - \frac{3T}{2P^2} \] (the product rule is required here)

b.) \( s(v,t) = \frac{1}{2} vt^2 + ve^{-v} \)

\[ \left( \frac{\partial s}{\partial v} \right)_t = \frac{1}{2} t^2 + e^{-v} - ve^{-v} \] (the product rule is required here)

\[ \left( \frac{\partial s}{\partial t} \right)_v = vt \]

c.) \( P(V,T) = \frac{RT}{V} (1 + bV) \)

\[ \left( \frac{\partial P}{\partial V} \right)_T = -\frac{RT}{V^2} \]

\[ \left( \frac{\partial P}{\partial T} \right)_V = \frac{R}{V} (1 + bV) \]

d.) \( u(r,\theta) = \frac{3}{2} r^2 \cos \theta - re^r \sin \theta \)

\[ \left( \frac{\partial u}{\partial r} \right)_\theta = 3r \cos \theta - e^r \sin \theta - re^r \sin \theta \] (the product rule is required here)

\[ \left( \frac{\partial u}{\partial \theta} \right)_r = -\frac{3}{2} r^2 \sin \theta - re^r \cos \theta \]
2. continued

e.) \[ H(T,P) = \frac{3}{2}RT + RT^2Pe^{-3P} \]
\[
\left( \frac{\partial H}{\partial T} \right)_P = \frac{3}{2}R + 2RTpe^{-3P}
\]
\[
\left( \frac{\partial H}{\partial P} \right)_T = RT^2e^{-3P} - 3RT^2Pe^{-3P} \quad \text{(the product rule is required here)}
\]

f.) \[ P(V,T) = RT + RT\ln V \]
\[
\left( \frac{\partial P}{\partial V} \right)_T = RT\ln V + RT \quad \text{(the product rule is required here)}
\]
\[
\left( \frac{\partial P}{\partial T} \right)_V = R + RV\ln V
\]

3. For each of the following functions of three variables, evaluate the requested partial derivatives.

a.) \[ r = \sqrt{x^2 + y^2 + z^2} \] ; evaluate \( \left( \frac{\partial r}{\partial x} \right)_{y,z} \).

We must use the chain rule in this case,
\[
\left( \frac{\partial r}{\partial x} \right)_{y,z} = \frac{1}{2} \left( x^2 + y^2 + z^2 \right)^{-1/2} \cdot 2x
\]
\[
= x \left( x^2 + y^2 + z^2 \right)^{-1/2} \quad \text{(which also equals} \quad \frac{x}{r} \quad \text{)}
\]

b.) \[ x = r \sin \theta \sin \phi \] ; evaluate \( \left( \frac{\partial x}{\partial \phi} \right)_{r,\theta} \).
\[
\left( \frac{\partial x}{\partial \phi} \right)_{r,\theta} = r \sin \theta \cos \phi
\]
4. Evaluate the following expressions using the ideal gas equation of state.

a.) \( \left( \frac{\partial T}{\partial P} \right)_{V_m} \)

In this case, the partial derivative required involves \( T \) and also requires \( V_m \) to be held constant. Therefore, the ideal gas equation of state should be solved for \( T \) and written in terms of \( V_m \) before the partial derivative is evaluated,

\[
T = \frac{PV}{nR} = \frac{PV_m}{R}.
\]

Then, the partial derivative may be evaluated,

\[
\left( \frac{\partial T}{\partial P} \right)_{V_m} = \frac{V_m}{R}.
\]

b.) \( \left( \frac{\partial T}{\partial V_m} \right)_{P} \)

The partial derivative required here involves \( T \) and also requires a derivative of \( V_m \) to be evaluated. Therefore, the ideal gas equation of state should be solved for \( T \) and written in terms of \( V_m \) before the partial derivative is evaluated,

\[
T = \frac{PV}{nR} = \frac{PV_m}{R}.
\]

Then, the partial derivative may be evaluated,

\[
\left( \frac{\partial T}{\partial V_m} \right)_{P} = \frac{P}{R}.
\]
5. The isothermal compressibility $\kappa$ is defined by the relation

$$\kappa = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T,$$

and the expansion coefficient $\alpha$ is given by

$$\alpha = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P.$$

Evaluate these quantities for an ideal gas (assume that $n$ is constant).

For the isothermal compressibility, the partial derivative required involves $V$. Therefore, the ideal gas equation of state should be solved for $V$ before the partial derivative is evaluated,

$$V = \frac{nRT}{P}.$$

Next, the partial derivative may be evaluated,

$$\left( \frac{\partial V}{\partial P} \right)_T = -\frac{nRT}{P^2}.$$

Finally, the partial derivative may be substituted into the expression for the isothermal compressibility and simplified,

$$\kappa = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T = -\frac{1}{V} \left( -\frac{nRT}{P^2} \right) = \frac{P}{nRT} \left( \frac{nRT}{P^2} \right)$$

$$\kappa = \frac{1}{P}.$$

For the expansion coefficient, the partial derivative required also involves $V$. Therefore, the ideal gas equation of state should be solved for $V$ before the partial derivative is evaluated,

$$V = \frac{nRT}{P}.$$

Next, the partial derivative may be evaluated,

$$\left( \frac{\partial V}{\partial T} \right)_P = \frac{nR}{P}.$$

Finally, the partial derivative may be substituted into the expression for the expansion coefficient and simplified,

$$\alpha = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P = \frac{1}{V} \left( \frac{nR}{P} \right) = \frac{P}{nRT} \left( \frac{nR}{P} \right)$$

$$\alpha = \frac{1}{T}.$$
6. The van der Waals equation for a real gas is defined as \( P + \frac{a}{V_m^2} \left( V_m - b \right) = RT \), where \( V_m \) is the molar volume, \( R \) is the gas constant, and \( a \) and \( b \) are van der Waals constants. For the van der Waals equation, determine

a.) \( \left( \frac{\partial P}{\partial V_m} \right)_T \)

In order to evaluate the partial derivative, we must first solve the van der Waals equation for \( P \),

\[
P + \frac{a}{V_m^2} \left( V_m - b \right) = RT
\]

\[
P = \frac{RT}{V_m - b} - \frac{a}{V_m^2}
\]

Using this expression, the partial derivative may be evaluated,

\[
\left( \frac{\partial P}{\partial V_m} \right)_T = -\frac{RT}{(V_m - b)^2} + \frac{2a}{V_m^2}
\]

b.) \( \left( \frac{\partial^2 P}{\partial V_m^2} \right)_T \)

This second partial derivative is related to the first partial derivative obtained in part (a),

\[
\left( \frac{\partial P}{\partial V_m} \right)_T = -\frac{RT}{(V_m - b)^2} + \frac{2a}{V_m^3}
\]

So now we take the derivative again with respect to \( V_m \) (holding \( T \) fixed),

\[
\left( \frac{\partial^2 P}{\partial V_m^2} \right)_T = \frac{2RT}{(V_m - b)^3} - \frac{6a}{V_m^4}
\]